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Noncommutative Geometry

by

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for his contribution to the theory of operator algebras, particularly the general classification and a structure theorem for factors of type III, classification of automorphisms of the hyperfinite factor, classification of injective factors, and applications of the theory of C^*-algebras to foliations and differential geometry in general.

Abstract: Through algebraic geometry we became familiar with the correspondence between geometrical spaces and commutative algebra. The aim of this talk is to show an analogous correspondence, in the domain of real analysis, between geometrical spaces and algebras of functional analysis, going beyond the commutative case. This theory is based on three essential points:

1. The existence of many examples of spaces which arise naturally, such as Penrose's space of universes, the space of leaves of a foliation, the space of irreducible representations of a discrete group, for which the classical tools of analysis lose their pertinence, but which correspond in a very natural fashion to a noncommutative algebra.
2. The possibility of reformulating the classical tools of analysis such as measure, topology and calculus in algebraic and Hilbertian terms, so that their framework becomes noncommutative, the commutative case being neither isolated nor closed in the general theory.
3. The relationship with physics, the spaces used by physicists being noncommutative in many cases.

Noncommutative Geometry

I would like to give a general survey of noncommutative geometry. I will explain the motivation and the general program. For this, I will rely mainly on two things. First, the oldest example in noncommutative geometry, which goes back to the discovery of quantum mechanics by Heisenberg; I will then continue with pure mathematics, and will end by coming back to physics (in fact, by coming back to what may be extracted from the actual phenomenology of elementary particles about the fine structure of space-time).

So let me begin by explaining what the general motivation for noncommutative geometry is. There is a well-known duality which occurs, for instance, in algebra and geometry between a space and a commutative algebra. Given a space, we want to study it by looking at the algebra of coordinates on the space, which has to satisfy a certain regularity. Of course, if we are doing algebraic geometry, we restrict ourselves to polynomial or algebraic functions; but when dealing with topology or differential geometry the regularity is less restrictive, and, for instance, we use continuous functions or smooth functions.

The basic theme of noncommutative geometry is that there are several quite important cases in which one is forced to replace a commutative algebra of coordinates with a noncommutative algebra.

The first instance of this goes back to Heisenberg, and the second example is the need to consider spaces or manifolds which are not simply connected and whose fundamental group fails to be Abelian (arbitrary finitely presented discrete groups may occur in this way). For these spaces, the ordinary use of the Pontrjagin dual of the group is, of course, inefficient. The third example comes from foliations: If the space of solutions of a differential equation is treated as a classical space, then most of the standard tools completely lose their pertinence; in fact, this space is precisely an example of a *quantum space*, in the sense that it is described by an algebra of coordinates that fails to be Abelian. Finally, the fourth example, which is now quite fashionable, is *quantum groups*. Let me explain in one word what quantum groups are in this framework: Quantum groups are the analog of Lie groups in noncommutative geometry.

A brief sketch of the general program is as follows. We wish to be able to transplant into the noncommutative setting the tools that we are accustomed to in the classical commutative framework.

When looking at a space in the classical way, there are a number of points of view which are like "finer and finer," and enable us to comprehend the space. The coarsest of these points of view is measure theory. If we know the space only up to measure theory, then essentially we know nothing, because most spaces are isomorphic in measure theory: They are isomorphic to the unit interval with the Lebesgue measure. Then we have topology and differential geometry (by "differential geometry" I mean only the differential geometry of differential forms, currents, characteristic classes, i.e., excluding the differential geometry which comes from a Riemannian metric). The fourth and most important point of view is thus Riemannian geometry. I will sketch at the end what the relevance of the noncommutative analog of Riemannian geometry is for the physics of elementary particles.

First Example: Quantum Mechanics

I will start now by explaining the origin of the subject, which is the discovery by Heisenberg of quantum mechanics. I would like to show how much this discovery relied on experiments and got rid of the usual framework of classical mechanics, forced by the experimental results. Let us go back to Heisenberg, at the time when he discovered quantum mechanics (which was not called "quantum mechanics," but *matrix mechanics* for a reason which will become clear when we look at the way it was found).

At that time, by a great deal of work, people had already realized that the atom was formed by an inner nucleus, around which there were revolving electrons that governed the chemical properties of the atom. Moreover, a fairly good way of observing atoms was by interaction with electromagnetic radiation. For instance, if one takes a prism and allows sunlight to pass through it, then this light will be decomposed into various rays, and, of course, these rays will form the colors of the rainbow. However, if one takes pure bodies like helium or hydrogen and looks at their emission spectrum, then this emission spectrum will not contain all the rays in the sunlight. It will only contain certain rays, which essentially form a sort of "signature" of the elements in question. Thus, it is extremely important to be able to understand the regularity of these rays. Now, if we try to apply classical mechanics in order to understand this, then we take for the atom a very simple model. Using mathematical language, this model will be described by the so-called *phase space*, which is known to be a symplectic manifold, and the functions on this space will be the *observable quantities* of the system. I have been thinking of the system as being the atom with the electrons and the nucleus, and all the observable quantities evolving with time according to the *Hamiltonian evolution*, which is given by the following equation:

$$\dot{f} = \{H, f\}, \tag{1}$$

where $\dot{f}$ is the time derivative of f, and $\{H, f\}$ is the Poisson bracket between f and a certain observable which is called the *energy*, which is the Hamiltonian of the system.

Now, for simple systems like the hydrogen atom, this equation will be totally integrable, which means that there are invariant tori describing the motion of the system, and in these tori the motion is almost periodic. This tells us that each observable quantity can be computed and expanded as a function of time

$$q(t) = \sum q_{n_1,\ldots,n_k} \exp(2\pi i \langle n, \nu \rangle t), \tag{2}$$

where the coefficients $q_{n_1,\dots,n_k}$ are complex numbers, and $\langle n, \nu\rangle = \sum n_j \nu_j$ is a combination of the basic frequencies with the same integers n_j that appear in $q_{n_1,\dots,n_k}$.

