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GROUP COHOMOLOGY WITH LIPSCHITZ CONTROL AND HIGHER SIGNATURES

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Introduction

Motivated by analytic aspects in the study of non-simply connected manifolds, we introduce and exploit in this paper a certain type of cohomology for finitely generated discrete groups Γ, which takes into account the metric structure of such a group given by a word-length metric. The basic objects used to produce this cohomology are families of Lipschitz contractions from Γ to finite-dimensional Euclidean spaces $\mathbf{R}^N$. The resulting cohomology classes, to be called *Lipschitz classes*, form a subspace $H_L^*(\Gamma)$ of the ordinary group cohomology with real coefficients $H^*(\Gamma) = H^*(B\Gamma, \mathbf{R})$. By imposing an additional properness condition, one obtains a more restricted variant, namely the *proper Lipschitz cohomology* $H^*_{L,\mathrm{pr}}(\Gamma)$.

Carrying a "Lipschitz structure" gives a significant advantage to a group cohomology class. In particular, as we show early on in the paper (cf. sec. I.10), every proper Lipschitz class gives rise to a homotopy invariant higher signature, for any closed oriented manifold mapping to $B\Gamma$. This, in fact, follows immediately from Mishchenko's higher signature theorem [M], once the construction of Lipschitz classes is given a K-theoretical counterpart by means of Kasparov's intersection product [K].

Thus, it becomes important to recognize the kinds of groups Γ for which the proper Lipschitz cohomology exhausts the cohomology of $B\Gamma$. After some familiarity with the concept is developed (sec. I.1-8), one can see fairly easily that $H^*_{L,\mathrm{pr}}(\Gamma) = H^*(\Gamma)$ if Γ is a discrete subgroup of the group of isometries of a complete, simply connected Riemannian manifold of non-positive sectional curvature (cf. sec. I.9). The same is true for discrete subgroups of an almost connected Lie group or of an algebraic group over a local field. The most difficult, but perhaps the most interesting, case is that of a general (word) hyperbolic group. The proof that all cohomology classes of any subgroup of such a group are properly Lipschitz occupies the entire chapter II. Its essential ingredients can be summarized as follows:

(1) a procedure of symmetrization at infinity, allowing the transformation of a proper map $\alpha : M \to \mathbb{R}^N$ into a Lipschitz map, provided there exists a proper self-contraction $f : M \to M$ with Lipschitz constant < 1;

(2) an "extension lemma" for Lipschitz maps, with control of the Lipschitz constant, which plays in the context of this paper a role analogous to the Hahn-Banach theorem for locally convex vector spaces;

(3) an estimate of the Lipschitz constant of the obvious candidate for a self-contraction on a δ-hyperbolic space, when restricted to a set of points whose mutual distances are sufficiently large with respect to the hyperbolicity constant δ.

Note that, at this stage, we have already recovered the proof of the Novikov conjecture on the homotopy invariance of higher signatures, in all cases which were previously known (compare [K], [CM], [KS]).

In chapter III we extend the homotopy invariance of higher signatures to non-proper Lipschitz cohomology classes. The proof, which is considerably more elaborate than in the proper case, is based on concepts and techniques developed in [C2]. Thus, in the spirit of loc. cit., we establish a "reverse index theorem" asserting that to each $\xi \in H^*_L(\Gamma)$ one can associate a suitable "analytic index map" $L \in \mathrm{Hom}_{\mathbb{Z}}\left(K_*(C^*(\Gamma), \mathbb{C}\right)$. This means that for any element $x \in K_*(B\Gamma)$ its "higher topological index" $\langle \mathrm{ch}_* x, \xi\rangle$ can be expressed as $L\big(\mu(x)\big)$, where $\mu : K_*(B\Gamma) \to K_*\big(C^*(\Gamma)\big)$ denotes the analytic assembly map (of [BC] and [K]).

Non-proper Lipschitz cohomology classes arise naturally in the context of continuous cohomology for topological groups made discrete. The general principle appears to be the following: if $\imath : \Gamma \to G$ is a homomorphism from a finitely generated discrete group to an almost connected topological group, then $\imath^*\big(H^*_{\mathrm{cont}}(G, \mathbb{R})\big) \subset H^*_L(\Gamma)$. In the present paper we prove the validity of this principle in two important cases:

(a) when G is locally compact, and

(b) when $G = \mathrm{Diff}^+(M)$ is the group of orientation preserving diffeomorphisms of a closed manifold M and $\imath^*$ is restricted to the subring of $H^*_{\mathrm{cont}}(G, \mathbb{R})$ generated by Gelfand-Fuchs classes via integration along the fiber.

I. Proper Lipschitz Cocyles

1. Multiproper maps $\Gamma \to \mathbf{R}^N$.

Let Γ be an abstract discrete group, consider a map $\alpha : \Gamma \to \mathbf{R}^N$, and let us try to pull back to Γ the fundamental class of $\mathbf{R}^N$ in the real cohomology with compact support, say $c \in H^N_{\text{comp}}(\mathbb{R}^N)$. We represent c by a (necessarily closed) N-form ω on $\mathbb{R}^N$ with compact support and we first define a non-Γ-invariant N-cocycle c^* on Γ by integrating ω over the N-simplices in $\mathbb{R}^N$ spanned by the α-images of $(N+1)$-tuples of points in Γ. That is,

$$c^*(\gamma_0, \ldots, \gamma_N) = \int_\Delta \alpha^*(\omega) ,$$

where Δ is the abstract N-simplex spanned by $\gamma_0, \gamma_1, \ldots, \gamma_N$ and the map α is extended (from $\gamma_0, \ldots, \gamma_N$) by linearity to a map of Δ to $\mathbb{R}^N$ also called α.

Next, in order to make c^* Γ-invariant we try to sum it over Γ and set

$$\overline{c}^* = \sum_{\gamma \in \Gamma} \gamma c^* ,$$

i.e.

$$\overline{c}^*(\gamma_0, \ldots, \gamma_N) = \sum_{\gamma \in \Gamma} c^*(\gamma\gamma_0, \ldots, \gamma\gamma_N) . \tag{1.1}$$

Notice that the above infinite sum would make perfect sense if there were only finitely many non-zero terms. This motivates the following definition.

2. DEFINITION:

A map $\alpha : \Gamma \to \mathbb{R}^N$ is called *multiproper* if for every finite subset $F \subset \Gamma$ and every compact subset $B \subset \mathbb{R}^N$ there exist at most finitely many $\gamma \in \Gamma$ such that the convex hull of the image of the γ-translate of F meets B, i.e.

$$B \cap \operatorname{Conv} \alpha(\gamma F) = \emptyset , \tag{1.2}$$

for almost all $\gamma \in \Gamma$.

Notice that "multiproper" reduces to the usual "proper" if (1.2) is required only for the one-point set $F = \{\text{id}\} \in \Gamma$.

Now, if α is multiproper, then the sum $\sum_\Gamma$ in (1.1) does have only finitely many non-zero terms and so the definition of the N-cocycle $\overline{c}^*$ on Γ is meaningful. Then we observe that the cohomology class of $\overline{c}^*$ in $H^N(\Gamma; \mathbb{R})$ depends only on $c = [\omega] \in H^N_{\text{comp}}(\mathbb{R}^N)$ and so we have our pull-back

$$\overline{\alpha}^*(c) \underset{\text{def}}{=} [\overline{c}^*] \in H^N(\Gamma) .$$

3. EXAMPLE:

Let Γ admit a compact smooth manifold X for a *classifying space.* This means that the universal covering $\widetilde{X}$ of X is contractible and the fundamental group $\pi_1(X)$ is isomorphic to Γ. Then every continuous proper map $A : \widetilde{X} \to \mathbb{R}^N$ induces a homomorphism

$$A^* : H^*_{\text{comp}}(\mathbb{R}^N) \to H^*_{\text{comp}}(\widetilde{X}) \ .$$

(If A is smooth, one can define A^* by pulling back forms ω from $\mathbb{R}^N$ to $\widetilde{X}$. If A is non-smooth, one can apply this to a smooth approximation of A.) Then, by summing over $\Gamma = \pi_1(X)$ which acts on $\widetilde{X}$ by deck transformations, one obtains a homomorphism

$$H^*_{\text{comp}}(\widetilde{X}) \to H^*(X) \ ,$$

whose composition with A^* is denoted by $\overline{A}^*$:

$$\overline{A}^* : H^*_{\text{comp}}(\mathbb{R}^N) \to H^*(X) \ .$$

If A is smooth, then the image $\overline{A}^*(c)$ of a class $c \in H^*(\mathbb{R}^N)$ given by a form ω on $\mathbb{R}^N$ can be represented by the form

$$\overline{\omega}^* = \sum_{\gamma \in \Gamma} \gamma A^*(\omega)$$

on $X = \widetilde{X}/\Gamma$.

Notice, that we have not used the contractibility of $\widetilde{X}$ so far. This becomes important only if we want to compare $\overline{A}^*$ to $\overline{\alpha}^*$. Recall that every isomorphism $\pi_1(X) \to \Gamma$ induces a homomorphism $h : H^*(\Gamma) \to H^*(X)$ and in the case where $\widetilde{X}$ is contractible, h is an isomorphism. So, with a fixed isomorphism between Γ and $\pi_1(X)$, we can identify the cohomology of Γ with that of X.

Next, by restricting A to the Γ-orbit of a base point $\widetilde{x}_0 \in \widetilde{X}$, we obtain a map $\alpha : \Gamma \to \mathbb{R}^N$ for

$$\alpha(\gamma) = A(\gamma \widetilde{x}_0) \ ,$$

and whenever α is multiproper we have the homomorphism

$$\overline{\alpha}^* : H^*_{\text{comp}}(\mathbb{R}^N) \to H^*(\Gamma) \ .$$

Then, under a very mild assumption on A, this homomorphism does not depend on the choice of $\widetilde{x}_0$ and, in fact, it equals $\overline{A}^*$ via the identification $H^*(\Gamma) = H^*(X)$. The "very mild assumption" is satisfied, for example, if A is a *Lipschitz* map with respect to the metric on $\widetilde{X}$ lifted from some Riemannian metric on X, i.e.

$$\mathrm{dist}_{\mathbb{R}^N}\left(A(x_1), A(x_2)\right) \leq c \,\mathrm{dist}_{\widetilde{X}}(x_1, x_2) ,$$

for all $x_1, x_2 \in \widetilde{X}$ and a fixed constant $c \geq 0$.

4.

Let us indicate a typical situation where the homomorphism $\overline{A}^*$ is *non-trivial.* Let the (compact) manifold X have no boundary, assume $\dim X = N$ and let the (proper) map $A : \widetilde{X} \to \mathbb{R}^N$ have a non-zero degree d with respect to some orientation chosen in $\widetilde{X}$. Then the fundamental class $c \in H^N_{\mathrm{comp}}(\mathbb{R}^N)$ goes to d times the fundamental class of $\widetilde{X}$ and $\overline{A}^*(c) \in H^N(X)$ equals d times the fundamental class of X.

EXAMPLE: Let $\Gamma = \mathbb{Z}^N$ and X equal the N-torus $\mathbb{T}^N$. Then $\widetilde{X} = \mathbb{R}^N$ and the identity map $A : \widetilde{X} \to \mathbb{R}^N$ induces the fundamental class of $\mathbb{T}^N$. Notice, that in this case, $\overline{A}^* = \overline{\alpha}^*$ and so the standard embedding $\alpha : \mathbb{Z}^N \to \mathbb{R}^N$ induces a non-zero element in

$$H^N(\mathbb{Z}^N) = H^N(\mathbb{T}^N) = \mathbb{R} .$$

(Recall that the cohomology here and below is taken with real coefficients.)

5.

There are definite limits to our construction of non-trivial elements in $H^*(\Gamma)$ of the form $\overline{\alpha}^*(c)$ for $c \in H^*_{\mathrm{comp}}(\mathbb{R}^N)$. First of all, the cohomology of $\mathbb{R}^N$ with compact support is non-zero only in dimension N, where it is one dimensional. Thus we may only produce non-trivial elements in $H^N(\Gamma)$. To see better what happens next, we assume that Γ is isomorphic to the fundamental group of a compact polyhedron X, such that the universal covering $\widetilde{X}$ is N-*connected*, i.e. $\pi_2(\widetilde{X}) = \pi_3(\widetilde{X}) = \ldots = \pi_N(\widetilde{X}) = 0$. (For example, the manifolds X with contractible $\widetilde{X}$ are good enough here.) Then the group $H^i_{\mathrm{comp}}(\widetilde{X})$ for $i \leq N$ is, in fact, independent of $\widetilde{X}$ and can be called $H^i_{\mathrm{comp}}(\Gamma)$. Furthermore, it is not hard to extend this definition to an arbitrary finitely generated group Γ without assuming the existence of X. (Notice that the existence of X amounts to a certain finite dimensionality property of the cohomology of Γ.) Now we observe that the homomorphism $\overline{\alpha}^*$ factors through

$$\alpha^* : H^*_{\text{comp}}(\mathbb{R}^N) \to H^*_{\text{comp}}(\Gamma) ,$$

and so it is zero in dimension N if $H^N_{\text{comp}}(\Gamma) = 0$. (It may be only fair to the reader if we admit at this point that our $H^*_{\text{comp}}(\Gamma)$ equals the cohomology of Γ with coefficients in the real group ring of Γ.) For example, if our X is a *closed* n-dimensional manifold, then the homomorphism $\overline{\alpha}^*$ is zero unless $n = N$.

To appreciate the third difficulty which may appear we suggest that the reader look at the free group Γ on two generators, where the space $H^1_{\text{comp}}(\Gamma)$ is infinite dimensional. The polyhedron X here is the figure ∞ and $\widetilde{X}$ is a regular infinite tree. It takes a minor effort to construct a *proper Lipschitz* map $A : \widetilde{X} \to \mathbb{R}$ which induces a given class c^* in $H^1_{\text{comp}}(\widetilde{X})$ from a class c in $H^1_{\text{comp}}(\mathbb{R})$. The problem becomes visibly more difficult for $N \geq 2$ where $\widetilde{X}$ is a kind of N-dimensional tree, e.g. an N-dimensional *Bruhat-Tits building*, whose first (and least interesting) representative is the Cartesian product of N trees. Here we have infinitely many N-dimensional branches getting in the way and arranging them into a proper Lipschitz map $A : \widetilde{X} \to \mathbb{R}^N$ with $A^* \neq 0$ on H^*_{comp} requires a certain amount of attention. We advise the reader to look at such maps of products of trees to $\mathbb{R}^N$.

6. Families of Maps.

We want to detect cohomology classes in $H^n(\Gamma)$ (or in $H^n(X)$ for $\Gamma = \pi_1(X)$) by using proper maps $\Gamma \to \mathbb{R}^N$ (or $\widetilde{X} \to \mathbb{R}^N$), where $N > n$. There are two somewhat different aspects to the problem, which we first discuss separately.

6.A Non-compact families. Suppose $\Gamma = \pi_1(X)$ where X is a closed connected *aspherical* n-dimensional manifold; "aspherical" means $\pi_1(X) = 0$ for $i \geq 2$, which is equivalent to contractibility of the universal covering $\widetilde{X}$ of X. The class we want to detect is $[X]^* \in H^n(X) = H^n(\Gamma)$, i.e. the fundamental cohomology class of X for some choice of orientation on X. If the dimension n of X equals $N = \dim \mathbb{R}^N$ then we have at our disposal, as before, proper maps $\widetilde{X} \to \mathbb{R}^N$ of *non-zero* degree d which "detect" $[X]^*$. Now let $N > n$. Then, of course, individual (proper) maps $\widetilde{X} \to \mathbb{R}^N$ are homologically insignificant and so we need families of such maps. Namely, we take a *parameter space* P which is an $(N-n)$-dimensional oriented manifold, e.g. $P = \mathbb{R}^{N-n}$, and use as our "detectors" proper maps $\widetilde{X} \times P \to \mathbb{R}^N$ of non-zero degree.

Remark: If we had no additional restrictions on our proper maps $\widetilde{X} \to \mathbb{R}^N$ there would be no need for extraneous parameters since every open n-dimensional manifold $\widetilde{X}$ admits a proper map of degree 1 onto $\mathbb{R}^n$. However, in our applications we are only allowed to use proper *Lipschitz* maps $\widetilde{X} \to \mathbb{R}^N$. Then, it may happen (though we do not have a convincing example) that a "Lipschitz detector" $\widetilde{X} \times P \to \mathbb{R}^N$ exists only for $N > n = \dim \widetilde{X}$. Anyhow, there are cases where the "parametric Lipschitz detector" is readily available while the non-parametric one (with $N = n$) is hard to come by.

Notice that the Lipschitz condition for maps $\widetilde{X} \times P \to \mathbb{R}^N$ applies to some product metric, where the metric on $\widetilde{X}$ must be Γ-invariant and the metric on P may be chosen as large as we want.

6.B Equivariant families. Let X be as above and suppose we want to "detect" a k-dimensional cohomology (or homology) class of X for $k < n = \dim X$. Imagine, for example, we have a k-dimensional submanifold $Y \subset X$ representing a class in $H_k(X)$ which we want to detect by maps into $\mathbb{R}^n$. There are cases where the fundamental group of Y constitutes a "k-dimensional piece" of $\pi_1(X)$, as happens, for example, if X is an n-torus and Y is a k-dimensional subtorus. For such a Y we can use an $(n-k)$-dimensional family of maps $\widetilde{Y} \to \mathbb{R}^n$ detecting $[Y]^* \subset H^k(Y) \subset H^*(X)$ as we did for $[X]^*$ in 6.A. However, in the general case the fundamental group of a connected submanifold Y representing a given class in $H_k(X)$ may be as big as $\pi_1(X)$. For example, let Y be a closed surface in the n-torus $\mathbb{T}^n$ whose fundamental group surjects onto that of $\mathbb{T}^n$. Then the lift $\widetilde{Y}$ of Y to $\mathbb{R}^n = \widetilde{\mathbb{T}}^n$ is a connected surface which is rather dense in $\mathbb{R}^n$ and there is no apparent family here of the previous kind $\widetilde{Y} \times P \to \mathbb{R}^n$ with non-compact P.