If we take this mechanical system that describes the atom and try to describe in classical terms its interaction with radiation, then the answer is given by the Maxwell theory. Maxwell theory tells us that, when the atom is in interaction with radiation, it emits plane waves and these plane waves can be completely described as follows. Take the observable quantity, which is called the *dipole moment.* (What we have are electrons which are revolving around the nucleus; these electrons have a certain charge, and so they form a dipole moment around the nucleus.) This defines a certain observable quantity $\vec{Q}(t)$ which has three components and can be expanded in an almost periodic series

$$\vec{Q}(t) = \sum \vec{q}_n \exp(2\pi i \langle n, \nu\rangle t). \tag{3}$$

Maxwell theory tells us that any of the components $\vec{q}_n$ provides a plane wave W_n which has frequency $\langle n, \nu\rangle$. Thus, in particular, the observable frequencies should form a subgroup of the real line, generated by the basic frequencies ν_j.

It turned out, however, that observation was already giving at that time a result which was contradictory to this fact. If one observes, for instance, the spectral rays of the hydrogen atom, then one finds that the wavelengths of these rays are certain precise numbers which are, as I said before, a sort of signature of hydrogen. The regularity of these rays was already found by Balmer long ago. He observed that the wavelengths of these rays were all simple rational multiples of a certain length L. They are of the form

$$H_\alpha = \frac{9}{5}L\,, \quad H_\beta = \frac{16}{12}L\,, \quad H_\gamma = \frac{25}{21}L\,, \quad H_\delta = \frac{36}{32}L\,, \quad \dots \tag{4}$$

so what we are dealing with is really

$$\lambda = \frac{n^2}{n^2-4}L. \tag{5}$$

The first thing that people realized then was that it was much more natural not to talk about wavelengths (it is the wavelengths one observes when looking at the spectral rays), but to talk about frequencies, which are calculated as the speed of light divided by wavelengths. When we look at frequencies, we get a simpler formula

$$\frac{1}{\lambda} = \frac{R}{m^2} - \frac{R}{n^2}\,, \tag{6}$$

where R is a constant called the *Rydberg constant*, and m and n are integers. Now, from the experimental results we find that the observed frequencies do not form a group; that is, they do not form a subgroup of the real line. What happens, however, is that if we look at them (see figure 1) we see that they combine together. For instance, we can take the first ray in the *Lyman series* (1–2) and combine it with the first Balmer ray (2–3), obtaining the second Lyman ray (1–3). If we combine it instead with the second Balmer ray, then we get the third Lyman ray, and so on. So what happens is that they do not form a group, but they combine according to the so-called *Ritz–Rydberg combination principle*, which is the following: One can label the frequencies by two indices, say ν_{ij} (these two

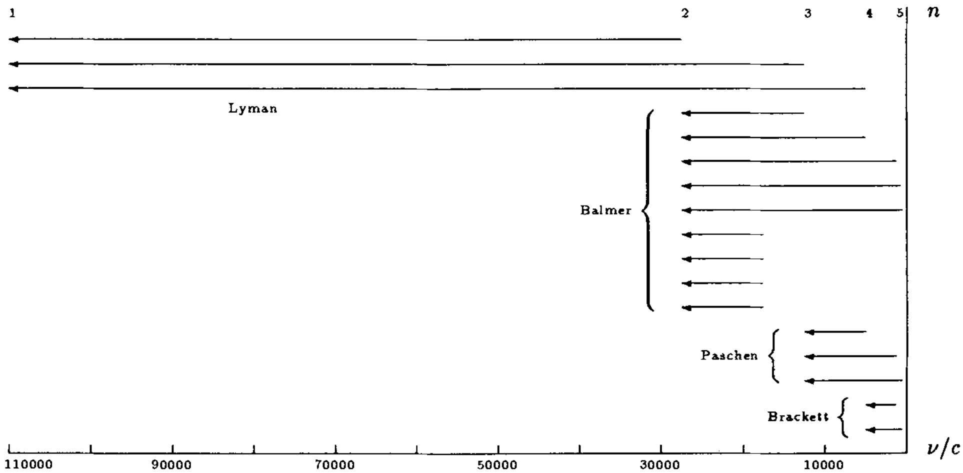


Figure 1

indices have nothing to do with integers; they may be whatever we want: Greek letters, colors, ...), and they combine according to the rule

$$\nu_{ij} + \nu_{jk} = \nu_{ik} \ . \tag{7}$$

This is what was found experimentally. Heisenberg used the following extremely pragmatic kind of reasoning: If one does a little bit of mathematics, then one finds that, in the classical case, there is an alternative way of describing the algebra of observable quantities. One takes almost periodic functions, which have the given frequencies, and multiplies them together by forming the convolution product

$$(qq')_{n''} = \sum_{n''=n+n'} q_n q_{n'} \ . \tag{8}$$

What is obtained is nothing other than the algebra of convolution of the group Γ, which is supposed to be the group of observable frequencies. However, as experiment shows, these do not form a group, so Γ is to be replaced by the set

$$\Delta = \{ \nu_{ij} = \nu_i - \nu_j \} \subset \mathbf{R} \tag{9}$$

of real numbers combining according to the Ritz–Rydberg combination principle. Heisenberg decided to follow the experimental results and to replace the commutative convolution algebra of the group Γ by the convolution algebra of the set Δ, and therefore to work out the convolution algebra of this set with its partially defined composition. It was found that the product of two observable quantities a and b was given by

$$(a \cdot b)_{ik} = \sum_j a_{ij} b_{jk} \ . \tag{10}$$

It is remarkable that Heisenberg invented this rule from experimental results, although he did not know about matrices. Later, he talked to Bohr and Jordan and found out that these things existed in mathematics and were called "matrices." This is why the theory was called *matrix mechanics.*

Then the law of evolution is quite simple; namely, it is given by

$$q_{ij}(t) = q_{ij}\exp(2\pi i \nu_{ij} t), \tag{11}$$

and something occurs which is even simpler than in the commutative case. Remember that the evolution was described by the Poisson bracket (the Poisson bracket was an additional structure which was coming from the symplectic structure of the phase space). Now, in the noncommutative case of matrices, this is not needed. It is replaced by the commutator

$$\frac{d}{dt}q(t) = \frac{i}{\pi}[H, q], \tag{12}$$

where H is a matrix which is zero outside the diagonal and whose diagonal entries are ν_i's such that $\nu_i - \nu_j = \nu_{ij}$. (This value ν_i is not unique; it is unique only up to a common addition of a constant.)

As a consequence of this, from the experimental results we cannot stick to a classical phase space; that is, we cannot stick to classical mechanics. Instead, we are obliged to replace the commutative algebra of functions of observable quantities of ordinary phase space with a noncommutative algebra.

It turns out that, in the case of Heisenberg, if we look at a system having finitely many levels (or even countably many) then the algebra that we get is not very complicated to analyse. But, for instance, as soon as we handle situations like quantum statistical mechanics —where one takes an assembly of atoms— then the noncommutative algebra that we are dealing with is much more difficult to analyse. But this would only pertain to the "measure theory" part of the discussion.

The Novikov Conjecture

After this motivating example of Heisenberg, I would like to enter the domain of pure mathematics, and deal with an example in which noncommutative geometry may be seen at work. The example is the following. We meet noncommutative objects as soon as we try to handle manifolds which are not simply connected. In fact, when we take a manifold M, to this manifold corresponds a group $\Gamma = \pi_1(M)$, its fundamental group, which measures the non-simply-connectedness of the manifold. Many of the results which are true for simply connected manifolds require more work when one tries to adapt them to the non-simply-connected world. Roughly speaking, the idea is that when one wants to adapt them to the non-simply-connected situation, one is no longer going to handle, for instance, vector spaces over the complex numbers, but modules over the group ring of a group. Basically, one is always taking into account the equivariance with respect to the action of the fundamental group, and, instead of doing things down in the manifold, one essentially has to work in the covering space. So I will try to show how noncommutative geometry works in examples, by dealing with the problem of the signature of a manifold.

I will first describe the signature theorem of Hirzebruch. If we take a manifold of dimension $4k$ which is compact and orientable, then there is an intersection form on the middle-dimensional cohomology and, by construction, the signature of this quadratic form turns out to be homotopy invariant, because it is defined in a homotopy invariant way. It is not clear at all whether it is possible to relate this quantity to other quantities which are computed, for instance, by characteristic classes of the tangent bundle of the manifold.

The following result is due to Hirzebruch:

$$\mathrm{Sign}(M) = \langle L(M), [M] \rangle; \tag{13}$$

that is, the signature can be computed by pairing the fundamental class $[M]$ of the manifold M with a universal polynomial $L(M) = P(p_1, \ldots, p_k)$ on the Pontrjagin classes of the tangent bundle of the manifold, which depends on the dimension of M.

There is a huge difference between the two sides of the equation (13), since the left-hand side is homotopy invariant by construction, and the right-hand side is essentially computable by local computations (by integration over the manifold). This is a fairly good answer for the simply connected case, and Novikov proved that this specific combination of characteristic classes is the only one that can be homotopy invariant.

Things get more interesting when the manifold is not simply connected. In the non-simply-connected case, there are quantities which are candidates for being homotopy invariant, but for which it is not obvious at all, at first sight, that they will be. These quantities are the *Novikov higher signatures*, which are defined as

$$\mathrm{Sign}_c(M) = \langle L(M) \cdot \varphi^*(c), [M] \rangle. \tag{14}$$

That is, we keep the same L-genus, but we have to be careful about one thing: Now the L-genus is not homogeneous, but it has several components. It has one component which is the top-dimensional component, and it also has components whose dimension differs from the dimension of M by multiples of 4. We multiply it by a group cocycle c of the fundamental group, after transferring it to a cohomology class on the manifold. (Since the manifold has a fundamental group Γ, the group cohomology of Γ maps very naturally to the cohomology of the manifold.) Then we compute the product $L(M) \cdot \varphi^*(c)$ and evaluate it on the fundamental class of the manifold M.

This is a fairly algebraic expression. For a more geometric definition, imagine that the so-called *classifying space* of the group Γ has been constructed. This is a certain space $B\Gamma$ that can be explicitly given in many cases. Essentially, we are taking a cocycle in $B\Gamma$, transversely oriented, considering a classifying map $\varphi: M \to B\Gamma$ transverse to the cocycle, and taking the inverse image $\varphi^*(c)$ of the cocycle.

The question is whether or not this signature is homotopy invariant. This is a purely geometric question, known as the *Novikov conjecture*. Novikov conjectured that in several cases, these quantities, which are called *higher signatures*, are homotopy invariant.

I would like to show how noncommutative geometry works in this case. Let me begin with the commutative case. I will first specifically discuss the situation when the fundamental group of the manifold is commutative, and we shall see how to make use of this commutativity. This is the proof given by Lusztig in this case. We shall see how many more tools we have when this group is commutative than in the noncommutative case.

If the fundamental group Γ is commutative, then it has a Pontrjagin dual $X = \hat{\Gamma}$, which is a compact space: the space of all linear characters of the group; that is, all homomorphisms from the group to the complex numbers of modulus one.

Now we can consider a product space $M \times X$ of the manifold and the Pontrjagin dual of its fundamental group. On this space we have a very canonical line bundle which is given by the fact that whenever we have a character, this character gives us a map from the fundamental group of M to the complex numbers of modulus one, and therefore a completely natural flat bundle on the manifold M with holonomy given by the character. Thus we get a family of flat bundles on M parametrized by the Pontrjagin dual X; that is, a natural line bundle on the product $M \times X$. It is not difficult now to consider the signature of this family (or the family of signature operators: For each of these flat bundles we have a signature with coefficients in the flat bundle, so that we can consider this family of operators). On one hand, the signature family is not just the difference between the dimension of the positive eigenvectors and the dimension of the negative eigenvectors; it is the subspace of the positive eigenvectors minus the subspace of the negative eigenvectors. What we really have is two vector bundles over the base X, and what we get in this way is not just a number, but an element of the so-called *K-theory* of X, which is denoted by $K(X)$. Once we have this element in the K-theory of the space X, it is not difficult to show, firstly, that this element is homotopy invariant (this is not much harder to prove than that the ordinary signature is homotopy invariant), and, secondly, that if one takes the Chern character of this family —just applying the Atiyah–Singer index theorem for families— one gets exactly the Novikov higher signature.