Let us indicate how to overcome this problem by using an action of $\Gamma = \pi_1(X)$ on the parameter space. Namely, we take some $(n-k)$-dimensional manifold $\widetilde{P}$ with a free Γ-action and look for (typically non-proper) maps $\widetilde{Y} \times \widetilde{P} \to \mathbb{R}^n$ which commute with the *diagonal action* of Γ on $\widetilde{Y} \times \widetilde{P}$. One can think of $\widetilde{\alpha} : \widetilde{Y} \times \widetilde{P} \to \mathbb{R}^n$ as a map of $P = \widetilde{P}/\Gamma$ into the space $\left(\mathrm{Map}(\widetilde{Y}, \mathbb{R}^n)\right)/\Gamma$ where the action of Γ on the space of maps is induced by the action of Γ on $\widetilde{Y}$. (Notice that the maps $\widetilde{X} \times P \to \mathbb{R}^N$ we looked at before correspond to maps $P \to \mathrm{Map}(\widetilde{X}, \mathbb{R}^N)$.)

We denote by Z the diagonal quotient space $\widetilde{Y} \times \widetilde{P}/\mathrm{dia}\Gamma$ and we insist on *properness* of the maps $Z \to \mathbb{R}^n$ corresponding to our diaΓ-invariant maps $\widetilde{Y} \times \widetilde{P} \to \mathbb{R}^n$. Such a map, by definition, detects $[Y]$ if the map $Z \to \mathbb{R}^n$ has non-zero degree.

Notice that we shall eventually use here only those maps $\widetilde{Y} \times \widetilde{P} \to \mathbb{R}^n$ which are Lipschitz with respect to the product metric in $\widetilde{Y} \times \widetilde{P}$ where $\widetilde{Y}$ must have Γ-invariant (i.e. coming from Y) metric, while on $\widetilde{P}$ we may choose arbitrarily large metric (in order to facilitate the Lipschitz condition).

EXAMPLES: (a) Let Y and P be closed, connected, oriented submanifolds in the n-torus $\mathbb{T}^n$, such that $\dim Y = k$, $\dim P = n-k$, and such that Y and P have a non-zero intersection index δ. We identify the universal covering of $\mathbb{T}^n$ with $\mathbb{R}^n$ and thus we obtain a proper map of $\widetilde{Y}$ to $\mathbb{R}^n$. Furthermore, each translate $Y+p$, $p \in P$ gives us a lift $\widetilde{Y}_p \subset \widetilde{\mathbb{T}}^n$ and hence a map of $\widetilde{Y}$ to $\mathbb{R}^n$ defined up to the action of $\Gamma = \mathbb{Z}^n$. Thus, we get our proper Lipschitz map $\widetilde{Y} \times \widetilde{P}/\Gamma \to \mathbb{R}^n$, which has degree $\delta \neq 0$ and which detects the class $[Y]^* \in H^k(\mathbb{T}^n)$ of Y (as well as the class $[P]^* \in H^{n-k}(T^n)$ of P).

(b) Let X be a closed manifold with a metric of *non-positive sectional curvature*. Then, for each point $x \in X$, the (geodesic) *exponential* map $\exp_x : T_x(X) \to X$ lifts to a *diffeomorphism* of the tangent space $T_x(X)$ onto the universal covering (by the Cartan-Hadamard theorem). This lift is uniquely defined by a choice of a point $\tilde{x} \in \widetilde{X}$ over x and denoted by $\widetilde{\exp}_x$. The collection of maps $\widetilde{\exp}_x$ for x running over X defines a fiberwise diffeomorphic map of the tangent bundle $T(X)$ to the manifold $Z(X) = \widetilde{X} \times \widetilde{X}/\mathrm{dia}\Gamma$ which is fibered over X with the fibers called $Z_x(X)$ $(= \widetilde{X})$. Since $\widetilde{\exp} : T(X) \to Z(X)$ is a diffeomorphism we have the inverse map, denoted

$$\widetilde{\log} : Z(X) \to T(X) \ ,$$

which diffeomorphically sends each fiber $Z_x(X)$ onto $T_x(X)$.

Now we take two intersecting cycles in X, realized by submanifolds Y and P as earlier. The embedding $\widetilde{Y} \times \widetilde{P} \subset \widetilde{X} \times \widetilde{X}$ induces an embedding of $Z = Z(Y, P) = \widetilde{Y} \times \widetilde{P}/\mathrm{dia}\Gamma$ into $Z(X)$ and the composition with $\widetilde{\log}$ gives us a map $Z \to T(X)$. Notice, that Y and P appear on an equal footing in this discussion, but we now distinguish between the two by making our choice of the projection $Z(X) \to X$. There are, a priori, two projections, corresponding to the projection of $\widetilde{X} \times \widetilde{X}$ on the first and on the second factor. Here we choose to project on the *second* factor, so that P goes to the base X of the fibration $Z(X) \subset X$ while $\widetilde{Y} \times \tilde{p}$, $\tilde{p} \in \widetilde{P}$ goes to the fibers. Thus, the image of our map $Z \to T(X)$ lies in the tangent bundle $T(X)$ restricted to $P \subset X$. One thinks of this as a family of maps of

$\widetilde{Y} = Z_p \subset Z_p(X)$ to the Euclidean space $\mathbb{R}^n = T_p(X)$ which now varies together with $p \in P$. This generalized family of maps $\widetilde{Y} \to \mathbb{R}^n$ reduces to an ordinary family (where $\mathbb{R}^n$ is independent of p) if the manifold X is *parallelizable*, i.e. if the tangent bundle $T(X)$ is trivial. In this case, there exists a fiber-isomorphic map $T(X) \to \mathbb{R}^n$, which we may also choose fiber isometric (for our Riemannian metric, thought of as a field of Euclidean structures on the fibers $T_x(X)$, $x \in X$). If we compose the above map $T(X) \to \mathbb{R}^n$ with the previously constructed $\widetilde{\log}$-map $Z \to T(X)$ we get a *proper* map $Z \to \mathbb{R}^n$ whose degree (clearly) equals the intersection index between Y and P. Besides, this map is Lipschitz on each fiber $Z_p = \widetilde{Y}$, $p \in P$, which is good enough for our future purposes.

If X is non-parallelizable, one cannot, in general, produce the desired map $Z \to \mathbb{R}^n$, but one can add extra parameters to obtain an $[X]^*$-detecting family of maps $\widetilde{Y} \to \mathbb{R}^N$ for some $N > n$ as we shall see later.

7. Families of maps $\Gamma \to \mathbb{R}^N$.

Here, as earlier, Γ is a discrete group with a fixed left-invariant metric and we study proper *Lipschitz* maps $\alpha : \Gamma \to \mathbb{R}^N$. Notice that every Lipschitz map with the *Lipschitz constant* λ, i.e. satisfying

$$\operatorname{dist}_{\mathbb{R}^N}\big(\alpha(\gamma_1), \alpha(\gamma_2)\big) \le \lambda \operatorname{dist}_\Gamma(\gamma_1, \gamma_2)\ , \qquad \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma\ ,$$

can be made Lipschitz with $\lambda = 1$ by composing with the scaling map $x \mapsto \lambda^{-1}x$ of $\mathbb{R}^N$. Lipschitz maps with $\lambda \le 1$ are also called *contracting*, since they are distance decreasing.

Similarly, given a family of *uniformly* Lipschitz maps, i.e. λ-Lipschitz for a fixed λ, we can make them all contracting, by the scaling $x \mapsto \lambda^{-1}x$. From now on we prefer to deal with *contracting* rather than uniformly Lipschitz families as it is somewhat easier on the terminology.

7.A Families with a fixed target. This means a continuous map $\widetilde{\alpha}$: $\Gamma \times \widetilde{P} \to \mathbb{R}^N$, where $\widetilde{P}$ is a locally finite polyhedron of finite dimension with a proper (but not necessarily free) action of Γ, and where $\widetilde{\alpha}$ satisfies the following three conditions.

(i) *Contracting property*. For each $\widetilde{p} \in \widetilde{P}$ the map $\widetilde{\alpha}$ on $\widetilde{\Gamma} = \widetilde{\Gamma} \times \widetilde{p}$ is contracting (for a left invariant metric on Γ chosen and fixed beforehand).

(ii) diaΓ-*invariance*. The map $\widetilde{\alpha}$ is invariant under the diagonal action of Γ on $\Gamma \times \widetilde{P}$ where Γ (isometrically) acts on itself by *left* translations.

(iii) *Properness.* For each $\gamma \in \Gamma$ the map $\tilde{\alpha}$ on $\widetilde{P} = \gamma \times \widetilde{P}$ is proper; this is equivalent to the properness of the map

$$\Gamma \times \widetilde{P}/\mathrm{dia}\Gamma \to \mathbb{R}^N$$

associated to Γ.

Let us translate the above conditions in terms of maps of the quotient space $\Gamma \times \widetilde{P}/\mathrm{dia}\Gamma$ to $\mathbb{R}^N$. First, we observe that this quotient space is canonically homeomorphic to $\widetilde{P}$, by the projection $\widetilde{P} = \mathrm{id} \times \widetilde{P} \to \Gamma \times \widetilde{P}/\mathrm{dia}\Gamma$, id = the identity element of Γ. Every diaΓ-invariant map $\tilde{\alpha} : \Gamma \times \widetilde{P} \to \mathbb{R}^N$ defines a map of $\widetilde{P} = \Gamma \times \widetilde{P}/\mathrm{dia}\Gamma$ to $\mathbb{R}^N$ by going to the quotient. Conversely, every map $\alpha : \widetilde{P} \to \mathbb{R}^N$ leads to $\tilde{\alpha}$ by $\tilde{\alpha}(\gamma, \tilde{p}) = \alpha(\gamma^{-1}\tilde{p})$; this $\tilde{\alpha}$ is Γ-invariant, as

$$\tilde{\alpha}(\beta\gamma, \beta\tilde{p}) = \alpha(\gamma^{-1}\beta^{-1}\beta\tilde{p}) = \alpha(\gamma^{-1}\tilde{p}) = \tilde{\alpha}(\gamma, \tilde{p}) \ .$$

Next we observe that

$$\mathrm{dist}_{\mathbb{R}^N}\left(\tilde{\alpha}(\mathrm{id} \times \tilde{p}), \tilde{\alpha}(\gamma \times \tilde{p})\right) = \mathrm{dist}_{\mathbb{R}^N}\left(\alpha(\tilde{p}), \alpha(\gamma^{-1}\tilde{p})\right)$$

and then we can easily see that the contracting property (i) is equivalent to the following

(i)′ *Displacement bound.* All $\tilde{p} \in \widetilde{P}$ and $\gamma \in \Gamma$ satisfy

$$\mathrm{dist}_{\mathbb{R}^N}\left(\alpha(\tilde{p}), \alpha(\gamma\tilde{p})\right) \le \|\gamma\| \ ,$$

where $\|\gamma\| \stackrel{\mathrm{def}}{=} \mathrm{dist}_\Gamma(\gamma, \mathrm{id})$. (Notice that $\|\gamma\| = \|\gamma^{-1}\|$ for every left invariant metric on Γ, as $\mathrm{dist}(\gamma^{-1}, \mathrm{id}) = \mathrm{dist}(\gamma\gamma^{-1}, \gamma)$.)

Summing up, we see that a family $\tilde{\alpha}$ satisfying (i)-(iii) amounts to a proper map $\alpha : \widetilde{P} \to \mathbb{R}^N$ satisfying (i)′.

7.B Families with a variable target. Here the Γ-space $\widetilde{P}$ comes along with an *Euclidean* Γ-*bundle* $\widetilde{T} \to \widetilde{P}$. This means that $\widetilde{T}$ is an oriented vector bundle over $\widetilde{P}$ with a continuous field of Euclidean metrics in the fibers $\widetilde{T}_p$, $\tilde{p} \in \widetilde{P}$ and Γ acts on $\widetilde{T}$ by fiber-wise linear, metric and orientation preserving homeomorphisms such that the resulting action on the set of the fibers coincides with the underlying action of Γ on $\widetilde{P}$.

Now, our families are maps $\tilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$, such that:

(i)* every Γ-slice $\Gamma \times \tilde{p}$ lands in the fiber $\widetilde{T}_{\tilde{p}}$ and the resulting map $\Gamma = \Gamma \times \tilde{p} \to \widetilde{T}_{\tilde{p}}$ is contracting;

(ii)* the map $\widetilde{\alpha}$ is diaΓ-equivariant;

(iii)* the function

$$(\widetilde{\gamma}, \widetilde{p}) \mapsto \|\widetilde{\alpha}(\gamma, \widetilde{p})\|_{\widetilde{T}}$$

is proper on $\widetilde{P} = \Gamma \times \widetilde{P}/\text{dia}\Gamma$ where $\| \quad \|_{\widetilde{T}}$ denotes the norm $\widetilde{T} \to \mathbf{R}_+$ corresponding to the Euclidean structure.

Reformulation in terms of $\widetilde{P}$ alone. If the action of Γ on $\widetilde{P}$ is *free*, or more generally, if the isotropy subgroup of every point $\widetilde{p} \in \widetilde{P}$ acts trivially on the fiber $\widetilde{T}_{\tilde{p}}$, then the bundle $\widetilde{T}$ is induced from a vector bundle T over the quotient space $P = \widetilde{P}/\Gamma$. In this case our families can be identified with *proper* continuous maps $\alpha : \widetilde{P} \to T$ which commute with projections of $\widetilde{P}$ and T to P, i.e. the diagram

$$\begin{tikzcd} \widetilde{P} \arrow[rr, "\alpha"] \arrow[dr] & & T \arrow[dl] \\ & P, & \end{tikzcd}$$

is commutative, and which satisfy the displacement bound

$$\text{dist}\left(\alpha(\gamma\widetilde{p}), \alpha(\widetilde{p})\right) \leq \|\gamma\| ,$$

for the implied Euclidean distance in the fibers T_p, $p \in P$.

In the general case, where the (finite) isotropy of $\widetilde{p}$ may act non-trivially on $\widetilde{T}_{\tilde{p}}$, the above reformulation is still possible but becomes somewhat awkward as it applies to the *orbibundle structure* on $\widetilde{T}/\Gamma$.

7.C Fixing the target. Every family with a fixed target space $\mathbf{R}^N$ can be formally (and obviously) reduced to the variable case with $\widetilde{T} = \widetilde{P} \times \mathbf{R}^N$ where Γ acts on the first factor. Conversely, if the Γ-bundle $\widetilde{T} \to \widetilde{P}$ is *trivial*, i.e. admits a continuous Γ-invariant field of orthonormal N-frames, for $N = \text{rank}\,\widetilde{T}$, in the fibers $\widetilde{T}_{\tilde{p}}$, $\widetilde{p} \in \widetilde{P}$, then the resulting identification of each fiber with $\mathbf{R}^N$ gives us a fiber isometric map $\widetilde{T} \to \mathbf{R}^N$. By composing this map with a family $\Gamma \times \widetilde{P} \to \widetilde{T}$ we obtain a family $\Gamma \times \widetilde{P} \to \mathbf{R}^N$ and we easily see that the properties (i)*-(iii)* of the former imply (i)-(iii) of the latter (compare Example (b) in 6.B).

More generally, let $\widetilde{T}$ be non-trivial, but just assume that the isotropy of each point $\widetilde{p} \in \widetilde{P}$ acts *trivially* on $\widetilde{T}_{\tilde{p}}$ (e.g. Γ acts *freely* on $\widetilde{P}$). In this case $\widetilde{T}$ is induced from the bundle $T = \widetilde{T}/\Gamma$ over $P = \widetilde{P}/\Gamma$. This T admits a *complementary* Euclidean bundle, say $S \to P$, such that $T \oplus S$ is trivial and then the lift $\widetilde{S} \to \widetilde{P}$ Γ-complements $\widetilde{T}$, i.e. $\widetilde{T} \oplus \widetilde{S}$ is Γ-trivial. (Notice that the existence of a Γ-complement to $\widetilde{T}$ *implies* that the isotropy subgroups act trivially on the fibers.)

Suspension. Let us *suspend* a given $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$ with an arbitrary (not necessarily complementary to $\widetilde{T}$) Euclidean Γ-bundle $\widetilde{S} \to \widetilde{P}$ as follows. We take the (total space of the) bundle $\widetilde{S}$ for the new (suspended) parameter space $\widetilde{P}'$, we pull back the bundle $\widetilde{T} \oplus \widetilde{S} \to \widetilde{P}$ to $\widetilde{S}$ and we denote the pulled back bundle by $\widetilde{T}' = \widetilde{T} \boxed{+} \widetilde{S} \to \widetilde{S} = \widetilde{P}'$. Now we define the suspension $\widetilde{\alpha}' = \widetilde{\alpha} \boxed{+} \widetilde{S} : \Gamma \times \widetilde{P}' \to \widetilde{T}'$

$$\widetilde{\alpha}'\big(\gamma, (\widetilde{p}, \widetilde{s})\big) = \widetilde{\alpha}(\gamma, \widetilde{p}) + \widetilde{s} , \tag{7.1}$$

where the points in $\widetilde{P}' = \widetilde{S}$ are represented by the pairs $(\widetilde{p}, \widetilde{s})$ for $\widetilde{p} \in \widetilde{P}$ and $\widetilde{s} \in \widetilde{S}_{\widetilde{p}}$ and where the sum on the right hand side is taken in the fiber $\widetilde{T}_{\widetilde{p}} \oplus \widetilde{S}_{\widetilde{p}}$ which is canonically identified with the fibers $\widetilde{T}'_{\widetilde{p}'}$ for all $p' \in \widetilde{P}'$ of the form $p' = (\widetilde{p}, \widetilde{s})$, $\widetilde{s} \in \widetilde{S}_{\widetilde{p}}$. In other words (7.1) defines a map $\Gamma \times \widetilde{P}' \to \widetilde{T} \oplus \widetilde{S}$ which is then interpreted as a map to the bundle $\widetilde{T}'$ over $\widetilde{P}'$ (which is induced from the bundle $\widetilde{T} \oplus \widetilde{S}$ over $\widetilde{P}$ by the projection $\widetilde{P}' = \widetilde{S} \to \widetilde{P}$).

It is immediate, that the properties (i)*-(iii)* for $\widetilde{\alpha}$ imply those for $\widetilde{\alpha}'$.

Fixing the target over the suspension. Now we assume that $\widetilde{S}$ is the bundle *complementary* to $\widetilde{T}$ and we use a trivializing map of $\widetilde{T}' = \widetilde{T} \boxed{+} \widetilde{S}$ to $\mathbb{R}^{N+M}$, for $N = \operatorname{rank} \widetilde{T}$ and $M = \operatorname{rank} \widetilde{S}$. This, composed with $\widetilde{\alpha}'$, gives us a family denoted

$$\widetilde{\alpha}^0 : \Gamma \times \widetilde{P}' \to \mathbb{R}^{N+M} ,$$

which satisfy our requirements (i)-(iii).

Remark: It will become clear later on (see (II) in 8.A) why we have to enlarge the parameter space ($\widetilde{P}' = \widetilde{S}$ instead of $\widetilde{P}$) rather than use the (more) obvious family $\Gamma \times \widetilde{P} \to \widetilde{T} \oplus \widetilde{S}$ obtained by composing the original $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$ with the embedding $\widetilde{T} \to \widetilde{T} \oplus \widetilde{S}$ by $\widetilde{t} \mapsto \widetilde{t} + 0$.

8. Proper Lipschitz cohomology.

We want to describe the cohomology classes of Γ with real coefficients which are "detected" by a given family $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$. In fact, we are going to construct a homomorphism, called $\widetilde{\alpha}_{\cap}$, from a certain homology of $\widetilde{P}$ to the cohomology $H^*(\Gamma)$ such that the "detectable" classes will be exactly those which lie in the image of this homomorphism. We start with the simplest case where $\widetilde{P}$ is a smooth oriented i-dimensional Γ-manifold (i.e. the action of Γ on $\widetilde{P}$ is smooth and orientation preserving) and we define the value of our homomorphism on the fundamental class $[\widetilde{P}]$ of $\widetilde{P}$ as follows. The homomorphism $\widetilde{\alpha}_\cap$, when applied to the i-dimensional homology, is going to land in $H^k(\Gamma)$ for $k = N - i$, where $N = \operatorname{rank} \widetilde{T}$. So, $\widetilde{\alpha}_\cap[\widetilde{P}]$ may be defined by a k-cochain on Γ, i.e. a Γ-invariant real valued function in the variables $\gamma_0, \dots, \gamma_k \in \Gamma$. Denote by $\Delta = \Delta(\gamma_0, \dots, \gamma_k)$ the simplex abstractly spanned by $\gamma_0, \dots, \gamma_k$, i.e. Δ equals the set of formal linear combinations $\sum_{j=0}^{k} \mu_j \gamma_j$, with $\mu_j \geq 0$, satisfying $\sum_{j=0}^{k} \mu_i = 1$. The map $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$ sends each $\Gamma = \Gamma \times \widetilde{p}$, $\widetilde{p} \in \widetilde{P}$ into the fiber $\widetilde{T}_{\widetilde{p}}$ and then this map extends by linearity to $\widetilde{\alpha}_\Delta : \Delta \times \widetilde{P} \to \widetilde{T}$,

$$\widetilde{\alpha}_\Delta\left(\sum_{j=0}^{k} \mu_j \gamma_j, \widetilde{p}\right) = \left(\widetilde{p}, \sum_{j=0}^{k} \mu_j \alpha(\gamma_j, \widetilde{p})\right) ,$$

where the points of $\widetilde{T}$ are represented by the pairs $(\widetilde{p}, \widetilde{t}) \in \widetilde{T}_{\widetilde{p}}$.

Let us take an exterior form ω on $\widetilde{T}$ whose support lies within bounded distance from the zero section $\widetilde{P} \hookrightarrow \widetilde{T}$, i.e. $\|s\|_{\widetilde{T}} \leq \text{const} < \infty$, for all s in the support of ω. We temporarily assume the map $\widetilde{\alpha}$ is smooth (on each $\gamma \times \widetilde{P}$) and set

$$c(\gamma_0, \dots, \gamma_k) = \int_{\Delta \times \widetilde{P}} \widetilde{\alpha}_\Delta^*(\omega) , \qquad (*)$$

for the induced N-form $\widetilde{\alpha}_\Delta^*(\omega)$ on the oriented manifold $\Delta \times \widetilde{P}$ (which has dimension $k + (N - k) = N$). Notice that the properness property of $\widetilde{\alpha}$ in conjunction with the contracting property (see (iii)* and (i)* in 7.B) insure that the form $\widetilde{\alpha}_\Delta^*(\omega)$ has compact support and so the integral in $(*)$ is indeed defined. Furthermore, if the form ω is Γ-invariant, then the diaΓ-equivariance of $\widetilde{\alpha}$ makes the function c invariant,

$$c(\gamma\gamma_0, \dots, \gamma\gamma_k) = c(\gamma_0, \dots, \gamma_k) .$$

Now we take a Γ-invariant closed form ω representing the (Γ-invariant) *Thom* class of $\widetilde{T}$ and take the cohomology class of c, $[c] \in H^k(\Gamma)$, for $\widetilde{\alpha}_\cap[\widetilde{P}]$.

Recall that the Thom class is the Poincaré dual of the infinite cycle represented by the zero section $\widetilde{P} \hookrightarrow \widetilde{T}$. Also notice that if $\widetilde{T}$ admits a (Γ-invariant) trivialization map $\widetilde{T} \to \mathbb{R}^N$, then the Thom class is induced by the fundamental class of $\mathbb{R}^N$ in $H^N_{\mathrm{comp}}(\mathbb{R}^N)$ and so one can use for ω the pull-back of some form with compact support on $\mathbb{R}^N$.

In the case where $\widetilde{\alpha}$ is non-smooth, one can always slightly (and Γ-equivariantly) perturb $\widetilde{\alpha}$ in order to make it smooth and then the above class in $H^k(\Gamma)$ is defined by means of the (smooth!) perturbed $\widetilde{\alpha}$. A trivial argument shows the result is independent of the perturbation.

8.A. Let us indicate two useful (albeit obvious) properties of the above construction.

(I) Functoriality. Let $\widetilde{P}$ and $\widetilde{P}'$ be two Γ-manifolds with i-dimensional Euclidean Γ-bundles $\widetilde{T}$ over $\widetilde{P}$ and $\widetilde{T}'$ over $\widetilde{P}'$ and let $f : \widetilde{T}' \to \widetilde{T}$ be a continuous fiber preserving and fiber isometric Γ-equivariant map, such that the underlying map $\overline{f} : \widetilde{P}' \to \widetilde{P}$ is proper. Let

$$\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T} \quad \text{and} \quad \widetilde{\alpha}' : \Gamma \times \widetilde{P}' \to \widetilde{T}'$$

be diaΓ-equivariant continuous maps, such that

$$f(\widetilde{\alpha}'(\gamma, \widetilde{p}')) = \widetilde{\alpha}(\gamma, \overline{f}(\widetilde{p}')) \ .$$

Then the contraction and properness properties (i)* and (iii)* (see 7.B) for $\widetilde{\alpha}$ imply those for $\widetilde{\alpha}'$, and if $\overline{f}$ (and hence f) is onto, then conversely (i)* + (iii)* for $\widetilde{\alpha}'$ imply those for $\widetilde{\alpha}'$. Furthermore, if the map $\overline{f}$ has certain degree d (i.e. $\overline{f}_*[P'] = d[P]$, which is always so for some d if the manifold P is connected), then $\widetilde{\alpha}'_\cap[\widetilde{P}'] = d\widetilde{\alpha}_\cap[\widetilde{P}]$.

EXAMPLE: Let f be an automorphism (gauge transformation) of $\widetilde{T}$, i.e. a Γ-equivariant fiber isometric map $\widetilde{T} \to \widetilde{T}$ sending each fiber into itself. Then each of the properties (i)*, (ii)* and (iii)* for $\widetilde{\alpha}$ implies the corresponding property for $\widetilde{\alpha}'$ and $\widetilde{\alpha}'_\cap(\widetilde{P}) = \widetilde{\alpha}_\cap(\widetilde{P})$.

(II) Suspension property. Consider a family $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$, let $\widetilde{S}$ be another Γ-bundle over $\widetilde{P}$ and take the suspension $\widetilde{\alpha}' = \widetilde{\alpha} \boxed{+} \widetilde{S} : \Gamma \times \widetilde{P}' \to \widetilde{T}'$ (where $\widetilde{P}'$ equals the total space of $\widetilde{S}$ and $\widetilde{T}' = \widetilde{T} \boxed{+} \widetilde{S}$, i.e. the bundle on $\widetilde{P}'$ induced by the projection $\widetilde{P}' \to \widetilde{P}$ from the Whitney sum $\widetilde{T} \oplus \widetilde{S}$). Then each of (i)*, (ii)* and (iii)* implies the corresponding property of $\widetilde{\alpha}'$ and

$$\widetilde{\alpha}'_\cap[\widetilde{P}'] = \widetilde{\alpha}_\cap(\widetilde{P}) \ .$$

8.B Extension of $\widetilde{\alpha}$ to $\mathbb{R}$-cycles. Now let $\widetilde{P}$ be an arbitrary (non-manifold) Γ-polyhedron and let us observe that the construction of $\widetilde{\alpha}_\cap(\widetilde{P})$ obviously generalizes to all infinite simplicial Γ-invariant cycles in $\widetilde{P}$ with real coefficients. Thus $\widetilde{\alpha}_\cap$ becomes a homomorphism from the real Γ-invariant homology of $\widetilde{P}$ with (infinite) non-compact support to the cohomology of Γ, i.e.

$$\widetilde{\alpha}_\cap : H_i(\widetilde{P}:\Gamma) \to H^{N-i}(\Gamma)\ , \qquad i = 0, 1, \ldots\ .$$

Notice that if the action of Γ is free and the quotient space P is compact then the above homology $H_*(\widetilde{P}:\Gamma)$ (with any, not only real, coefficients) is, obviously, the same as the ordinary homology of $P = \widetilde{P}/\Gamma$. If P is non-compact, then $H_*(\widetilde{P}:\Gamma)$ equals the homology of P with non-compact (infinite) supports (again, for arbitrary coefficients). In general, the natural homomorphism

$$H_*(\widetilde{P}:\Gamma) \to H_*(\widetilde{P}/\Gamma)$$

may not be an isomorphism. Yet if the coefficient field is $\mathbb{R}$ and if there are only finitely many conjugacy classes of $\gamma \in \Gamma$ which act non-freely on P, then this homomorphism *is* an isomorphism. (Observe that the above finiteness condition is satisfies if $\widetilde{P}/\Gamma$ is compact.)

8.C Definition: A cohomology class c in $H^*(\Gamma)$ with real coefficient is called *proper Lipschitz* (with the variable target space $\mathbb{R}^N$) if there exist a proper Γ-space $\widetilde{P}$, $\widetilde{T} \to \widetilde{P}$ and $\widetilde{\alpha}$ as above, such that $c = \widetilde{\alpha}_\cap(b)$ for some $b \in H_*(\widetilde{P}:\Gamma)$. (Recall, that for $N = \operatorname{rank}\widetilde{T}$ and $b \in H_i$ we get $c \in H^{N-i}$). Here, one may distinguish the case of $c = \widetilde{\alpha}_\cap[\widetilde{P}]$ for a *manifold* $\widetilde{P}$, and also the case of the *fixed* target space $\mathbb{R}^N$, which means $\widetilde{T} = \widetilde{P} \times \mathbb{R}^N$ and the map $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$ reduces to a Γ-invariant map $\Gamma \times \widetilde{P} \to \mathbb{R}^N$.

The following proposition shows that the target space $\mathbb{R}^N$ can be fixed.

8.D Proposition. *Every proper Lipschitz class c can be represented as $\widetilde{\alpha}'_\cap(b)$ where the implied Γ-bundle $\widetilde{T} \to \widetilde{P}$ is trivial and so $\widetilde{\alpha}'$ reduces to a map $\Gamma \times \widetilde{P} \to \mathbb{R}^N$ satisfying* (i)-(iii) *of* 7.A.

Proof: Let us first assume that Γ acts freely on $\widetilde{P}$. Then there exists a (complementary) bundle $\widetilde{S} \to \widetilde{P}$ such that $\widetilde{T} \oplus \widetilde{S}$ is trivial and the proposition follows by taking the suspension $\widetilde{\alpha}' = \widetilde{\alpha} \boxed{+} \widetilde{S}$ for a given $\widetilde{\alpha}$ with $\widetilde{\alpha}(b) = c$. Here it is worth noticing that the projection, say $\pi : \widetilde{P}' = \widetilde{S} \to \widetilde{P}$ induces a natural *suspension homomorphism* (which is, in fact, an isomorphism) $\pi^\cap : H_i(\widetilde{P}:\Gamma) \to H_{i+M}(\widetilde{P}':\Gamma)$, $i = 0, 1, \ldots$, as every i-cycle in $\widetilde{P}$

pulls back to an $(i+M)$-cycle in $\widetilde{P}' = \widetilde{S}$ for $M = \operatorname{rank}\widetilde{S}$. The suspension homomorphism clearly commutes with $\widetilde{\alpha}_\cap$ (compare (II) in 8.B), i.e.

$$\widetilde{\alpha}'_\cap\big(\pi^\cap(b)\big) = \widetilde{\alpha}_\cap(b) \quad \text{for all} \quad b \in H_*(\widetilde{P} : \Gamma) \ ,$$

which is exactly what we use in the proof.

Now let us drop the freeness assumption on the action of Γ on $\widetilde{P}$ and then invoke the following elementary fact.

LEMMA. *For every homology class $b \in H_i(\widetilde{P} : \Gamma)$ there exist an i-dimensional Γ-polyhedron $\widetilde{P}'$ on which the action of Γ is free and a proper Γ-invariant map $\sigma : \widetilde{P}' \to P$, such that $b = \sigma(b')$ for some $b' \in H_*(P' : \Gamma)$.*

Proof: One should think of $\widetilde{P}'$ as a kind of blow-up of (the support of) a cycle representing b at the fixed point locus. In fact, one may first blow up all of $\widetilde{P}$ by replacing every point $\widetilde{p}$ with non-trivial isotropy subgroup $\Gamma_{\widetilde{p}}$ by the classifying space of $\Gamma_{\widetilde{p}}$. Since the groups $\Gamma_{\widetilde{p}}$ are finite, this process does not change the real (co)homology of $\widetilde{P}$ and so every Γ-cycle in $\widetilde{P}$ lifts to the blown up space. The details of the proof are left to the reader.

Now we use the functoriality of $\widetilde{\alpha}_\cap$ (which was stated in (I) of 8.A in the special case of $b = [\widetilde{P}]$ and which obviously holds in general) and conclude that

$$\widetilde{\alpha}_\cap(b) = \widetilde{\alpha}'_\cap(b') \ ,$$

where $\widetilde{\alpha}' : \Gamma \times \widetilde{P} \to \widetilde{T}'$ is induced by σ from a given $\widetilde{\alpha} : \Gamma \times \widetilde{P} \to \widetilde{T}$ as follows. The bundle $\widetilde{T}' \to \widetilde{P}'$ is induced from $\widetilde{T}$ by σ (in the usual sense) and $\widetilde{\alpha}'$ sends every "slice" $\Gamma \times \widetilde{p}'$ to $\widetilde{T}_{\widetilde{p}'} = T_{\widetilde{p}}$, for $\widetilde{p} = \sigma(\widetilde{p}')$ by $\widetilde{\alpha}'(\gamma, \widetilde{p}') = \widetilde{\alpha}\big(\gamma, \sigma(p)\big)$. Thus we reduce the general case of the Proposition to the free case which has been already settled. □

8.E *Remark*: Suppose the class c we want to "fix" is of the form $\widetilde{\alpha}_\cap[\widetilde{P}]$ for an i-dimensional *manifold* $\widetilde{P}$. Then we may look for another manifold $\widetilde{P}'$ with a *trivial* $\widetilde{T}' \to \widetilde{P}'$ (i.e. with the fixed target space $\mathbb{R}^{N'}$) and for $\widetilde{\alpha}' : \widetilde{P}' \to \widetilde{T}'$ such that

$$\widetilde{\alpha}'_\cap(\widetilde{P}') = c \ .$$

If the action of Γ on $\widetilde{P}$ is free, then the suspension construction with $\widetilde{S}$ complementary to $\widetilde{P}$ works perfectly within the manifold framework. Also there is no problem with the blow-up if there are only finitely many conjugacy classes of isotropy subgroups $\Gamma_{\widetilde{p}}$. But in the general case there is no apparent simple procedure for fixing the target without passing from manifolds to $\mathbb{R}$-cycles.