Now, the problem which I really would like to deal with is what replaces the Pontrjagin dual X, the K-theory of X, the Chern character, the index theorem, and so forth, when the group Γ (the fundamental group of the manifold) is no longer commutative.

So far, we have used the commutativity in an essential way. It was used in order to define the Pontrjagin dual, and to deal with this Pontrjagin dual in the standard way of commutative spaces.

The Group K_0

What is K-theory? K-theory is essentially doing linear algebra with parameters that vary continuously in a base space X. If we view things algebraically, K-theory is just doing linear algebra where the ground ring has been replaced from the complex numbers $\mathbf{C}$ to the ring $\mathcal{C}(X)$ of continuous functions on X. There is a purely algebraic definition of K-theory in terms of classification of finite projective modules. The fact that modules are finite corresponds to the fact that fibres are finite-dimensional, and *projective* is a translation of the fact that bundles are locally trivial. So, in fact, the meaning of doing K-theory over an algebra can be formulated in a purely algebraic way. Moreover, we find out very quickly that the commutativity of the algebra —the ground ring we are dealing with— has nothing to do with the problem. Therefore, we are free from the hypothesis of commutativity of $\mathcal{C}(X)$. As soon as we are dealing with finite projective modules, we need to represent them inside matrices over the algebra, as idempotents. But matrices over an algebra do not commute, so commutativity has nothing to do with the problem.

The second point is that if we take a discrete group Γ, then the construction of the Pontrjagin dual gives a noncommutative C^*-algebra rather than a commutative one. Let me explain how this is constructed. One takes the regular representation of the group in the space $\ell^2(\Gamma)$, the Hilbert space with orthonormal base formed by the elements of the group. In this Hilbert space, the group is acting by the left regular representation and so the group ring —the linearization $\mathbf{C}\Gamma$— is also acting. We simply take the norm closure of this group ring. If the group were Abelian, what we would get would be precisely the continuous functions on the Pontrjagin dual of the space (this is not difficult; it is just Fourier analysis). So, in general, we have a good replacement for this, except that it is not a commutative algebra.

There is a natural way to define the signature of the covering space of a manifold. If we look at the universal cover of the manifold, then on this universal cover we have the fundamental group acting, and we can mimic the usual construction of the signature on the universal cover. We can still consider the differential forms with a certain growth at infinity, and the cup product, which gives us a pairing and hence a quadratic form. It turns out that this quadratic form can be defined as an element of the so-called *Witt group*. The Witt group is a group of abstract quadratic forms over the group ring $\mathbf{C}\Gamma$. But the trouble with the Witt group is that it is defined abstractly by a presentation of quadratic forms (we want them to be equal if they differ by a change of variables, or if they are stably equal, and so forth). So it is difficult to analyse.

Now it should be clear why we take C^*-algebras. Precisely because C^*-algebras are the only algebras for which the spectrum of the self-adjoint elements is real. Why not take for example the algebra $\ell^1(\Gamma)$ of summable functions on Γ? This is a Banach algebra. But if we take a self-adjoint element in it, in general its spectrum will fill in the whole corona. It is not true for an involutive algebra in general that the spectrum of a self-adjoint element is real; this is precisely the characterization of C^*-algebras. Therefore, we take C^*-algebras precisely in order to be able to say that an element of the Witt group —a self-adjoint quadratic form $H = H^*$ that belongs to the ring of $q \times q$ matrices over an algebra $\mathcal{A}$— determines a positive eigenspace and a negative eigenspace. How do we get these positive and negative eigenspaces? When the spectrum is real, we do a Cauchy integral over a closed curve C enclosing the positive spectrum of H

$$\frac{1}{2\pi i}\int_C R_\lambda \, d\lambda, \tag{15}$$

where R_λ is the resolvent of the quadratic form. In doing this, by general results, we know that we get an idempotent projection. So this enables us to say that the Witt group in this situation maps to the K-theory (and, in fact, the Witt group is equal to the K-theory). Thus for C^*-algebras the main simplification is that the K-theory is the same as the Witt group, and K-theory is far simpler since it is just linear algebra.

Putting all these things together and using results of Wall–Mishchenko, we obtain that the signature of the universal cover $\tilde{M}$, taken equivariantly with respect to the fundamental group, is in fact an element of the K-theory of the C^*-algebra of the group

$$\mathrm{Sign}_c(\tilde{M}) \in K(C^*(\Gamma)). \tag{16}$$

The problem is as follows. If we were in the Abelian case, then this C^*-algebra would be the continuous functions on the Pontrjagin dual of Γ, and the next step would be trivial; it would just be to take the Chern character of this signature. (Of course, it would be nontrivial to compute this Chern character; here is where the Atiyah–Singer index theorem for families would come in. Nevertheless, there would be no need to define a new theory of signatures; we could just take the Chern character, and compute it.)

If the group is non-Abelian, we do not have the space. We would like to say that this C^*-algebra of Γ is like continuous functions on some space, but we do not have the Pontrjagin dual because the algebra can drastically fail to be Abelian. It turns out that what is needed in order to replace the Chern character is, first, to think about the theory of characteristic classes and to be able to understand the theory in such a way that it will still hold in the non-Abelian case. This gives *cyclic cohomology*, with which I will now deal. This theory is motivated very strongly by the example, in the sense that there is a need for a replacement for the calculations of curvature, characteristic classes, and so on, in this non-Abelian situation, where we cannot use the usual setting.