9. Manifolds with $K \leq 0$.

Let $\widetilde{P}$ be a complete simply connected Riemannian manifold with non-positive sectional curvature and let Γ be a discrete subgroup in the isometry group $\operatorname{Iso}\widetilde{P}$. Take some point $\widetilde{p}_0 \in \widetilde{P}$ (at which we may assume the action of Γ is free if we wish so) and define the map $\widetilde{\alpha}$ from $\Gamma \times \widetilde{P}$ to the *tangent* bundle $\widetilde{T} = T(\widetilde{P})$ as the inverse to the exponential maps at all $\widetilde{p} \in \widetilde{P}$, restricted to the orbit of $\widetilde{p}_0$, i.e.

$$\widetilde{\alpha}(\gamma, \widetilde{p}) = \log_{\tilde{p}}\left(\gamma(\widetilde{p}_0)\right) ,$$

where

$$\log_{\tilde{p}} \underset{\text{def}}{=} \exp_{\tilde{p}}^{-1} : \widetilde{P} \to T_{\tilde{p}}(\widetilde{P})$$

(compare 6.B, Ex. (b)). The condition $K \leq 0$ makes $\log_{\tilde{p}}$ contracting and the conditions (ii)* and (iii)* in 7.B are trivially satisfied for this $\widetilde{\alpha}$. Now, we claim that *the homomorphism*

$$\widetilde{\alpha}_\cap : H_*(\widetilde{P} : \Gamma) \to H^*(\Gamma)$$

is an isomorphism.

Proof: First, assume the action of Γ to be free. Then

$$H_*(\widetilde{P} : \Gamma) = H_*^{\inf}(P) \quad \text{for} \quad P = \widetilde{P}/\Gamma$$

and

$$H^*(\Gamma) = H^*(P) ,$$

since $\widetilde{P}$ is *contractible* (in fact, $\exp_{\tilde{p}}(\widetilde{P}) \to \widetilde{P}$ is a homeomorphism for every $\widetilde{p} \in \widetilde{P}$ when $K \leq 0$). Furthermore, the exponential map pulls back the Thom class of $T = T(P)$ to the Poincaré dual of the diagonal in $P \times P$. It becomes clear at this point that our $\widetilde{\alpha}_\cap$ now amounts to the Poincaré duality *isomorphism*

$$D : H_i^{\inf}(P) \to H^{n-i}(P) .$$

In the general (non-free) case, the relevant Poincaré duality isomorphism applies to $H_i(\widetilde{P} : \Gamma)$ and lands in the cohomology of the deRahm complex of Γ-invariant forms, say,

$$\widetilde{D} : H_i(\widetilde{P} : \Gamma) \to H^{n-i}(\widetilde{P} : \Gamma) ,$$

and $\widetilde{\alpha}_\cap$ is the composition of this $\widetilde{D}$ with the homomorphism

$$H^*(\widetilde{P} : \Gamma) \to H^*(\Gamma) \ ,$$

corresponding to the orbit map

$$\Gamma \to \widetilde{P} \ , \qquad \gamma \mapsto \gamma(\widetilde{p}_0) \ .$$

Notice that since $\widetilde{P}$ is contractible, the orbit map extends Γ-equivariantly to the simplices abstractly spanned by the $(k+1)$-tuples $(\gamma_0, \dots, \gamma_k)$, which defines the above homomorphism. (Since $K \leq 0$, there is a particularly nice extension of the orbit map to simplices, which assigns to $\sum_{j=0}^{k} \mu_j \gamma_j$ the Riemannian "center of map" of the weighted points $\mu_0\gamma_0(p_0), \dots, \mu_k\gamma_k(p_0)$ in $\widetilde{P}$.)

Finally, we observe that the homomorphism $H^*(\widetilde{P} : \Gamma) \to H^*(\Gamma)$ is an isomorphism for the cohomology with *real* coefficients which concludes the proof of our claim.

9.A *Remark on contractible manifolds* $\widetilde{P}$: If we drop the assumption $K(\widetilde{P}) \leq 0$ but only assume $\widetilde{P}$ is contractible, much of the above remains valid. Namely, here one has a proper Γ-equivariant map $\widetilde{A} : \widetilde{P} \times \widetilde{P} \to T(\widetilde{P})$, such that every slice $\widetilde{P} \times \widetilde{p}$ goes to the fiber $T_{\widetilde{p}}(\widetilde{P})$ and the map $\widetilde{A}_p : \widetilde{P} \times \widetilde{p} \to T_p(\widetilde{P})$ has degree one for all $\widetilde{p} \in \widetilde{P}$. This follows by elementary homotopy theory. What the homotopy theory is unable to provide is the contracting property of $\widetilde{A}$ on the slices $\widetilde{P} \times \widetilde{p}$. Yet, even without this property, one can study the restriction $\widetilde{\alpha}$ of $\widetilde{A}$ to a Γ-orbit, $\Gamma(\widetilde{p}_0) \times \widetilde{P} \to T(\widetilde{P})$, define the homomorphism $\widetilde{\alpha}_\cap : H_*(\widetilde{P} : \Gamma) \to H^*(\Gamma)$, and then prove $\widetilde{\alpha}_\cap$ is an isomorphism. Unfortunately, the lack of the contracting property makes $\widetilde{A}$ and $\widetilde{\alpha}$ unsuitable for our purposes, at least at the present state of the art. On the other hand, there is no counter-example in sight of the above contractible Γ-manifold $\widetilde{P}$, where one cannot find $\widetilde{A}$ with the contraction property. (If $\widetilde{P}/\Gamma$ is non-compact one should allow a preliminary modification of the metric in $\widetilde{P}$.)

10. Proper Lipschitz cocycles and the Novikov conjecture.

In this section we shall show that for any discrete group Γ and any *proper Lipschitz* cohomology class $\omega \in H^*(\Gamma, \mathbb{R})$, the Novikov conjecture is satisfied, i.e. the following expression is a homotopy invariant of pairs

(M, ψ), where M is a compact oriented manifold and $\psi : M \to B\Gamma$ a continuous map:

$$\textit{Higher Signature } (M, \psi) = \langle L(M)\psi^*(\omega), [M] \rangle \ .$$

The proof of this fact will be a simple application of Kasparov's Γ-equivariant KK-theory [K].

10.A THEOREM. *Let Γ be a discrete group. Every proper Lipschitz cohomology class $\omega \in H^k(\Gamma, \mathbb{R})$ satisfies the Novikov conjecture.*

We first recall that a *Fredholm representation* of the group Γ is given by a unitary representation π of Γ on a Hilbert space $\mathfrak{h}$ and an operator F on $\mathfrak{h}$ such that the following are compact operators (for any $g \in \Gamma$):

$$\text{a) } F^2 - 1 \ , \qquad \text{b) } F - F^* \ , \qquad \text{c) } \big[\pi(g), F\big] \ .$$

More specifically, these data define an *odd* Fredholm representation. An *even* one is given by the same data together with a $\mathbb{Z}/2$ grading γ, $\gamma^2 = 1$, $\gamma = \gamma^*$ of $\mathfrak{h}$, which *commutes with* $\pi(g)$, $\forall\, g \in \Gamma$, and *anticommutes* with F.

Using the Hilbert bundle on $B\Gamma$ obtained from the representation π of Γ on $\mathfrak{h}$ and a continuous family of Fredholm operators $(F_x)_{x \in B\Gamma}$ obtained from F (cf. [M]) one associates to every even Fredholm representation of Γ a virtual bundle on $B\Gamma$. This yields a map:

$$\mu^t : KK(C^*(\Gamma), \mathbb{C}) \to K^*(B\Gamma)$$

of the K-homology of the C^*-algebra of the group Γ to the K theory of the classifying space $B\Gamma$. Here $C^*(\Gamma) = C^*_{\max}(\Gamma)$ is the enveloping C^*-algebra of the involutive Banach algebra $\ell^1(\Gamma)$ of ℓ^1-functions on Γ, with convolution, and with the involution $f^*(g) = \overline{f}(g^{-1})$. The K-homology is defined for any C^*-algebra, as is the bivariant functor $KK(A, B)$ of Kasparov ([K]).

It follows from the work of Miscenko [M] that:

10.B Lemma. *Any $\omega \in H^*(B\Gamma, \mathbb{R})$ which is of the form $\mathrm{ch}\big(\mu^t(\beta)\big)$ for some $\beta \in KK\big(C^*(\Gamma), \mathbb{C}\big)$, satisfies the Novikov conjecture.*

As mentioned above, K-homology is defined for any C^*-algebra A. Moreover, in the commutative case, i.e. when $A = C_0(X)$ is the algebra of continuous functions vanishing at ∞ on a locally compact space X, one has a natural rational isomorphism:

$$\mathrm{ch}_* : KK(A, \mathbb{C}) \simeq H_*(X) ,$$

where $H_*(X)$ is the homology we considered in section 8 above when X is a locally finite polyhedron.

A bit more generally, this Chern character in K-homology still makes sense in the case of *proper* actions of a discrete group Γ on a locally compact space Y and one gets a natural rational isomorphism:

$$\mathrm{ch}_* : KK_\Gamma\big(C_0(Y), \mathbb{C}\big) \simeq H_*(Y : \Gamma) ,$$

which reduces to the above for $X = Y/\Gamma$ when the action of Γ is free. Here we used the equivariant KK-theory of Kasparov, whose definition is recalled below.

All this shows that, in order to prove Theorem 10.A, it is enough to lift to K-theory the construction of the map $\alpha_\cap : H_*(\widetilde{P} : \Gamma) \to H^*(B\Gamma)$ which was defined and explored in the previous sections. That is, it is enough to construct a map φ from the group $KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big)$ to $KK\big(C^*(\Gamma), \mathbb{C}\big)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big) & \xrightarrow{\phi} & KK\big(C^*(\Gamma), \mathbb{C}\big) & \xrightarrow{\mu^t} & K^*(B\Gamma) \\ \Big\downarrow \mathrm{ch}_* & & & & \Big\downarrow \mathrm{ch}^* \\ H_*(\widetilde{P} : \Gamma) & & \xrightarrow{\qquad\quad \alpha_\cap \qquad\quad} & & H^*(B\Gamma) \end{array} \tag{10.1}$$

Our data here is, exactly as above, a proper Γ-space $\widetilde{P}$ and a proper map $\alpha : \widetilde{P} \to \mathbb{R}^N$ which satisfies the displacement bound:

$$\mathrm{dist}_{\mathbb{R}^N}\,\big(\alpha(\widetilde{p}), \alpha(\gamma\widetilde{p})\big) \leq \|\gamma\| \qquad \forall\, \widetilde{p} \in \widetilde{P}\ ,\ \forall\, \gamma \in \Gamma\ .$$

It then follows that any class of the form $\omega = \alpha_\cap(x)$ for some $x \in H_*(\widetilde{P}; \Gamma)$ can (rationally, which is enough for our purpose) be written as $\omega = \mathrm{ch}^*\big(\mu^t \phi(y)\big)$ for some $y \in KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big)$ such that $\mathrm{ch}_*(y) = x$. Thus, any proper Lipschitz class is (rationally) in the range of $\mathrm{ch}^* \circ \mu^t$ and hence, by Lemma 10.B, satisfies the Novikov conjecture.

Thus, our proof is now reduced to two main steps:

1) construction of the map ϕ;
2) check that the diagram (10.1) is commutative.

Let us begin with 1). Parallel to the homological sitation, we shall define the map ϕ as an intersection product, i.e. we shall construct an element $K(\alpha) \in KK_\Gamma(\mathbb{C}, C_0(\widetilde{P}))$ of the Γ-equivariant K-theory of $\widetilde{P}$ and let

$$\phi(y) = K(\alpha)\#_\Gamma y \in KK_\Gamma(\mathbb{C}, \mathbb{C}) \simeq KK(C^*(\Gamma), \mathbb{C}) \ ,$$

where the intersection product in Γ-equivariant KK theory has been written $\#_\Gamma$.

For the convenience of the reader, we shall recall the definition of Kasparov of the group $KK_\Gamma(A, B)$, where A, B are two C^*-algebras on which the discrete group Γ acts by automorphisms. It will then be obvious that a proper map $\alpha : \widetilde{P} \to \mathbb{R}^N$ defines an element

$$K(\alpha) \in KK_\Gamma(\mathbb{C}, C_0(\widetilde{P})) \ .$$

In general, the group $KK_\Gamma(A, B)$ is constructed as the group of equivalence classes of Kasparov $A - B$ bimodules $(\mathcal{E}, F, \gamma)$. We have to explain what $\mathcal{E}, F, \gamma$ are and what conditions they have to satisfy.

First, $\mathcal{E}$ is a C^*-module over B. This notion extends the commutative notion (i.e. when $B = C_0(X)$, X locally compact) of a continuous field of Hilbert spaces $(\mathfrak{H}_x)_{x\in X}$, which itself contains as a special case the Hermitian complex vector bundles E over X. Given such a bundle E on X, the space $\mathcal{E} = C_0(X, E)$ of continuous sections of E vanishing at ∞ has the following structure:

- $\mathcal{E}$ is a right module over $C_0(X)$.
- The map $\xi, \eta \in \mathcal{E} \to \langle \xi, \eta \rangle \in C_0(X)$, $\langle \xi, \eta \rangle (x) = \langle \xi(x), \eta(x) \rangle$ (which uses the inner product in each fiber E_x of E) verifies, besides its obvious sesquilinearity (antilinear in ξ):
  - α) $\langle \xi a, \eta \rangle = a^* \langle \xi, \eta \rangle b \qquad \forall a, b \in C_0(X)$
  - β) $\langle \xi, \xi \rangle \geq 0$
  - γ) gifted with norm $\|\xi\| = \big\| \langle \xi, \xi \rangle \big\|^{1/2}$, $\mathcal{E}$ is a Banach space.

All these conditions make sense when $C_0(X)$ is replaced by an arbitrary C^*-algebra B and define the notion of C^*-module over B. Even when B is commutative, the notion is more flexible than that of the Hermitian bundle since it allows fiber dimensions which vary in a semicontinuous discontinuous manner.

To the usual notion of endomorphism of a Hermitian bundle, corresponds in general the notion of *endomorphism* of a C^*-module. They form a C^*-algebra:

$$\mathrm{End}_B(\mathcal{E}) ,$$

whose elements are pairs T, T^* of B-linear continuous maps from $\mathcal{E}$ to $\mathcal{E}$ such that:

$$\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle \qquad \forall\, \xi, \eta \in \mathcal{E} .$$

Any pair ξ, η of elements of $\mathcal{E}$ gives rise to the endormorphism:

$$|\xi >< \eta| \in \mathrm{End}_B(\mathcal{E}) ,$$

with:

$$(|\xi >< \eta|)\zeta = \xi \langle \eta, \zeta\rangle \in \mathcal{E} \qquad \forall\, \zeta \in \mathcal{E} .$$

The linear span of these special "rank one" endomorphisms, is a two sided ideal in $\mathrm{End}_B(\mathcal{E})$, and its elements are called *compact* endomorphisms.

Now a Kasparov $A - B$ bimodule is given by a C^*-module $\mathcal{E}$ over B, a representation of A in $\mathcal{E}$ (i.e. a $*$-homomorphism π of A in $\mathrm{End}_B(\mathcal{E})$), and an element F of $\mathrm{End}_B(\mathcal{E})$ such that the following are compact endomorphisms:

$$\text{a) } (F^2 - 1)\pi(a) ,\ \forall\, a \in A ; \qquad \text{b) } (F - F^*)\pi(a) ,\ \forall\, a \in A ;$$

$$\text{c) } \big[\pi(a), F\big] ,\ \forall\, a \in A .$$

Finally, in the Γ-equivariant case, in which Γ acts by automorphisms on both C^*-algebras A and B, one requires that Γ also acts on $\mathcal{E}$, that is one has an action ρ of Γ on $\mathcal{E}$ which is compatible with the action of Γ on A and B (i.e. $\rho(g)(a\xi b) = g(a) \cdot \rho(g)\xi \cdot g(b)$ for $a \in A$, $\xi \in \mathcal{E}$, $b \in B$ and $\langle \rho(g)\xi, \rho(g)\eta\rangle = g\, \langle \xi, \eta\rangle\ \forall\, \xi, \eta \in \mathcal{E}$) and verifies:

$$\text{d) } \rho(g)F\rho(g)^{-1} - F \text{ is compact for any } g \in \Gamma .$$

The essential feature of the Kasparov theory is the existence of the composition or intersection product:

$$KK_\Gamma(A, B) \times KK_\Gamma(B, C) \to KK_\Gamma(A, C) ,$$

which satisfies bilinearity and associativity relations. We refer to [K] for the precise description of the equivalence relation giving rise to $KK_\Gamma(A, B)$

out of classes of Kasparov Γ equivariant $A-B$ bimodules, and for the intersection product.

We shall now proceed to define the element $K(\alpha)$ of $KK_\Gamma(\mathbb{C}, C_0(\widetilde{P}))$, given the proper Γ-space $\widetilde{P}$ and the proper map $\alpha : \widetilde{P} \to \mathbb{R}^N$ satisfying the displacement bound. We thus construct a Γ-equivariant Kasparov $A-B$ bimodule where $A = \mathbb{C}$ is trivial and will be ignored, while $B = C_0(\widetilde{P})$ with the action of Γ coming from Γ's action on $\widetilde{P}$.