Cyclic Cohomology

Let me try to present cyclic cohomology as simply as possible. It is just a generalization of the notion of trace. If we have a noncommutative algebra $\mathcal{A}$, then there is a simple equality on a functional —on a linear form of this algebra— which enables us to erase the noncommutativity, i.e., which enables us to do many things as if the algebra were commutative. This is the notion of a *trace*

$$\tau: \mathcal{A} \to \mathbf{C}. \tag{17}$$

The trace satisfies the following cocycle condition:

$$\tau(a^0 a^1) - \tau(a^1 a^0) = 0. \tag{18}$$

A cyclic cocycle, in general, is just a higher trace. By *higher trace* I mean that it is again a functional, but on several variables in the algebra, and satisfying the following two conditions

$$\begin{aligned} &\tau(a^0 a^1, a^2, \ldots, a^{n+1}) - \tau(a^0, a^1 a^2, \ldots, a^{n+1}) + \cdots \\ &\quad \cdots + (-1)^n \tau(a^0, a^1, \ldots, a^n a^{n+1}) + (-1)^{n+1} \tau(a^{n+1} a^0, a^1, \ldots, a^n) = 0, \end{aligned} \tag{19}$$

$$\tau(a^1, a^2, \ldots, a^n, a^0) = (-1)^n \tau(a^0, a^1, \ldots, a^n). \tag{20}$$

A simple example of a cyclic cocycle appears in the situation where the algebra is the algebra of functions on a manifold. Assume given a de Rham current (recall that a *de Rham current of dimension k* is a linear form on differential forms of degree k). When I say that it is *closed* I mean that when it is paired with a closed form it yields 0.

If we start with a closed current c, then we can indeed define a multilinear functional on the algebra by the following formula:

$$\tau_c(a^0, \ldots, a^k) = \langle c, a^0 da^1 \wedge \ldots \wedge da^k \rangle, \tag{21}$$

and it is not difficult to show that it satisfies conditions (19) and (20) above. Condition (19) is just the fact that the differential of a product is given by the Leibniz rule, and condition (20) tells us that the current is closed, so we can integrate by parts in the current and this enables us to cyclically permute the variables.

In order to extend the previous functional to matrices by multilinearity, one simply has to extend it on tensor products of functions by matrices. There is only one natural formula that can be applied:

$$\tau_c'(a^0 \otimes \mu^0, a^1 \otimes \mu^1, \ldots, a^k \otimes \mu^k) = \tau_c(a^0, \ldots, a^k)\,\mathrm{Tr}(\mu^0 \cdots \mu^k), \tag{22}$$

where $a^0, \ldots, a^k$ are functions, $\mu^0, \ldots, \mu^k$ are $q \times q$ matrices, and Tr denotes the ordinary trace. Observe that this new expression is not invariant under all permutations, because the trace of a product is only invariant under cyclic permutations. It is precisely this small fact which forces us to consider only cyclic permutations.

Why are traces important? The trace on an algebra is important because the trace automatically gives a *dimension* to any finite projective module. If we have a finite projective module over an algebra, this module can be viewed as an idempotent in matrices. The trace extends to matrices, and when we evaluate the trace on the corresponding idempotent, it does not depend upon any choice.

A higher trace (i.e., a cyclic cocycle) gives us an invariant, exactly like the Chern character, for finite projective modules. We shall see by very simple examples that this reduces to the Chern character in the example of a current given above.

It turns out that the evaluation of a cyclic cocycle τ of even dimension on a diagonal element $\tau(e, e, \ldots, e)$, for $e \in \mathrm{Proj}(M_q(\mathcal{A}))$, is homotopy invariant. In other words, if we move the idempotent by deformation among idempotents, then this quantity does not change. How does one prove this? The point is the following: If you move an idempotent among idempotents, then, of course, it is a nice spectral deformation, because the spectrum of an idempotent is only formed by 0's and 1's, so there has to be a nice spectral equation to satisfy. This equation is

$$\dot{e}_t = [x_t, e_t] \tag{23}$$

for some element x_t. This equation is easily obtained by differentiating the equation $e_t^2 = e_t$. Now, when we differentiate $\tau(e, e, \ldots, e)$, we get an $\dot{e}$ appearing only once at a time, and then, by a little algebraic manipulation using the cocycle identity, we can prove that we get 0. So this is invariant under deformations, and, moreover, it is not difficult to prove that it only depends upon the isomorphism class of the finite projective module defined by e. Moreover, it is additive, so that if we take the direct sum of two finite projective modules —even if we have a monomial which is not linear— what we get is a sum of the corresponding traces.

This means that the so-called *cyclic cohomology*, where elements are cyclic cocycles modulo an obvious relation, pairs with K-theory, so each cocycle class defines a map from the K_0 of the algebra, $K_0(\mathcal{A})$, to the scalars.

Let me show with this example of currents, first, how this computation reduces to the Chern–Weil computation by connections and curvature for vector bundles. I showed that if we have a closed de Rham current on M, then we have a cyclic cocycle. Of course, if we want to know the Chern character pairing, it is enough to know how the Chern character pairs with any closed de Rham current, because the closed de Rham currents generate the homology of the manifold. So what we have to do is to show the equality

$$\langle \tau_c, [E] \rangle = \tau_c(e, \ldots, e) = \langle \mathrm{ch}(E), c \rangle, \tag{24}$$

where $[E]$ is the finite projective module of the vector bundle E, and c also denotes the homology class of the current. How does one prove this? The finite projective module of a vector bundle is given by an idempotent. A more geometric way of formulating this is to say that the vector bundle is the pull-back of the canonical vector bundle on the Grassmannian by a map from the manifold to the Grassmannian, because when we take an idempotent in $n \times n$ matrices over the algebra, just by a matter of translation, this is exactly a map from the space to the set of idempotents of $n \times n$ matrices, which is the Grassmannian. On the Grassmannian we have a canonical connection, which comes from the orthogonal projection from one fibre (i.e., from one vector space) to the nearby vector space. Now we can pull back this canonical connection.

If we compute the curvature of this connection, we will find that it is given as a matrix of differential forms $edede$, where de is the differential of this map e. And so, when we pair the curvature to some power with the current, we immediately see that we get $\tau_c(e, e, \ldots, e)$.

What we have done is to translate algebraically the pairing in such a way that, firstly, it is now completely free of the commutativity hypothesis; and, secondly, that it relates in fact to cohomology which is well defined, because, if one looks at the definition of a cyclic cocycle, one easily understands that condition (19) is the condition of being closed. Observe that the sum appearing in this condition is nothing other than the Hochschild coboundary of the cochain we are dealing with. And condition (20) is a restriction to cyclic cochains, which turns out to be stable under coboundary, so what we get is a complex, and out of this complex we get, of course, a cohomology theory which is cyclic cohomology.

The Equivariant Index

Let us look at other examples. Take the group ring of a discrete group, $\mathbb{C}[\Gamma]$, and suppose given a group cocycle $c(g_1, \ldots, g_n) \in \mathbb{C}$, $g_i \in \Gamma$.

Remember that when we were considering the Pontrjagin dual, we wanted to compute the Chern character. The problem was, of course, only to be able to pair this Chern character with group cocycles. Now we still have the group, but it is not Abelian, so we cannot talk about the Pontrjagin dual. However, we have the group ring, and the claim is that the following extremely simple formula

$$\tau_c(g^0, \ldots, g^n) = \begin{cases} 0 & \text{if } g^0 g^1 \cdots g^n \neq e \\ c(g^1, \ldots, g^n) & \text{otherwise} \end{cases} \tag{25}$$

assigns to every group cocycle a cyclic cocycle on the group ring.

Now the main trouble is that we do not have the Atiyah–Singer index theorem. Remember that when we were doing the calculation in the case of the Pontrjagin dual of an Abelian group, we used the Atiyah–Singer index. So we need a replacement for it. This replacement is a theorem which will not only handle the signature operator, but in fact will handle an arbitrary elliptic Γ-invariant operator on the covering space $\tilde{M}$.

If we are given a differential operator D which is elliptic on the manifold M, we can always lift it (because it is local) to the covering space, into an operator $\tilde{D}$ which is Γ-invariant and still elliptic. It turns out that, while an operator downstairs has a *parametrix* (an inverse modulo smoothing operators) the operator on the covering also has a parametrix, but this one is not an inverse modulo smoothing operators: It is an inverse modulo $R\Gamma$, the group ring of Γ extended by the smoothing operators. (R denotes the ring of smoothing operators on the base, which does not depend on the manifold.)

In fact, the index for the operator is an element of the K-theory of the group ring of Γ extended by the smoothing operators, $\mathrm{Ind}_\Gamma \tilde{D} \in K_0(R\Gamma)$, which is called *$\Gamma$-equivariant index.* Now the following theorem holds, which is exactly a higher analog of the Atiyah–Singer index theorem, in the same way as the Novikov higher signature is the analog of the ordinary signature.

THEOREM (CONNES–MOSCOVICI). *If c is a group $2q$-cocycle on the group Γ, then*

$$\langle \tau_c, \mathrm{Ind}_\Gamma \tilde{D}\rangle = \frac{1}{(2\pi i)^q}\frac{q!}{(2q)!}\langle \mathrm{ch}\sigma_D \cdot \mathrm{Td}(M)\cdot [c], [M]\rangle. \tag{26}$$

Here Td stands for the *Todd genus* of the complexified tangent bundle. This formula contains two new terms with respect to the Atiyah–Singer formula; namely, the numerical constant $q!/(2q)!$, which takes care of the dimension of the cocycle, and the factor $[c]$, which is the class of the group cocycle c viewed on the manifold M.

Now we apply this to the signature operator on the covering and get the following:

COROLLARY. *If D is the signature operator, then*

$$\langle \tau_c, \mathrm{Ind}_\Gamma \tilde{D}\rangle = \textit{Novikov higher signature} = \langle L[M]\cdot \varphi^*(c), [M]\rangle. \tag{27}$$

We could say that now the Novikov conjecture is solved in general; but there is still one technical problem. (Nevertheless, the conjecture is solved for a generic family of groups, namely *Gromov hyperbolic groups*. I will not give the technical definition of these groups, but I will mention the technical reason which still restricts the proof to these groups.)

When dealing with the Abelian case, one has smooth functions, and smooth functions have the following two rather important properties. The first property is that the K-theory of the algebra of smooth functions is the same as the K-theory of continuous functions

$$K(\mathcal{C}^\infty(M)) \xrightarrow{\cong} K(\mathcal{C}(M)). \tag{28}$$

The second is that the cyclic cocycles that we get automatically extend from the group ring $\mathbf{C}\Gamma$, which is like Laurent polynomials, to smooth functions. It turns out that, when we take non-Abelian groups, then this problem —which is quite trivial in the Abelian case— has analytical difficulties. However, this technicality can be solved for Gromov hyperbolic groups.

I will now explain in what sense these groups are generic. By a result of Gromov, if we look at (finitely presented) groups given by generators and relations, pick a finite number of generators, bound the length of the relations, and count among the obtained groups those which are hyperbolic, then the percentage of these tends to 100% as length tends to infinity.

We would like to have the Novikov conjecture true in general. It might be that the above technical problem is indeed essential, and the conjecture is only true for groups for which one has a sort of analytical control. It is very important to deal with such questions, because it is not only for the Novikov conjecture that they are relevant. What one is really dealing with in this situation is analysis on the dual of a discrete group. This is a quantum space (it is not Abelian) and this analysis is much more complicated and rich than in the Abelian case. Essentially, the Abelian case is a sort of finite-dimensional case, while in the non-Abelian case, because of the growth of the groups, we have phenomena which are infinite-dimensional in nature.

Riemannian Geometry

At the beginning I explained the original motivation of Heisenberg. Then I showed by means of some examples how the idea of noncommutative geometry can be used in specific examples. In the foregoing discussion, I have been dealing only with topology, K-theory, differential forms, and characteristic classes. Now I want to discuss the very essential part of geometry that deals with the measurement of lengths, i.e., Riemannian geometry. This is by no means finished. Although only a small part of it is complete, I think it will have applications in physics, so I want to return to physics in the last part of my talk.