As a C^*-module we take $\mathcal{E} = C_0(\widetilde{P}, S)$, the space of continuous maps vanishing at ∞ from $\widetilde{P}$ to the fixed (finite dimensional) Hilbert space $S \simeq \mathbb{C}^{2^{N/2}}$ of spinors, associated to the Euclidean space $\mathbb{R}^N$. That means that S is a Hilbert space equipped with a linear map $\gamma : \mathbb{R}^N \to \operatorname{End}(S)$ such that

α) $\gamma(X) = \gamma(X)^*$, $\quad \forall X \in \mathbb{R}^N$

β) $\gamma(X)^2 = \|X\|^2$, $\quad \forall X \in \mathbb{R}^N$.

An endomorphism $T \in \operatorname{End}_B(\mathcal{E})$ is given by a continuous family $T_x \in \operatorname{End}(S)$, $x \in \widetilde{P}$. We can now use the proper map $\alpha : \widetilde{P} \to \mathbb{R}^N$ in order to define the endomorphism F we are looking for. Specifically, we take

$$F_x = \gamma\big(\alpha_1(x)\big) \ , \quad \alpha_1(x) = \big(1 + \big\|\alpha(x)\big\|\big)^{-1}\alpha(x) \ , \tag{10.2}$$

for any $x \in \widetilde{P}$.

The action of the group Γ on $\mathcal{E} = C_0(\widetilde{P}, S)$ is the obvious one, coming from the action on $\widetilde{P}$; that is

$$\big(\rho(g)\xi\big)(x) = \xi(g^{-1}x) \qquad \forall x \in \widetilde{P} \ , \ g \in \Gamma \ .$$

We are now all set to check that the triple $(\mathcal{E}, F, \rho)$ is a Γ-equivariant Kasparov $A-B$-bimodule, i.e. that conditions a),b),c),d) above are fulfilled. Since $A = \mathbb{C}$ acts by $\pi(\lambda) = \lambda \cdot \mathrm{id}_{\mathcal{E}}$, condition c) is automatic and we can replace $\pi(a)$ in a) and b) by $\mathrm{id}_{\mathcal{E}}$. By construction (condition α) above), the operator F is selfadjoint so that b) is clear.

To check a) we have to show that $F^2 - 1$ is a compact endomorphism of $\mathcal{E}$. Here $\mathcal{E}$ is the space of sections of a (trivial) Hermitian bundle with *finite dimensional* fiber S; thus, the compactness of an endomorphism, $(T_x)_{x\in\widetilde{P}} = T \in \operatorname{End}_B(\mathcal{E})$, is equivalent to the condition: $\|T_x\| \to 0$ when $x \to \infty$. This is immediately implied for $F^2 - 1$ by the conjunction of β) $(\gamma(X)^2 = \|X\|^2)$ and of the properness of α which shows that $\big\|\alpha_1(x)\big\| \to 1$ when $x \to \infty$ in $\widetilde{P}$.

Thus, it remains to check d), i.e. to show that for each element $g \in \Gamma$ one has:

$$\|F_{gx} - F_x\| \to 0 \quad \text{when} \quad x \to \infty\ .$$

Again, this follows immediately from the displacement bound together with the properness of α, which show that:

$$\big\|\alpha_1(gx) - \alpha_1(x)\big\| \to 0 \quad \text{when} \quad x \to \infty\ .$$

This shows that the triple $(\mathcal{E}, F, \rho)$ defines an element $K(\alpha) \in KK_\Gamma\big(\mathbb{C}, C_0(\widetilde{P})\big)$ of the Γ-equivariant K-theory group of $\widetilde{P}$. Note that this element is odd or even according to the parity of $N = \dim(\mathbb{R}^N)$; this means that in the even case the spinors S have a natural $\mathbb{Z}/2$ grading γ which makes F odd and everything else even. We can now define the map $\phi : KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big) \to KK_\Gamma(\mathbb{C}, \mathbb{C}) = KK\big(C^*(\Gamma), \mathbb{C}\big)$ by:

$$\phi(y) = K(\alpha) \#_\Gamma y\ .$$

In order to prove Theorem 10.A, it remains to show that with this choice of ϕ the diagram (10.1) is commutative.

We let Γ act on the space $E\Gamma$ of formal convex combinations $\sum\limits_0^k \mu_j \gamma_j$; $\mu_j \geq 0$, $\sum\limits_0^k \mu_j = 1$, of elements $\gamma_j \in \Gamma$. The space $E\Gamma$ is contractible and, "rationally", we can identify the quotient $B\Gamma = E\Gamma/\Gamma$ with the classifying space of Γ.

We let $\widetilde{P}$ be a proper Γ-space and $\alpha : \widetilde{P} \to \mathbb{R}^N$ be a proper map satisfying the displacement bound. We may assume (cf. 8.D) that the action of Γ on $\widetilde{P}$ is free. In this case, parallel to the homological situation discussed above, one has a natural isomorphism:

$$KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big) \simeq KK\big(C_0(\widetilde{P}/\Gamma), \mathbb{C}\big)\ .$$

To see that, one observes that, by construction, the group $KK_\Gamma\big(C_0(\widetilde{P}), \mathbb{C}\big)$ of Γ-equivariant Kasparov $C_0(\widetilde{P}) - \mathbb{C}$ bimodules is identical with the group $KK\big(C_0(\widetilde{P}) \rtimes \Gamma, \mathbb{C}\big)$ where the C^*-algebra $C_0(\widetilde{P}) \rtimes \Gamma$ is the crossed product of $C_0(\widetilde{P})$ by the action of Γ. Indeed, in both cases one deals with a covariant representation π of $\big(C_0(\widetilde{P}), \Gamma\big)$ on a Hilbert space $\mathfrak{h}$

together with an operator F on $\mathfrak{h}$ such that for any $f \in C_0(\widetilde{P})$ and $\gamma \in \Gamma$ the following are compact operators in $\mathfrak{h}$:

a) $\pi(f)(F^2-1)$; b) $\pi(f)(F-F^*)$; c) $[\pi(f), F]$; d) $[\pi(\gamma), F]$.

This gives the equality:

$$KK_\Gamma(C_0(\widetilde{P}), \mathbb{C}) = KK(C_0(\widetilde{P}) \rtimes \Gamma, \mathbb{C}) .$$

Now the action of Γ on $\widetilde{P}$ being free and proper, one has the Morita equivalence $C_0(\widetilde{P}) \rtimes \Gamma \cong C_0(\widetilde{P}/\Gamma)$ and hence the isomorphism:

$$KK(C_0(\widetilde{P}) \rtimes \Gamma, \mathbb{C}) \simeq KK(C_0(\widetilde{P}/\Gamma), \mathbb{C}) .$$

This implies in particular that we can restrict to triples $(\mathfrak{h}, \pi, F)$ as above such that $[\pi(\gamma), F] = 0$, $\forall \gamma \in \Gamma$. Let such a triple $\psi = (\mathfrak{h}, \pi, F)$ be given, and consider the space $Z = \widetilde{P} \times_\Gamma E\Gamma$. It is the total space of fibration $Z \xrightarrow{p} B\Gamma$ with fiber $\widetilde{P}$, canonically associated to the action of Γ on $\widetilde{P}$. The fibers $p^{-1}(x) \simeq \widetilde{P}$ are locally compact by construction. The exact Γ-invariance of $(\mathfrak{h}, \pi_0, F)$ where π_0 is the restriction of π to $C_0(\widetilde{P})$ shows that the class y can be used to integrate over the fibers in K theory, i.e. as a map:

$$K^*_{fc}(Z) \to K^*(B\Gamma) ,$$

where the left-hand side means K-theory with fiberwise compact support. More specifically, for any compact subset M of $B\Gamma$, the subspace $p^{-1}(M)$ of Z is locally compact and the class y defines an element of $KK(C_0(p^{-1}(M)), C(M))$, and hence a map:

$$K^*_c(p^{-1}(M)) \xrightarrow{y \times 1} K^*(M) .$$

Here the C^*-module over $C(M)$ is the space $\mathcal{E} = C(M, \widetilde{\mathfrak{h}})$ of continuous sections of the flat bundle of Hilbert spaces on $B\Gamma$ induced by the representation π of Γ. In other words, if we let $\widetilde{M}$ be the Γ-covering of M given by the pull-back to M of $E\Gamma \to B\Gamma$, an element ξ of $\mathcal{E}$ is a Γ-invariant continuous section of the trivial bundle with constant fiber $\mathfrak{h}$ on $\widetilde{M} : \mathcal{E} = C(\widetilde{M}, \mathfrak{h})^\Gamma$.

Since the operator F is Γ-invariant it defines an endomorphism of $\mathcal{E}$ by:

$$(F\xi)_x = F\xi_x , \qquad \forall \xi \in C(\widetilde{M}, \mathfrak{h})^\Gamma .$$

The action ρ of $C_0(p^{-1}(M))$ on $\mathcal{E}$ is given by:

$$\big(\rho(f)\xi\big)(\widetilde{x}) = \pi(f_{\tilde{x}})\xi_{\tilde{x}} \ , \qquad \forall \widetilde{x} \in \widetilde{M} \ , \ \xi \in \mathcal{E} \ ,$$

where $f_{\tilde{x}} \in C_0(\widetilde{P})$ is the restriction of f to the fiber $\widetilde{P}$ in the identification $\widetilde{P} \times_\Gamma \widetilde{M} = p^{-1}(M)$.

Now the element $\mu^t\big(\phi(y)\big) \in K^*(B\Gamma)$ is equal to $(y \times 1)(\sigma)$, where $\sigma \in K^*_{fc}(Z)$ is constructed as follows. One first extends, as we did above when dealing with homology, the map $\alpha : \widetilde{P} \to \mathbb{R}^N$ to a map α' from $\widetilde{P} \times_\Gamma E\Gamma \to \mathbb{R}^N$. Here $\widetilde{P} \times_\Gamma E\Gamma$ can be thought of as the space of formal convex combinations $\sum \mu_j \widetilde{x}_j$, where $\widetilde{x}_j \in \widetilde{P}$ are on the same Γ-orbit; then

$$\alpha'\left(\sum \mu_j \widetilde{x}_j\right) = \sum \mu_j \alpha(\widetilde{x}_j) \ .$$

This map is *fiberwise proper* on the fibration

$$\widetilde{P} \times_\Gamma E\Gamma = Z \xrightarrow{p} B\Gamma \ ,$$

and thus the pull-back by α' of the Bott elements $\beta \in K_c(\mathbb{R}^N)$ (i.e. the fundamental class of $\mathbb{R}^N$ in K-theory with compact supports) gives us an element $\alpha'^*(\beta) \in K_{fc}(Z)$.

Our claim, that $\mu^t\big(\phi(y)\big) = (y \times 1)\alpha'^*(\beta)$, follows from the standard description of the Bott element from spinors and Clifford multiplication [ABS].

It then follows that for any compact subset $M \subset B\Gamma$ and any K-homology class $z \in K_*(M)$ one has the equality:

$$\big\langle \mu^t\big(\phi(y)\big), z\big\rangle = \langle \alpha'^*(\beta), y \times z\rangle \ .$$

Thus, passing to Chern characters, we get:

$$\big\langle \mathrm{ch}^*\big(\mu^t\phi(y)\big), \mathrm{ch}_*(z)\big\rangle = \big\langle \mathrm{ch}^*\big(\alpha'^*(\beta)\big), \mathrm{ch}_*(y) \times \mathrm{ch}_*(z)\big\rangle \ .$$

The commutativity of the diagram (10.1) follows now from the equality:

$$\mathrm{ch}^*(\beta) = \text{Fundamental class of } \mathbb{R}^N \ (\text{in } H^*_{\mathrm{comp}}(\mathbb{R}^N)$$

and the naturality property of the Chern character.

II. Internal Criteria for the Existence of Contracting Maps into $\mathbb{R}^N$

Let Y be an n-dimensional Riemannian manifold, e.g. the universal covering $\widetilde{X}$ of a compact manifold X. We are interested in proper, contracting (i.e. distance decreasing) maps (and families of maps) $Y \to \mathbb{R}^N$, which may be used in the case $Y = \widetilde{X}$ for the construction of Lipschitz (co)homology classes of the fundamental group $\Gamma = \pi_1(X)$. For example, we seek a geometric criterion for the existence of a single proper, contracting map $Y \to \mathbb{R}^n$ of degree 1 (Y is assumed oriented at this point) and we want this criterion to be formulated in terms of Y itself without explicit reference to the external Euclidean space $\mathbb{R}^n$.

1. Selfcontracting manifolds and spaces.

A proper, continuous, selfmap $f : Y \to Y$ is called *selfcontracting* if

(a) the Lipschitz constant $\lambda = \lambda(f)$ is < 1, which means

$$\operatorname{dist}\left(f(y_1), f(y_2)\right) \leq \lambda \operatorname{dist}(y_1, y_2) ,$$

for all y_1, y_2 in Y and a fixed $\lambda < 1$;

(b) the map f is homotopic to the identity by a homotopy of *proper* maps $f_t : Y \to Y$.

In what follows we assume the Riemannian manifold is *complete* and *connected*. In this case the map f has a unique fixed point, denoted $y_0 \in Y$, and every Riemannian ball $B(r) \subset Y$ arround y_0 satisfies

$$f^{-1}\left(B(r)\right) \supset B(\lambda^{-1} r) ,$$

for the above $\lambda < 1$; therefore, the iterated pull-backs

$$b_1 = f^{-1}\left(B(r)\right) , \quad B_2 = f^{-1}(B_1), \ldots, B_i = f^{-1}(B_{i-1}), \ldots,$$

exhaust Y (provided $r > 0$). The complements $D_i = B_i - B_{i-1}$ do not pairwise intersect, their union $\bigcup_{i=1}^{\infty} D_i$ covers the complement $Y - B(r)$, and f maps D_{i+1} onto D_i for all $i = 1, 2, \ldots$ Thus, for every point y in Y outside $B(r)$ there exists a unique $i = i(y) = 0, 1 \ldots$ such that the i-th power (iterate) f^i of f brings y to $D_1 = f^{-1}\left(B(r)\right) - B(r)$.

2. Symmetrization of proper maps $Y \to \mathbb{R}^N$.

The word "*symmetry*" refers to equivariance of maps $Y \to \mathbb{R}^N$ with respect to f acting on Y and some self-similarity $x \mapsto \mu x$, μ for $0 < \mu < 1$, on $\mathbb{R}^N$. According to this, a map $\alpha : Y \to \mathbb{R}^N$ is called *μ-symmetric* (equivariant) *at infinity* if

$$\alpha(f(y)) = \mu\alpha(y) ,$$

for all y outside a compact subset in Y.

2.A LEMMA. *Let Y be a complete connected Riemannian manifold with a selfcontraction f and let μ be a number in the interval $0 < \mu < 1$. Then an arbitrary proper continuous map $\alpha_0 : Y \to \mathbb{R}^N$ is properly homotopic to a map $\alpha : Y \to \mathbb{R}^N$ symmetric at infinity.*

Proof: Let f_t denote the implied homotopy between $f = f_1$ and $\mathrm{id} = f_0$ and let $\mu_t = 1 - t(1-\mu)$ interpolate between $\mu_0 = 1$ and $\mu_1 = \mu$. Then we compose the homotopies f_t in Y and $x \mapsto \mu_t^{-1}x$ in $\mathbb{R}^N$ and set

$$\alpha_t = \mu_t^{-1}\alpha_0 \circ f_t .$$

Observe that $\alpha_{t=0} = \alpha_0$ and that $\alpha_1 = \mu^{-1}\alpha_0 \circ f$. Thus the deviation of α_1 from α_0 measures the asymmetry of α_0 in our sense. We observe that the homotopy α_t is *proper*, as it is a composition of proper homotopies. We choose a sufficiently large Riemannian ball $B = B(r) \subset Y$ around the fixed point y_0 of f, such that the complement $Y - B$ stays away from the origin $0 \leq \mathbb{R}^N$ in the course of the homotopy α_t, i.e. $\alpha_t(Y - B) \subset \mathbb{R}^N$, $t \in [0,1]$, does not meet a fixed open ball in $\mathbb{R}^N$ around the origin. Then we modify the homotopy α_t by making it constant on B and without changing it outside a small neighbourhood of B. This is done using the standard homotopy extension lemma (of Borsuk) which provides us with a new homotopy of proper maps $\alpha'_t : Y \to \mathbb{R}^N$, $t \in [0,1]$, such that

(i) $\alpha'_0 = \alpha_0$,
(ii) $\alpha_t \mid B = \alpha_0$, $t \in [0,1]$,
(iii) α'_1 equals α_1 on the complement $Y = f^{-1}(B)$,
(iv) the images $\alpha'_t(Y - B) \subset \mathbb{R}^N$ keep away from the origin for all $t \in [0,1]$.

Observe that conditions (ii) and (iii) imply, in view of the continuity of α_1, that α'_1 is symmetric in our sense on the boundary of the region

$D_1 = f^{-1}(B) - B$. Recall that $f^{-1}(B) \supset B$ and notice that the boundary ∂D_1 consists of two disjoint parts: the interior part, where D_1 meets B, i.e.

$$\partial_{\text{in}} = \partial B \ ,$$

and the exterior part, where D_1 is adjacent to $Y = f^{-1}(B)$,

$$\partial_{\text{ex}} = \partial\big(Y - f^{-1}(B)\big) \ .$$

The map f sends D_1 into B, such that ∂_{ex} goes to ∂_{in}, while the rest of D_1 goes strictly inside B. The map α_1' on ∂_{ex} equals $\alpha_1 = \mu^{-1}\alpha_0 \circ f$, which implies the symmetry condition for α_1' on ∂D_1,

$$\alpha_1'(y) = \mu^{-1}\alpha_1'\big(f(y)\big) \ , \qquad y \in \partial_{\text{ex}} \ ,$$

since $\alpha_1' \mid \partial_{\text{in}} = \alpha_0$.

Now we observe that the map α_1' on $f^{-1}(B)$ uniquely extends to a map α which is symmetric outside B by

$$\alpha(y) = \mu^{-i}\alpha_1' f^i(y)$$

for the integer $i = i(y)$, $y \in Y - B$, such that $f^i(y) \in D_1 = f^{-1}(B) - B$. The symmetry of α_1' on ∂D_1 insures the continuity of α on Y and the above property (iv) shows that α is properly homotopic to α_1 and hence to α_0. In fact, that property allows a homotopy between α_0 and α which keeps the infinity of Y away from the origin and then such a homotopy can be made proper by an obvious radial deformation in $\mathbb{R}^N$.

2.B COROLLARY. *Every proper map $\alpha_0 : Y \to \mathbb{R}^N$ is properly homotopic to a contracting map $Y \to \mathbb{R}^N$.*

Proof: An obvious smoothing operation makes the above α smooth, keeping it symmetric. Then the smoothed α is necessarily (and obviously) Lipschitz, provided $\mu^{-1}\lambda < 1$, where $\lambda = \lambda(f) < 1$ is the contracting (Lipschitz) constant of $f : Y \to Y$. (In fact, the Lipschitz constant of the smoothed α on D_i is bounded by $\text{const}(\mu^{-1}\lambda)^i$.) Since μ could be chosen arbitrarily, the corollary follows, as every Lipschitz map can be made contracting (i.e. with the Lipschitz constant < 1) by composing it with an appropriate self-similarlity of $\mathbb{R}^N$.

2.C *Remark on families of maps*: Our symmetrization of proper maps was completely canonical at the homotopy level and thus it works perfectly for arbitrary families of maps in so far as the relevant properties of these maps are uniform with respect to the parameters. This will be specified later on when we turn to specific cases.

2.D EXAMPLES: The above corollary shows in particular that every complete connected Riemannian manifold Y which admits a selfcontraction also admits a proper contracting map $\alpha : Y \to \mathbb{R}^n$, $n = \dim Y$, of degree 1. Notice that the existence of a selfcontraction $f : Y \to Y$ which is (by definition) homotopic to the identity, implies that Y is *contractible* and hence an *orientable* manifold. Then, for any choice of the orientation in Y, one has a proper map $\alpha_0 : Y \to \mathbb{R}^n$ of degree one (since Y is non-compact, as also immediately follows from the existence of a selfcontraction on Y), and Corollary 2.B allows a proper homotopy of α_0 to a *contracting* map α.

One can somewhat relax the assumptions on f needed for the existence of a proper contracting map $\alpha : Y \to \mathbb{R}^n$ of degree one. Namely one may only assume f has degree one, without insisting that it be homotopic to the identity. (Here, the manifold Y does not have to be contractible but we assume it is orientable.) Then our symmetrization process can be applied (we leave it to the interested reader) to some power f^k of f which is good enough for the Lipschitz corollary. Unfortunately, the homotopies involved in this more general symmetrization are non-canonical which makes the construction unsuitable for familes of maps.

Finally, one may ask what happens if one starts with a proper contracting map $Y \to Y$ of degree $d \geq 2$. Probably, such a Y does not, in general, admit any proper map to $\mathbb{R}^n$ of positive degree but we have not worked out a specific counterexample.

3. Γ-equivariant diagonal selfcontraction.

Let us describe the most important example of a family of selfcontractions of a Riemannian manifold Y with a given discrete isometric action of a group Γ. The manifold Y here may have a boundary but is still complete as a metric space.

3.A DEFINITIONS: A Γ-*dia-selfcontraction* of Y is a family of selfcontractions $f_y : Y \to Y$ parametrized by $y \in Y$ which is represented by a continuous map $F : Y \times Y \to Y \times Y$ for $F : (y_1, Y_2) \mapsto (y_1, f_{y_1}(y_2))$, such that the following four conditions are satisfied:

(i) The map F fixes the diagonal $Y = \Delta \subset Y \times Y$, i.e. $f_y(y) = y$.

(ii) The map F commutes with the diagonal action of Γ on $Y \times Y$, i.e. in terms of f_y,

$$\gamma f_y \gamma^{-1} = f_{\gamma y} \text{ for all } \gamma \in \Gamma \text{ and } y \in Y .$$

(iii) Every map f_y, $y \in Y$, is a proper λ-Lipschitz map (i.e. Lipschitz, with Lipschitz constant λ) of Y into itself, for a fixed $\lambda < 1$ independent of y.

(iv) There exists a homotopy F_t between $F_0 = \mathrm{Id}$ and $F_1 = F$, such that:

(iv)′ F_t is Γ-equivariant for the diagonal action of Γ;

(iv)″ F_t fixes the diagonal $Y = \Delta \subset Y \times Y$ and is *fiber preserving*, i.e. F_t maps $y \times Y$ into itself for all $y \in Y$;

(iv)‴ F_t is uniformly proper which means the following: there exists a function $R(d)$, such that $R(d) \to +\infty$ for $d \to +\infty$ and such that every point in $Y \times Y$ which is d-far from the diagonal $Y = \Delta \subset Y \times Y$ (for the product metric) remains $R(d)$-far from the diagonal in the course of the homotopy, i.e. the function $\delta(y_1, y_2) = \operatorname{dist}\big((y_1, y_2), \Delta\big)$ satisfies

$$\delta\big(F_t(y_1, y_2)\big) \geq R\big(\delta(y_1, y_2)\big) ,$$

for all $t \in [0,1]$ and $(y_1, y_2) \in Y \times Y$.

Notice that the uniformity is automatic if the action of Γ on Y is *cocompact*, i.e. if Y/Γ is compact.

3.B Example: Let Y be a complete simply connected manifold of non-positive sectional curvature $K \leq 0$. Here if there is a boundary we insist it is *convex* and then every two points y_1 and y_2 are joined by a unique geodesic segment. Let $f_{y_1}(y_2) \in Y$ be the center of this segment. Then this is a Γ-dia-selfcontraction with the contraction constant $\lambda = \frac{1}{2}$, as follows from the elementary properties of $K \leq 0$. Notice that here we do not need the metric to be smooth but may allow singular spaces with $K \leq 0$ (see [G], [D-G]).

4. Fiber contracting maps $Y \times Y \to T(Y)$.

Let Y be a contractible manifold with a smoth proper Γ-action. Then there exists a smooth map α_0 of $Y \times Y$ to the tangent bundle $T(Y)$ with the following five properties.

(a_0) α_0 is Γ-equivariant for the diagonal action of Γ on $Y \times Y$ and the obvious action on $T(Y)$ (i.e. the differential of the action of Γ on Y).

(b_0) Each "fiber" $y \times Y \subset Y \times Y$, $y \in Y$, is sent by α_0 to the tangent space $T_y(Y)$ and the map $y \times Y \to T_y(Y)$ is proper for all $y \in Y$.

(c_0) At every boundary point $y \in \partial Y$ the map α_0 send $y \times Y$ to the half-space in $T_y(Y)$ formed by the *inward* looking tangent vectors.

(d_0) At each diagonal point $(y, y) \in Y \times Y$ the differential of α_0 restricted to the tangent spece $T_{(y,y)}(y \times Y) = T_y(Y)$ equals the identity map $T_y(Y) \to T_y(Y)$.

(e_0) The α_0-pullback of the zero section $Y \hookrightarrow T(Y)$ equals the diagonal in $Y \times Y$.

The existence of such α_0 is a trivial exercise in algebraic topology and it is also clear that such α_0 is unique up to a homotopy in the class of maps satisfying (a_0)–(e_0).

An important (albeit obvious) property of α_0 is that it pulls back the Γ-invariant Thom class of $T(Y)$ to the Poincaré dual of the diagonal of $Y \times Y$, provided Y has no boundary. (A similar property remains valid in the presence of the boundary but we shall bypass the boundary problem in our cases of interest.)

Now let us assume that Y is a Riemannian manifold which admits a Γ-dia-selfcontraction.

4.A PROPOSITION. *There exists a continuous map $\alpha : Y \times Y \to T(Y)$ which satisfies the above properties* (a_0)–(d_0) *and also the following three additional properties:*

(1) *For each $y \in Y$ the map $\alpha : y \times Y \to T_y$ is contracting (i.e. λ-Lipschitz with $\lambda < 1$).*

(2) *The map α is uniformly proper, i.e.*

$$\left\|\alpha(y_1, y_2)\right\| > R(\operatorname{dist}(y_1, y_2)) ,$$

for some function $R(d)$ satisfying $R(d) \to +\infty$ for $d \to +\infty$ and the norm $\| \quad \|$ on $T(Y)$ defined by the Riemannian metric on Y.

(3) *The map α is homotopic to α_0 in the class of maps $Y \times Y \to T(Y)$ satisfying* (a_0)–(d_0).

Proof: One can trivially homotope α_0 in order to achieve (2) and so we may assume that α_0 is uniformly proper to start with. Then we apply the symmetrization process to the maps $\alpha : y \times Y \to T_y$ for all y and thus obtain a map α_1 which is symmetric, and hence contracting at infinity. Then it can be made contracting everywhere by the discussion in II.5.G below (which collapses to a triviality in the present case).

4.B COROLLARY. *The cohomology of* Γ *is proper Lipschitz.*

Proof: If Y has no boundary the proof is immediate with the map α restricted to $Y \times \Gamma(y_0)$ for some y_0.

Now, let the boundary ∂Y be non-empty and let us adjust our discussion to this case. First we introduce the manifold

$$Y_+ = Y \cup (\partial Y \times \mathbb{R}_+) ,$$

where ∂Y is identified with $\partial(\partial Y \times \mathbb{R}_+) = \partial Y \times 0$ in the obvious way. Notice that Y_+ is diffeomorphic to the interior of Y but for us it appears as the extension of Y by $\partial Y \times \mathbb{R}_+$. For each point $y_+ = (y,t) \in \partial Y \times \mathbb{R}_+ \subset Y_+$ we denote by $\beta_+ : T_y(Y) \to T_{y_+}(Y_+)$ the obvious isomorphism and we denote by $\delta = \delta(y_+)$ the tangent vector field $-t\frac{\partial}{\partial t}$ on $\partial Y \times \mathbb{R}$. Finally, we define the map $\alpha_+ : Y_+ \times Y \to T(Y_+)$ by

(i) $\alpha_+ \mid Y \times Y = \alpha$.

(ii) $\alpha_+(y_+, y') = \beta_+(\alpha(y,y')) + \delta(y_+)$,

for $y_+ = (y,t) \in \partial Y \times \mathbb{R}_+ \subset Y_+ - Y$. Then this α_+ restricted to $Y_+ \times \Gamma(y_0)$, $y_0 \in Y$ does the job, as a straightforward verification shows.

4.C Singular spaces with self-contractions. We want to extend the discussion in the previous section to singular (i.e. non-manifolds) Γ-spaces which can be regularized by embeddings into manifolds.

DEFINITION: A metric space Y with a Γ-action is called Γ-*regularizable* if there exist a Riemannian manifold Y' with a proper isometric action of Γ, a Γ-equivariant Lipschitz embedding $Y \subset Y'$ and a Γ-equivariant retraction $p : Y' \to Y$ which is homotopic to the identity by a homotopy of Γ-equivariant maps $Y' \to Y'$.

EXAMPLE: If Y is a finite dimensional polyhedron and the action of Γ is *free* or if it is *cocompact* then (Y,Γ) is well known to be regularizable (compare 6.F′).

4.C′. *If* (Y,Γ) *is* Γ-*regularizable and if* Y *admits a* Γ-*dia-selfcontraction then the cohomology of* Γ *is proper Lipschitz.*

Proof: One has as earlier in II.4 an equivariant map $Y' \times Y' \to T(Y')$ satisfying (a_0)–(e_0) of II.4, which can be symmetrized on $Y' \times Y$ such that the resulting map $\alpha : Y' \times Y \to T(Y')$ becomes contracting on the fibers $y' \times Y$. The details here are straightforward and left to the reader.

Remark: The regularity assumption on (Y, Γ) can be relaxed by allowing embeddings into certain infinite dimensional spaces as will be shown somewhere else.

EXAMPLE: We have already mentioned that the spaces of nonpositive curvature $K \leq 0$ admit Γ-dia-selfcontractions and whenever they are regularizable our proposition applies. Important instances of such Y are Bruhat-Tits buildings where the regularity condition restricts the structure of the isotropy subgroups Γ_y, $y \in Y$. For example, if the orders of Γ_y are bounded by a constant independent of $y \in Y$, then (Y, Γ) is regularizable (compare [KS]).

5. Interpolation of selfcontracting maps.

A subset in a metric space, say $Y_0 \subset Y$, is called a *net* if

$$\sup_{y \in Y} \operatorname{dist}(y, Y_0) < \infty .$$

Often one says "ε-net" for an $\varepsilon > 0$ if the above sup is $< \varepsilon$. This terminology suggests that ε is small but in our discussion the nets may be quite *rare*, which corresponds to large ε. A typical example is where Y is isometrically acted upon by a (discrete) group Γ with the compact quotient space Y/Γ and our Y_0 is a Γ-orbit $\Gamma(y_0) \subset Y$.

The problem we address in this section is as follows. Given a contracting (i.e. Lipschitz with the Lipschitz constant < 1) map $f_0 : Y_0 \to Y$, when does it extend to a contracting map $Y \to Y$?

What we are really interested in are contracting maps $Y_0 \to \mathbb{R}^N$ for $Y_0 = \Gamma(y_0)$ and these will eventually be constructed in three steps starting from a contracting map $f_0 : Y_0 \to Y$. The first step, which will be accomplished in this section, consists of an extension of f_0 to a contracting map $f : Y \to Y$. The second step is a construction (by symmetrization) of a contracting map $\alpha : Y \to \mathbb{R}^N$ starting from f. The third and final step is trivial, as contracting maps to $\mathbb{R}^N$ restrict from Y to Y_0. Thus the role of $Y \supset Y_0$ is purely auxiliary but it seems impossible to achieve our goal without bringing Y explicitly into the picture.

5.A Generalities on extension of Lipschitz maps. The basis of all extension results is the following well known and almost trivial

PROPOSITION. *Let Y_0 be a subset in an arbitrary metric space Y and $\alpha_0 : Y_0 \to \mathbb{R}$ be a Lipschitz function. Then α_0 extends to a Lipschitz function $\alpha : Y \to \mathbb{R}$, such that the Lipschitz constant of α equals that of α_0.*

Proof: First, let the complement $Y - Y_0$ consist of a single point y_1. Then the value $x_1 = \alpha(y_1) \in \mathbb{R}$ must lie in the intersection of the closed balls (intervals) $I(\alpha_0(y), r_y) \subset \mathbb{R}$, $y \in Y_0$, of radii $r_y = \lambda_0 \operatorname{dist}_Y(y, y_1)$, where λ_0 denotes the Lipschitz constant of f_0. Every two such intervals (balls) do intersect, since the extension problem is (obviously) solvable for the case where Y_0 consists of two points, and by the (trivial one-dimensional case of) Helly Theorem all intervals intersect. Thus the extension of α_0 is possible for

$$Y = Y_0 \cup \{y_1\} .$$

In the general case, the proof is concluded by well ordering the complement of Y_0. i.e. by writing

$$Y = Y_0 \cup \{y_1\} \cup \{y_2\} \cup \ldots,$$

and by using transfinite induction. (This looks slightly less ridiculous if the complement $Y - Y_0$ contains a countable dense subset as we only need the extension construction on this subset.)

EUCLIDEAN COROLLARY. *Every Lipschitz map $\alpha_0 : Y_0 \to \mathbb{R}^N$ extends to a Lipschitz map $\alpha : Y \to \mathbb{R}^N$ with the Lipschitz constant*

$$\lambda(\alpha) \leq \sqrt{N}\lambda(\alpha_0) .$$

Proof: Apply the above proposition to the coordinate functions of α_0.

Remark: One would not have had the loss in the constant if one had used the metric in $\mathbb{R}^N$ corresponding to the sup-norm

$$\left\|(x_1, \ldots, x_N)\right\| = \sup_{i=1,\ldots,N} |x_i| ,$$

instead of the Euclidean norm $\sqrt{\sum_i x_i^2}$.

RIEMANNIAN COROLLARY. *Let V be a contractible Riemannian manifold and $V_0 \subset V$ be a compact subset in V. Then every Lipschitz map $\alpha_0 : Y_0 \to V_0$ extends to a Lipschitz map $\alpha : Y \to V$ such that*

$$\lambda(\alpha) \leq C\lambda(\alpha_0) ,$$

where the constant C depends on V and V_0 (but not on Y, Y_0 or α_0).

Proof: Take a smooth embedding $V \subset \mathbb{R}^N$ (for $N = 2\dim V$) and observe that the contractibility of V implies the existence of a (smooth) Lipschitz map $p : \mathbb{R}^N \to V$ which fixes V_0. (In fact one only needs here the contractibility of the embedding $V_0 \hookrightarrow V$ and one may choose p with the image in a compact submanifold $V_1 \subset V$ containing the implied contracting homotopy.) Then the required extension is obtained by first extending α_0 to a Lipschitz map $Y \to \mathbb{R}^N \supset V_0$ and then by composing this with p.