In doing noncommutative geometry, one arrives after many examples at the following notion of what might replace the notion of Riemannian manifold. This notion, on one hand, will cover the finite-dimensonal case; but in Riemannian geometry it will do something more: It will mix the discrete and the continuum. (Riemannian geometry, as it is usually known, deals only with the continuum and does not handle the discrete.) It will also make it possible to handle nonintegral dimensions, like Hausdorff dimensions. For instance, if we have a circle which is winding in a set of higher Hausdorff dimension in the plane, it will enable this to be handled exactly as it would be in Riemannian geometry, but the functions will not be differentiable.

Let me come back to the fundamentals of Riemannian geometry. Riemannian geometry deals with a certain metric space where the distance is computed as the infimum of the arc-lengths, $d(p,q) = \mathrm{Inf}\ \{ \int_p^q ds \}$, where the length of an infinitesimal arc is given by the square root of a quadratic form

$$ds = \sqrt{\sum g_{\mu\nu} dx^\mu dx^\nu}\ . \tag{29}$$

This geometry is amazingly relevant for two reasons. One is that it has a wide variety of examples, and the second is that many tools are available. In particular, all the tools of differential and integral calculus are available and make computations possible.

At first, Riemannian geometry was meant to be a generalization of Euclidean and non-Euclidean geometries, so there was the temptation of restricting it to extremely special spaces, like the ones in which rigid motion is possible. General relativity has shown that this would be a mistake, because in general relativity one is obliged to consider all possible spaces of a certain kind, and one is obliged to vary among them.

Let me now turn from this to a more algebraic standpoint. We take the algebra $\mathcal{A}$ of functions on the manifold M —no regularity is assumed— and this algebra is supposed to act on a Hilbert space. This Hilbert space is the space of L^2-spinors $\mathcal{H} = L^2(M, S)$. Moreover, I take as given the Dirac operator D; that is, what we are given is an algebra of functions together with a representation. But if we were just handling representations, we would have nothing to work with. I want to add some finiteness condition. This finiteness condition is given by the Dirac operator D, which is finite in the sense that its inverse is compact, or in the sense that its eigenvalues go to infinity. And it is compatible with the algebra of functions, in the sense that if we permute the Dirac operator with functions, they do not commute, but what we get is bounded. (The Dirac operator is not bounded.)

I will next show how to recover the manifold M, the geodesic distance $d(p,q)$ on M, the Riemannian volume, the integration of functions, the gauge potential, and the Yang–Mills action, out of the purely operator-theoretic data $(\mathcal{A}, \mathcal{H}, D)$. And this will be done in such a way that we will not be limited to Riemannian manifolds, but after a while will be able to handle discrete spaces as well.

Let me go very briefly through the way the manifold M is recovered. We have the algebra, yet we do not quite have the regularity. We recover the regularity by asking that the commutator be bounded. Then by closing we get the algebra of continuous functions. By the well-known duality between the algebra of continuous functions and the points, we recover the points as a compact topological space

$$\begin{aligned} M &= \text{Spectrum of the } C^*\text{-algebra } \overline{\mathfrak{a}}; \\ \mathfrak{a} &= \left\{ a \in \mathcal{A} \;\middle|\; [D,a] \text{ is bounded } \right\}. \end{aligned} \tag{30}$$

Let us look at the distance, which is much more interesting. The usual formula for the distance is the infimum over all arcs. I will replace this formula with a formula which will give the same answer (i.e., the geodesic distance), but which will be dual; instead of considering arcs embedded in the manifold, I will consider coordinates. I want to measure the distance between two points as follows:

$$d(p,q) = \operatorname{Sup}\left\{ \; |a(p) - a(q)| \;\middle|\; \|\,[D,a]\,\| \leq 1 \; \right\}. \tag{31}$$

Let us check that this is true. When we compute the commutator $[D,a]$, we find that this is Clifford multiplication by the gradient ∇a of the function a. To say that this operator has norm less than or equal to one is precisely to say that, at each point, this gradient has a length less than or equal to one. By a simple argument, this is precisely to say that the function is Lipschitz for the geodesic distance, with Lipschitz constant equal to one:

$$\frac{\operatorname{Sup} |a(p) - a(q)|}{d(p,q)} \leq 1. \tag{32}$$

Thus we immediately see that one inequality is indeed given. To get the other inequality, we just take the function which is the geodesic distance to a given point p. This function is Lipschitz, so we can put it on the right-hand side and we are done.

What we get here is the same geodesic distance as usual. However, the measurement has been different, and, in fact, when we are doing measurements —not of long lengths, but of very small lengths— we are perfectly unable to use a path. It could be said, for instance, that a photon has a trajectory which is a path going from one point to another point. Yet this is not true: The photon in quantum mechanics is a plane wave having a definite momentum, so that there is no path of a photon, and, in fact, we are not measuring the distance by the formula of infimum of arc-length, but precisely by the formula (31).

Having this formula does not account for much, because we need to be able to integrate functions. There is an analysis of the residue —that is, what is called the *Dixmier trace* of operators on the Hilbert space— which enables us to write down the volume form in the Riemannian case purely operator-theoretically from the Dirac operator:

$$\int_M f\,dv = \mathrm{Tr}_\omega(fD^{-p}), \tag{33}$$

where p is the dimension, i.e., the order of growth of the eigenvalues of D: $\lambda_n \sim n^{1/p}$. This is related to the Tauberian theorem, in the sense that if we take the functional Tr_ω —the Dixmier trace, which is not the ordinary trace— then it is related to the residue of the zeta-function of the operator at the point 1.

This is a trace which was discovered by Dixmier in 1966. Essentially, his paper was never read: It remained completely hidden in the literature for a very long time; but from the work of Manin–Wodzicki and Guillemin I noticed that the residue of pseudo-differential operators was the same trace, except that the Dixmier trace exists in general. It is not particular to the case of differential operators, or the set up of a manifold. So it could be used in general to perform integration in this general Riemannian-theoretic situation. And now there is this quite amazing fact that the Hausdorff measure (for instance, on the boundary of quasi-Fuchsian groups) is also given precisely by the Dixmier trace, although we are now in the non-integral-dimensional situation.

Thus one constructs first the integration of functions, the distance, and then proceeds to construct gauge theory. To construct gauge theory, one uses the Dirac operator, defines connections, vector bundles, curvature, and so on.