5.B Uniformly Lipschitz contractible (ULC) spaces. A metric space V is called *C-contractible* for some positive function $C = C(\delta)$, $\delta \in [0,\infty)$, if for an arbitrary metric space Y, a subspace $Y' \subset Y$ and a Lipschitz map $\alpha' : Y' \to V$ with $\operatorname{Diam}\alpha'(Y') \leq \delta$, there exists an extension of α' to a Lipschitz map $\alpha : Y \to V$, such that the Lipschitz constants of α and α' satisfy

$$\lambda(\alpha) < C(\delta)\lambda(\alpha') .$$

In view of the proof of the above corollary a sufficient condition for the C-contractibility is as follows:

(5.1) for every subset $V' \subset V$ with $\operatorname{Diam} V' \leq \delta$, there exist a Lipschitz embedding $q : V' \to \mathbb{R}^N$ and a Lipschitz map $p : \mathbb{R}^N \to V$, such that $p \circ q = \mathrm{Id} : V' \to V'$ and the Lipschitz constants of p and q satisfy

$$\sqrt{N}\lambda(p)\lambda(q) \leq C(\delta) .$$

We say that V is *uniformly Lipschitz contractible* if it is C-contractible for some function $C = C(\delta)$.

5.C Basic Examples. Let V be a *contractible* Riemannian manifold which admits an isometric action of a group Γ with a compact quotient V/Γ. Then V is ULC as immediately follows from the above criterion (5.1) and the preceeding proof of the Riemannian Corollary.

A special case of this example is when V is the universal covering of a compact aspherical manifold and Γ is the Galois group of the covering.

A more general example of a similar nature is when V appears as a leaf of the foliation of some compact space, such that all leaves (including V) lying in the closure of V are contractible.

5.D LOCALIZATION OF THE LIPSCHITZ INEQUALITY NEAR A NET. *Let $Y_0 \subset Y$ be an ε_0-net and let $\alpha : Y \to V$ be a map such that*

(a) *$\alpha \mid Y_0$ is λ_0-Lipschitz, i.e. α is Lipschitz on Y_0 with the implied Lipschitz constant $\leq \lambda_0$,*

(b) *α satisfies the λ-Lipschitz inequality for all pairs of points y_1 and y_2 in Y within distance ε_0, i.e.*

$$\mathrm{dist}_V\left(\alpha(y_1),\alpha(y_2)\right) \le \lambda\,\mathrm{dist}_Y(y_1,y_2)\ ,$$

whenever $\mathrm{dist}_Y(y_1,y_2) \le \varepsilon_0$. *Then α is λ'-Lipschitz for*

$$\lambda' = 5\max(\lambda_0,\lambda)\ .$$

Proof: In order to estimate $\mathrm{dist}_V\left(\alpha(y_1),\alpha(y_2)\right)$ in the case where $\mathrm{dist}_Y(y_1,y_2) \ge \varepsilon_0$ we move y_1 and y_2 with $\mathrm{dist}(y_i,y_i') \le \varepsilon_0$ for $i = 1,2$, and observe that by the triangle inequality in V

$$\mathrm{dist}\left(\alpha(y_1),\alpha(y_2)\right) \le$$

$$\le \mathrm{dist}\left(\alpha(y_1),\alpha(y_1')\right) + \mathrm{dist}\left(\alpha(y_1'),\alpha(y_2')\right) + \mathrm{dist}\left(\alpha(y_2'),\alpha(y_2)\right)\ .$$

On the other hand, the triangle inequality in Y shows that

$$\mathrm{dist}(y_1',y_2') \le \mathrm{dist}(y_1,y_2) + 2\varepsilon_0$$

and so

$$\mathrm{dist}\left(\alpha(y_1),\alpha(y_2)\right) \le 2\lambda\varepsilon_0 + \lambda_0\left(\mathrm{dist}(y_1,y_2) + 2\varepsilon_0\right)\ .$$

This obviously implies for $\mathrm{dist}(y_1,y_2) \le \varepsilon_0$ that

$$\mathrm{dist}\left(\alpha(y_1),\alpha(y_2)\right) \le \lambda\,\mathrm{dist}(y_1,y_2)$$

for the required $\lambda' = 5\max(\lambda_0,\lambda)$. □

5.E Uniform local boundedness (ULB). A metric space Y is called *uniformly locally bounded* if it can be covered by subsets $B_i \subset Y$, $i \in I$, such that B_i are *uniformly bounded*, i.e.

$$\sup_{i\in I}\mathrm{diam}\,B_i \le \beta < \infty\ ,$$

and each ball $B(R) \subset Y$ of radius R contains at most $\nu < \infty$ subsets B_i where the number ν depends only on R (but not on the center of $B(R)$), i.e.

$$B_{i_j} \subset B(R)\ , \qquad j = 1,\dots,k, \Rightarrow k \le \nu = \nu(R)\ .$$

EXAMPLE: Let Y be the universal covering of a compact Riemannian manifold X. Then Y is ULB and the implied covering of Y can be obtained as a lift of a finite open covering of X by simply connected subsets.

Δ-coverings. A covering of Y by (finitely many) subsets, say $Y_1, \ldots, Y_m$, in Y, is called a Δ-*cover* if each Y_j can be decomposed into a union of *uniformly bounded* subsets,

$$Y_j = \bigcup_i B_j^i , \qquad i = 1, 2, \ldots,$$

which are mutually Δ-*separated*, i.e.

$$\operatorname{dist}(B_j^i, B_j^{i'}) > \Delta \quad \text{for all } j = 1, \ldots, m$$

and $i' \neq i$, where

$$\operatorname{dist}(A, B) \underset{\text{def}}{=} \inf \operatorname{dist}(a, b) \quad \text{over all } a \in A \text{ and } b \in B .$$

The following proposition is obvious.

A metric space Y is ULB *if and only if for every $\Delta > 0$ it admits a finite Δ-covering.*

EXAMPLE: Let Y be a discrete δ-*separated* space for some $\delta > 0$, which means

$$\operatorname{dist}(y_1, y_2) \geq \delta$$

for every two distinct points y_1 and y_2 in Y. Then the ULB property means that every R-ball in Y contains at most $\nu(R)$ points and a Δ-covering amounts to a partition of Y into a union of Δ-separated subsets.

It is also worth noticing that if some ε-net $Y_0 \subset Y$ is ULB then so is Y. Since every metric space Y contains a δ-separated ε-net Y_0 for arbitrary $\varepsilon > 0$ and $\delta < \varepsilon$, the above discussion applies via Y_0 to non-discrete spaces Y.

Finally we mention the following well known geometric criterion for ULB which will not, however, be explicitly used in this paper.

A complete Riemannian manifold with Ricci curvature bounded from below,

$$\text{Ricci} \geq -\rho > -\infty ,$$

is ULB.

5.F Lipschitz interpolation. Let Y and V be metric spaces, $Y_0 \subset Y$ be an ε_0-net and let $\alpha_0 : Y_0 \to V$ be a Lipschitz map.

INTERPOLATION LEMMA. *If V is* ULC *(Uniformly Lipschitz Contractible) and either Y or V is* ULB *(Uniformly Locally Bounded), then α_0 extends to a Lipschitz map $\alpha : Y \to V$.*

Proof: First we consider the case where Y is ULB and we take a finite Δ-covering of Y for some $\Delta > \varepsilon_0$, say

$$Y = \bigcup_{j=1}^{m} Y_j \ .$$

Now the extension is achieved in steps, by induction, as follows. Suppose, we have already obtained a Lipschitz map α_k on $\bigcup_{j=1}^{k} Y_j$ and we want to extend it to $\bigcup_{j=1}^{k+1} Y_j$. We decompose Y_{k+1} into a union of uniformly bounded and mutually Δ-separated subsets, $Y_{k+1} = \bigcup_i B^i_{k+1}$ and let $Y^i_{k+1}(\varepsilon_0) \subset \bigcup_{j=1}^{k} Y_j$ consist of the points y in $\bigcup_{j=1}^{k} Y_j$ which lie ε_0-close to B^i_{k+1}, i.e.

$$\operatorname{dist}(y, B^i_{k+1}) \leq \varepsilon_0 \ .$$

Now, we use ULC and extend the map $\alpha_k \mid Y^i_{k+1}(\varepsilon_0)$ to the union of $Y^i_{k+1}(\varepsilon_0)$ with B^i_{k+1}. The Lipschitz constant λ^i_{k+1} of the extended map depends only on $\lambda(\alpha_k)$ and so

$$\sup_i \lambda^i_{k+1} \leq \lambda_{k+1} < \infty \ .$$

Thus the above extensions, for all $i = 1, 2, \ldots,$ define a map $\alpha_{k+1} : \bigcup_{j=1}^{k+1} Y_j \to V$, such that the Lipschitz property (inequality) with the constant λ_{k+1} is satisfied for all pairs of points y_1, y_2 in $\bigcup_{j=1}^{k+1} Y_j$ within distance $\leq \varepsilon_0$. (This follows from $\Delta > \varepsilon_0$ and the definition of $Y^i_{k+1}(\varepsilon_0)$.) Then the Lipschitz localization (see II.2.D) implies that α_{k+1} is Lipschitz with $\gamma(\alpha_{k+1}) \leq 5\lambda_{k+1}$ as we may assume $\lambda_{k+1} \geq \lambda_k(\alpha_k)$.

Now we turn to the second case when V is ULB and we take a Δ-covering $V = \bigcup_{j=1}^{m} V_j$ for $\Delta > 3\varepsilon_0\lambda(\alpha_0)$. Then we pull back each V_j to Y_0 and take the ε_0-neighborhood of the pull-back $\alpha_0^{-1}(V_j) \subset Y$ for Y_j. Clearly $\bigcup_{j=1}^{m} Y_j = Y$ and each Y_j decomposes into a union of mutually Δ'-separated subset B_j^i for $\Delta' \geq (\lambda(\alpha_0))^{-1}\Delta - 2\varepsilon_0 > \varepsilon_0$, where the diameters of the images $\alpha_0(Y_0 \cap B_j^i) \subset V$ are uniformly bounded. It suffices to apply the above step-by-step extension argument which concludes the proof in the second case.

Remark: Notice that the Lipschitz constant $\lambda(\alpha)$ of the extension $\alpha : Y \to V$ is bounded in terms of the following data:

(1) The Lipschitz constant $\lambda_0 = \lambda(\alpha_0)$.

(2) The "net" number ε_0.

(3) The UCL-function $C(\delta)$, $\delta > R$.

(4) The number m of the Δ-covering.

(5) The supremum D of the diameters of the bounded subsets in the Δ-covering.

More precisely, in the first case, where Y is ULB, the Δ-covering depends on ε_0 and so the numbers Δ and D depend on ε_0. Then the first step of the extension process works at the scale $\delta_0 = \lambda(\alpha_0)(D + 2\varepsilon_0)$ and so $\lambda(\alpha_1)$ is bounded by $5\lambda(\alpha_0)C(\delta_0)$. Then $\delta_1 = \lambda(\alpha_1)(D + 2\varepsilon_0)$ and $\lambda(\alpha_2) \leq 5\lambda(\alpha_1)C(\delta_1)$ and so on. Here the final $\lambda(\alpha)$ depends on the geometries of both spaces Y and V.

On the contrary, in the second case, where V is ULB, the geometry of Y and Y_0 affects the final constant $\lambda(\alpha)$ only via ε_0 as we take a Δ-covering of V with $\Delta > 3\varepsilon_0\lambda_0$. In particular, if $\varepsilon_0\lambda_0$ is a priori bounded by a fixed constant, then $\lambda(\alpha)$ depends only on λ_0 and (the geometry of) V.

5.F′ CONTRACTING COROLLARY. *Let V be a metric space which is ULC and ULB and let λ and δ be positive constants. Then there exists a positive number $\lambda_0 = \lambda_0(V, \lambda, \delta)$, such that every λ_0-Lipschitz (i.e. with the Lipschitz constant $\leq \lambda_0$) map $\alpha_0 : Y_0 \to V$ extends to a λ-Lipschitz map $\alpha : Y \to V$, where Y is an arbitrary metric space and $Y_0 \subset Y$ is an ε_0-net with $\varepsilon_0 \leq \lambda_0^{-1}\delta$.*

We whall use this corollary for a fixed $\lambda < 1$ and we express it in words by saying that *every sufficiently strongly contracting map $\alpha_0 : Y_0 \to V$ extends to a contracting map $\alpha : Y \to V$, provided $\varepsilon_0\lambda(\alpha_0)$ is bounded* *by a fixed constant independent of the needed "strength of contraction"* $(\lambda(\alpha_0))^{-1}$.

5.G Interpolation of families of maps. Here we are interested in families of Lipschitz maps $\alpha_p : Y \to V$ where p runs over some parameter space P which typically is a manifold or a locally compact polyhedron. We want to have a family α_p which is continuous in $(y,p) \to Y \times P$ and such that the Lipschitz constants of α_p are uniformly bounded, i.e. $\lambda(\alpha_p) \leq \lambda < \infty$, $p \in P$. There is a simple reduction of this problem to the case of an individual map. Namely, every continuous family of λ-Lipschitz maps $\alpha_p : Y \to V$ for a given $\lambda > 0$ can be regarded as a λ-Lipschitz map $Y \times P \to V$ for the following metric d_α on $Y \times P$. To construct d_α we start with an arbitrary metric d_0 on $Y \times P$, whose restriction to each $Y = Y \times p$, $p \in P$, equals the original metric of Y. Then we denote by d^* the pull-back of the metric dist_V to $Y \times P$ by the map $\alpha : (y,p) \mapsto \alpha_p(y)$. Finally, we set

$$d_\alpha = \max(d_0, \lambda^{-1} d^*) \ .$$

(Notice that d^* is not quite a metric as it vanishes at some pairs of points in $Y \times P$, namely at those pairs of points which are identified by α, but d_α is a fully-fledged metric.) Since the maps $\alpha_p = \alpha \mid Y \times p$ are λ-Lipschitz for the metric $d_0 \mid Y \times p$, the metric d_α on $Y \times p$ equals d_0 (which is our original metric on Y). It is also clear that α is λ-Lipschitz with respect to d_α.

In order to apply the above considerations to the extension of maps from $Z \subset Y \times P$ to $Y \times P$ we first need an extension of metrics. To simplify the presentation we assume in the following Lemma that the spaces Y and P are *locally compact.*

METRIC EXTENSION LEMMA. *Let d be a metric on Z such that $d \mid Y \times p = d_Y$ for all $p \in P$ and a given metric d_Y on $Y = Y \times p$. Then there exists a metric $\bar{d}$ on $Y \times P$ such that*
(i) $\bar{d} \mid Y \times p = d_Y$ *for all* $p \in P$ *and*
(ii) $\bar{d} \mid Z \geq d$.

Proof: We start with the case when $Y \times P$ is compact and we look for a metric d' on P such that the *Cartesian product* (or *sum* $\bar{d}$ of d' and d_Y satisfies (ii), where the Cartesian product (sum) $\bar{d}$ is defined as the supremum of those metric δ on $Y \times P$ such that

$$\delta \mid Y \times p = d_Y \ , \qquad p \in P$$

and

$$\delta \mid y \times P = d_P , \qquad y \in Y .$$

Notice that with this definition the property (i) of $\bar{d}$ is automatic.

In order for $\bar{d}$ to satisfy (ii) the metric d' on P must be bounded from below by the function $\varepsilon(p_1, p_2)$ defined as follows. Let z_1 and z_2 be two points in Z over p_1 and p_2, i.e. of the form $z_1 = (y_1, p_1)$, $z_2 = (y_2, p_2)$. Set

$$\varepsilon(z_1, z_2) = d(z_1, z_2) - d_Y(y_1, y_2)$$

and then let

$$\varepsilon(p_1, p_2) = \sup \varepsilon(z_1, z_2)$$

where the sup is taken over all pairs of points z_1 and z_2 in Z lying over p_1 and p_2.

The function ε may be negative (even equal $-\infty$) for some p_1 and p_2 and we rectify the matter by taking

$$\varepsilon_+ = \max(0, \varepsilon) .$$

The function ε_+ obviously vanishes on the diagonal $\Delta = \{p_1 = p_2\} \subset P \times P$ and since Y and P are compact ε_+ is *uniformly* continuous at Δ. Therefore, there exists a non-negative *continuous* function ε' on $P \times P$ which dominates ε_+, i.e. $\varepsilon' \geq \varepsilon_+$ and still vanishes on Δ. Then, by compactness of P, the function ε' can be dominated by a metric d' on P, for example by

$$d'(p_1, p_2) = \sup_{p \in P} \left| \varepsilon'(p_1, p) - \varepsilon'(p_2, p) \right| .$$

This concludes the proof in the compact case.

If $Y \times P$ is non-compact, we take a locally finite cover of it by compact product subsets $Y_i \times P_i$, $i \in I$, equip each of them with a metric $\bar{d}_i$ satisfying the conclusion of the Lemma on $Y_i \times P_i$ and then define $\bar{d}$ on $Y \times P$ as the supremum of the metrics δ on $Y \times P$ satisfying the following two conditions:

(i)$'$ $\delta | Y \times p = d_Y$ for all $p \in P$.

(ii)$'$ $\delta | Y_i \times P_i \leq \bar{d}_i$ for all $i \in I$.