Let us go to the key point. This gauge theory has exactly the same features as the ordinary gauge theory. In particular, it is only in dimension 4 that one has a general theorem which relates the second Chern class with the Yang–Mills action. This follows from a completely general theorem using the Dixmier trace, which, in fact, justifies the Dixmier trace and provides an inequality showing that the gauge theory is not trivial when the second Chern class is not trivial.

THEOREM. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a triple with $D^{-1} \in \mathcal{L}^{2n+1}$. Then:*

1. *The equality $\varphi(a^0, \ldots, a^{2n}) = \mathrm{Tr}_\omega(\gamma a^0[D, a^1] \cdots [D, a^{2n}]D^{-2n})$ defines a Hochschild cocycle on $\mathcal{A}$.*

2. *The class of φ is the same as the class of the Chern character of the K-homology class of $(\mathcal{A}, \mathcal{H}, D)$.*

Now I would like to show what happens because of the fact that the theory is not limited to the continuum. We may consider a space which is a product space of a continuum (the ordinary four-dimensional continuum) by a discrete space, and the simplest discrete space we can take is a two-point space. One translates algebraically the meaning of taking a product by

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 , \\ \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 , \\ D &= D_1 \otimes 1 + \gamma_1 \otimes D_2 . \end{aligned} \tag{34}$$

Let us do gauge theory for this two-point space. The two-point space is described by the algebra $\mathcal{A} = \mathbf{C} \oplus \mathbf{C}$, since functions on the two-point space are just given by two complex numbers $(f(a), f(b))$. What is the Dirac operator there? By a general theory which is called *K-homology*, it can be shown that it reduces to the following form: The Hilbert space is of the form $\mathcal{H} = \mathbf{C}^N \oplus \mathbf{C}^N$; the algebra will act by the matrices

$$\begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} \tag{35}$$

and the Dirac operator D will be off-diagonal and, of course, self-adjoint:

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} \tag{36}$$

for a certain $N \times N$ matrix M. So this is the structure that we want to consider on the space.

Now, the first thing we have to do is to compute the distance. If we take for the distance the formula given by the infimum over the arc-lengths, then, since we have a two-point space, we will get nothing, because there is no arc in the two-point space. But we have the other formula, and we can compute the distance between our two points. Using the formula (31) we get

$$d(a,b) = \mathrm{Sup}\left\{ \, |f(a) - f(b)| \, \middle| \, \| [D,f] \| \leq 1 \, \right\} = \frac{1}{\lambda} , \tag{37}$$

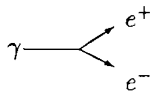
where λ is the norm of the matrix M, that is, the square root of the largest eigenvalue of M^*M. Then we compute the gauge potential, the Yang–Mills action, and we find a term that is precisely the so-called *symmetry-breaking term*, which physicists were obliged to introduce in order to assign masses to elementary particles.

Then we go a little further and ask ourselves what are vector bundles over a two-point space. Of course this is very trivial: A vector bundle is given by two fibres, $\mathbf{C}^k$ and $\mathbf{C}^{k'}$, and a nontrivial bundle is one in which $k \neq k'$. We pick the simplest nontrivial bundle, which has fibres of dimension 1 in one point and of dimension 2 in the other point.

Once it is seen in detail what is the Riemannian case in dimension 4 and the two-point discrete case, we can look at the product case. When we take the product of these two spaces and compute what is the gauge theory, we find exactly what physicists have been given, too, by elementary particle physics in the so-called *Glashow–Weinberg–Salam model.* One finds a Lagrangian which comprises many more terms than the usual Lagrangian. Ordinarily, from Maxwell theory and Dirac theory, we know that the theory of quantum electrodynamics is described by one Lagrangian

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_f + \mathcal{L}_{fB} , \tag{38}$$

which has a pure gauge potential part $\mathcal{L}_B$, a fermionic part $\mathcal{L}_f$, and an interaction between fermions and bosons given by the diagram



which tells us how a photon can give a positron and an electron, for instance. This Lagrangian is that of quantum electrodynamics.

In this century, it has been understood that quantum electrodynamics was not enough to describe the so-called *electroweak interaction*. In fact, it has been discovered that there is a nuclear beta decay, that there is radioactivity (which was discovered at the end of the last century), and, gradually, with a lot of experiments, people have been led to the following experimental Lagrangian:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_f + V(H) + \mathcal{L}(B, H) + \mathcal{L}(f, H). \tag{39}$$

where $V(H)$ is the *Higgs potential*, $\mathcal{L}(B, H)$ is the *minimal coupling*, and $\mathcal{L}(f, H)$ is the *Yukawa coupling*.

By doing a small calculation in noncommutative Riemannian geometry, I have shown that if one alters the space a little bit by crossing it with a discrete set of two points, then space-time becomes like a product of ordinary space-time by two points, and these two points are extremely close: If one computes their distance, one finds something like 10^{-16} cm.

The idea consists of not just introducing new dimensions, but to pick a discrete fibre. Now, when we compute the Lagrangian as explained above, for this new Riemannian space we find exactly the standard model with all its five terms (39).

At the moment, in order to incorporate quarks, one has to do a little more. There are two copies of the space, i.e., there are two sides: one is *left-handed* and the other is oriented the other way. In order to incorporate quarks, instead of considering only scalar-valued functions on the left-handed copy, we have to consider quaternionic-valued functions. The algebra of quaternions is slightly noncommutative (by "slightly noncommutative" I mean that they satisfy polynomial identities; they are not something which is of high dimension with respect to matrices).

The general idea is that in order to understand space-time, it may be important not to be limited to ordinary Riemannian connected manifolds and to allow a more general notion of space-time —a more general notion of Riemannian geometry— based on operator-theoretic data and which makes it possible to talk about "effective" space-time. I am by no means saying that this is the final answer on space-time. What I am saying is that if we take this space-time and compute the analog of quantum electrodynamics on it, then we get precisely the complicated Lagrangian above (39). So this gives us a better geometric understanding of the finest existing effective model of elementary particle physics.

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