Conclusion: Extension of Lipschitz families. Suppose we are given a continuous map $\alpha : Z \to V$ which is λ-Lipschitz on the intersections $Z \cap (Y \times p)$, $p \in P$, and which we want to extend continuously to all of $Y \times P \supset Z$ with a controlled Lipschitz constant on all $Y \times p$. This is done by first bringing in the metric $d = d_\alpha$ on Z, defined as at the beginning of this section by

$$d_\alpha = \max(d_0, \lambda^{-1} d^*) ,$$

where d_0 is the original (product) metric on $Y \times P$ restricted to Z and d^* is induced by α from dist_V. Then d extends with the above Lemma (whenever that applies) to a metric $\overline{d}$ and then the extension problem for continuous families of Lipschitz maps $\alpha_p : Y \to V$ reduces to that for individual maps $Y \times P \to V$ which are Lipschitz with respect to $\overline{d}$.

6. Selfcontracting of hyperbolic spaces.

Let us start with a general geometric contraction which sometimes leads to a selfcontraction.

6.A Geodesic similarity map. Let Y be a metric space with a fixed point y_0 and λ be a number in the interval $0 \le \lambda \le 1$. Then λ-*selfsimilarity* $f = f_{y_0,\lambda}$ is the following set-valued selfmapping of Y : $f(y) \subset Y$ consists of those $y' \in Y$ which satisfy

$$\mathrm{dist}(y_0, y') = \lambda \, \mathrm{dist}(y_0, y) \quad \text{and} \quad \mathrm{dist}(y', y) = (1 - \lambda) \, \mathrm{dist}(y_0, y) \ .$$

If the subset $f(y) \subset Y$ is *non-empty* for all y_0, λ and y, then we say that Y is a *geodesic* space. If Y is complete then this geodesic property obviously implies the existence of a *geodesic segment* $[y_0, y] \subset Y$ between every pair of points y_0, y in Y, i.e. a subset in Y isometric to the real segment $[0, d]$ for $d = \mathrm{dist}(y_0, y)$, such that $0 \mapsto y_0$ and $d \mapsto y$ under the implied isometry $[0, d] \to [y_0, y_1]$. Notice that the selfsimilarity $f = f_{y_0,\lambda}$ reduces to $t \mapsto \lambda t$ on the geodesic segments issuing from y_0.

6.A$'$. Another useful "geodesic" construction is the radial (normal) projection of Y to the sphere of radius R in Y around y_0. This projection p applies to the points $y \in Y$ with $\mathrm{dist}(y_0, y) \ge R$ by

$$p(y) = \left\{ y' \in Y \mid \mathrm{dist}(y', y_0) = R \ , \ \mathrm{dist}(y, y') = \mathrm{dist}(y, y_0) - R \right\} .$$

6.B Hyperbolicity. A geodesic metric space Y is called *δ-hyperbolic* for some $\delta \geq 0$ if for every $y_0 \in Y$, every $R \geq 0$ and every two points with $r_i = \operatorname{dist}(y_0, y_i) \geq R$, $i = 1, 2$, any two points $y_i' \in p(y_i)$, $i = 1, 2$, satisfy the following *δ-inequality*

$$\operatorname{dist}(y_1', y_2') \leq \max\left(\delta, \operatorname{dist}(y_1, y_2) - \Delta_1 - \Delta_2\right)$$

for $\Delta_i = r_i - R$, $i = 1, 2$. We say that Y is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

EXAMPLES: (a) The real line $\mathbb{R}$ is obviously δ-hyperbolic for $\delta = 0$ but the Euclidean spaces $\mathbb{R}^k$ for $k \geq 2$ are not hyperbolic.

(b) Trees with geodesic metrics are 0-hyperbolic, as a simple argument shows. In fact every 0-hyperbolic space is a (generalized) tree.

(c) Every complete simple connected Riemannian manifold Y with strictly negative curvature, $K(Y) \leq -\kappa < 0$ is δ-hyperbolic for $\delta < 10\kappa^2$. This follows from the Cartan-Alexandrov-Toponogov inequality for $K < 0$.

A detailed account of basic properties and examples of hyperbolic spaces can be found in [G] and [D-G]. Here we only mention the following (easy but not completely trivial) statement.

6.C LIPSCHITZ STABILITY OF HYPERBOLICITY. *Let Y_1 and Y_2 be geodesic metric spaces (e.g. complete Reimannian manifolds), such that Y_1 is hyperbolic. If Y_1 and Y_2 admit nets (in the sense of* II.5 *$Y_1' \subset Y_1$ and $y_2' \subset Y_2$ which are Lipschitz equivalent (i.e. there exists a bijective Lipschitz map $Y_1' \to Y_2'$ whose inverse is also Lipschitz), then Y_2 is also hyperbolic.*

Γ-COROLLARY. *Let Y_1 and Y_2 be geodesic metric spaces which admit discrete cocompact isometric actions of some group Γ. If Y_1 is hyperbolic then so is Y_2.*

The Lipschitz stability and the Γ-Corollary are proved in [G] and [D-G]. An important consequence is that the hyperbolicity of a cocompact Γ-space Y (cocompact means Y/Γ is compact) depends only on Γ and is called the *word hyperbolicity* of Γ ("word" refers to the notion of a word metric in Γ). Every word hyperbolic group is finitely presented and if such Γ is realized by $\pi_1(X)$ for a compact manifold X, then the universal covering $\widetilde{X}$ is hyperbolic for the Riemannian metric on $\widetilde{X}$ induced from a Riemannian metric on X (see [G], [D-G]).

An important geometric class of word hyperbolic groups is constituted by the fundamental groups of closed Riemannian manifolds with negative curvature $K < 0$ but some of (the known examples of) hyperbolic groups do not come from negative curvature. In fact, one does not know if every hyperbolic group admits a discrete isometric action on some manifold with negative curvature. Yet many results extend from $K < 0$ to the hyperbolic case. An instance of this, the Novikov conjecture, is treated in the present paper (see II.6.E and compare [C-M]).

6.D Hyperbolic contraction of nets. Let Y be a δ-hyperbolic space and $f = f_{\lambda,y_0}$ the geodesic selfsimilarity of Y defined in II.6.A. Take two points y_1 and y_2 within a certain distance d in Y and let y_1' and y_2' be two points in the images $f(y_1)$ and $f(y_2)$, respectively. We want to estimate the distance $d' = \operatorname{dist}(y_1', y_2')$ and to do that we observe that the map f acts on each $y \in Y$ as the radial projection to the sphere of radius $\lambda \operatorname{dist}(y_0, y)$. We assume

$$\operatorname{dist}(y_0, y_1) \le \operatorname{dist}(y_0, y_2)$$

and let y_2'' be some radial projection of y_2' to the sphere of radius $R_1' = \operatorname{dist}(y_0, y_1') = \lambda \operatorname{dist}(y_0, y_1)$, i.e. $y_2'' \in p(y_1')$ for the implied radial projection p (see II.6.A$'$). By the triangle inequality

$$d' = \operatorname{dist}(y_2', y_2') \le \operatorname{dist}(y_1', y_2'') + \operatorname{dist}(y_2'', y_2') ,$$

where the first summand on the right hand side is estimated by the δ-inequality applied to the projection p to the sphere of radius R_1' and where the second summand equals $\lambda(R_2 - R_1)$ for $r_i = \operatorname{dist}(y_0, y_i)$, $i = 1, 2$, and where

$$R_2 - R_1 \le d$$

by the triangle inequality. Therefore, $d' \le m' + \lambda d$, for

$$\begin{aligned} m' &= \max\left(\delta, d - (1-\lambda)R_1 - (R_2 - \lambda R_1)\right) = \\ &= \max\left(\delta, d - (1-\lambda)(R_1 + R_2) - \lambda(R_2 - R_1)\right) \le \max(\delta, \lambda d) , \end{aligned}$$

and so

$$d' \le \max(\delta, 2\lambda d) .$$

Now let Y_0 be a Δ-separated ε-net in Y (i.e. every two distinct points in Y_0 have dist $\ge \Delta$) for $\Delta \ge 2\lambda^{-1}\delta$ and let us assign to each $y \in Y_0$ some point $y' \in f(y)$ for $f = f_{\lambda,p}$. Then the map $y \mapsto y'$ clearly is (by (5.1)) a 2λ-contracting map $Y_0 \to Y$.

6.E Selfcontraction of Y. Notice that the above ε-net Y_0 can be chosen with $\varepsilon \leq 2\lambda^{-1}\delta$ and so $\lambda\varepsilon \leq 2\delta$ remains bounded for $\lambda \to 0$. Thus we are in a position to find a Lipschitz extension of the above map, call it $f_0 : Y_0 \to Y$ to all of Y, according to (II.5.F′). Namely:

Let Y be ULB *and* ULC *then, for a sufficiently small $\lambda > 0$, the map f_0 extends to a self contraction of Y.*

Recall that the ULC and ULB properties are satisfied if Y is contractible and admits a discrete cocompact action of an isometry group Γ and that the hyperbolicity of Y in this (cocompact) case is equivalent to the word hyperbolicity of Γ. Also recall that the major role of selfcontractions $Y \to Y$ is to provide (sufficiently many) proper Lipschitz maps $\Gamma \to \mathbb{R}^N$, and so we want to realize a given group Γ by isometries of a contractible manifold. This is achieved in the hyperbolic case with the following theorem.

6.F THEOREM OF RIPS. *Every word hyperbolic group Γ admits a discrete cocompact simplicial action on some locally compact simplicial polyhedron P.*

Proof: Let Y be a metric space and $d > 0$. Denote by $P_d(Y)$ the simplicial complex whose k-simplices are abstractly spanned by those $(k+1)$-tupes of points $y_0, y_1, \ldots, y_k \in Y$ which satisfy

$$\operatorname{dist}(y_i, y_j) \leq d \ , \qquad i, j = 0, 1, \ldots, k \ .$$

LEMMA. *If Y is δ-hyperbolic and $d \geq d_0 = d_0(\delta)$ then the polyhedron $P_d(Y)$ is contractible. Furthermore, if Y_p is an ε-net in Y then $P_d(Y_0)$ is contractible for $d \geq d_0 = d_0(\delta, \varepsilon)$.*

See [G] and [D-G] for the proof.

Now, with this Lemma the proof of Rip's theorem is immediate. Take an arbitrary Riemannian manifold with cocompact isometric action of Γ, e.g. the universal covering of Y of a compact Riemannian manifold X with $\pi_1(X) = \Gamma$. Then choose some Γ-orbit $Y_0 = \Gamma(y_0) \subset Y$ and take $P = P_d(Y_0)$ for a sufficiently large d.

6.F′ COROLLARY. *Every word hyperbolic group Γ admits a faithful discrete isometric cocompact action on some contractible manifold Y_{co} with boundary.*

Proof: Every locally compact polyhedron P with a cocompact Γ-action is Γ-regularizable (see II.4.C) as it admits a piece-wise linear Γ-equivariant embedding into some Γ-manifold with boundary, say $i : P \to Y_{\mathrm{co}}$, so that the embedding i is a Γ-homotopy equivalence. For example, if Γ acts freely on P, one starts with a P.L. embedding of $P/\Gamma \to \mathbb{R}^N$, then one chooses a regular neighborhood $X \subset \mathbb{R}^N$ of the image and one finally takes the universal covering of X for Y_{co}. If the action of Γ on P is non-free this argument should be preceeded by locally equivariant embeddings of small neighborhoods of points $p \in P$ acted upon by the isotropy subgroups $\Gamma_p \subset \Gamma$. This is easy and well known, and in our case, of a contractible P, this gives us the desired contractible Γ-manifold Y_{co}.

6.F″ *Remarks*: (a) The above discussion yields the following more general conclusion.

Let Y be a hyperbolic ULB *space with an isometric action of some group Γ, such that the action is uniformly discrete in the following sense: for every $D \geq 0$ there exists an integer $k \geq 0$, such that for every subset $B \subset Y$ with* $\operatorname{Diam} B \leq 0$ *there exists at most k elements $\gamma \in \Gamma$ for which $B \cap \gamma(B) \neq \emptyset$. Then Γ admits a uniformly discrete isometric action on some* ULB *and* ULC *Reimannian manifold Y_{co} with boundary.*

(b) *Blowing away the boundary of Y_{co}*. Let us take the interior $Y_{\mathrm{in}} = Y_{\mathrm{co}} - \partial Y_{\mathrm{co}}$ and introduce a complete Riemannian metric g^+ on Y_{in} as follows. Denote by g_{co} the original Riemannian metric on Y_{co} and let $\varphi(y) = \inf\big(1, \operatorname{dist}(y, \partial Y_{\mathrm{co}})\big)$ for $y \in Y_{\mathrm{in}}$. Then we set

$$g^+ = \varphi^{-2} g_{\mathrm{co}} \quad \text{on} \quad Y_{\mathrm{in}}\ ,$$

and $Y^+ = (Y_{\mathrm{in}}, g^+)$ (compare II.4.B). Clearly, Y^+ is a *complete* manifold as the g^+-length of each curve C equals the integral of φ^{-1} over C with g_{co}-length (measure) element. In fact, the geometry of (Y_{in}, g^+) near the boundary ∂Y_{co} is close to that of the standard hyperbolic space $H^n - \operatorname{Int} B^n$, $n = \dim Y$, with the Poincaré metric.

To obtain a better picture of g^+ we assume that the manifold $Y_{\mathrm{co}} = (Y_{\mathrm{co}}, g_{\mathrm{co}})$ has *bounded geometry* in the following sense:

There exist constants $\varepsilon > 0$ and $\lambda > 0$, such that every point $y \in Y_{\mathrm{co}}$ admits a neighborhood U which is λ-Lipschitz equivalent to the intersection of an open ε-ball in $\mathbb{R}^n$ with a closed half-space, where "λ-Lipschitz" refers to a λ-Lipschitz homeomorphism whose inverse is also λ-Lipschitz.

Notice that if the action of Γ on $Y_{\rm co}$ is cocompact, then, obviously, the geometry is bounded. Also observe that under the assumptions of the above Remark (a) one can insure that $Y_{\rm co}$ has bounded geometry.

Now it is easy to see that in the bounded geometry case the following implications take place:

(1) $Y_{\rm co}$ is hyperbolic $\Leftrightarrow Y^+$ is hyperbolic,

(2) $Y_{\rm co}$ is ULB $\Leftrightarrow Y^+$ is ULB,

(3) $Y_{\rm co}$ is ULC $\Leftrightarrow Y^+$ is ULC.

It follows that, by replacing $Y_{\rm co}$ by Y^+, we can, under the assumptions of (a), produce a *complete hyperbolic Riemannian* Γ*-manifold without boundary* which is ULB and ULC. (In fact, if one wishes, one may have *bounded geometry* which is stronger than ULB.)

6.G Γ-nets. A Γ-net is, by definition, a subset $Z \subset Y \times Y$ invariant under the diagonal action of Γ and such $Z_y = Z \cap (y \times Y) \subset Y = y \times Y$ is a net in Y for every $y \in Y$ and such that the implied *density constants* $\varepsilon(Z_y) = \sup\limits_{y' \in Y} \operatorname{dist}(y', Z_y)$ are bounded from above by some $\varepsilon = \varepsilon(Z)$.

In what follows we often need Z to be Δ-*separated* for some large $\Delta > 0$ which means such separation for all $Z_y \subset Y$. It is also often convenient to have the diagonal of $Y \times Y$ contained in Z which can always be achieved by just adding the diagonal to Z and removing other points in Z close to the diagonal in order to retain the separation property of Z.

Locally constant nets. Suppose we are given a *covering* of Y by some subsets Y_i, $i \in I$, such that for every $\gamma \in \Gamma$ the translated set $\gamma(Y_i)$ equals some Y_j, $j = \gamma(i) \in I$, such that either $i = j$ or Y_j is disjoint from Y_i. Notice, that in our case of a *discrete* isometric action of Γ on a *locally compact* space there always exists an arbitrary fine locally finite covering by open subsets Y_i with the above property, such that the subgroup Γ_i moving Y_i into itself equals the isotropy subgroup Γ_{y_i} of some point $y_i \in Y_i$. Next let us take a net $Z(i) \subset Y$ for each $i \in I$, such that $Z(i)$ is invariant under the subgroup Γ_i and $Z(j) = \gamma Z(i)$ whenever $\gamma(Y_i) = Y_j$. If the densities $\varepsilon\big(Z(i)\big)$ are bounded by a fixed constant ε_0 then, obviously, the union

$$Z = \bigcup_{i \in I} Y_i \times Z(i)$$

is a Γ-net which has the implied density constant $\varepsilon(Z)$ bounded by ε_0 and which is called a *locally constant* Γ-net. Notice that each net $Z_y =$

$Z \cap (y \times Y) \subset Y = y \times Y$ in this case equals the union $\cup Z(i)$ over those $i \in I$ for which $Y_i \ni y$. Thus the Δ-separation property of Z essentially reduces to that for $Z(i)$. Namely if all $Z(i)$ are Δ-separated and

$$\operatorname{dist}\big(Z(i), Z(j)\big) > \Delta \tag{6.1}$$

whenever $Y_i \cap Y_j \neq \emptyset$ for $Y_i \neq Y_j$ then Z is Δ-separated. Notice that (6.1) imposes a lower bound on $\varepsilon = \varepsilon(Z)$ in terms of Δ and the multiplicity of the covering $Y = \bigcup_i Y_i$. On the other hand, if the multiplicity of the covering is bounded by some k, then for every $\Delta > 0$ one can find Δ-separated nets $Z(i)$ satisfying (6.1) and having $\varepsilon \leq C(k)\Delta$ for some universal function $C(k)$. This takes care of the free action and, in general, we have the following simple lemma.

6.G′ Lemma. *Suppose the space Y is finite dimensional and assume that for every $\Delta_0 > 0$ and every (necessarily finite) subgroup $\Gamma_0 \subset \Gamma$ fixing some point in Y the union of Δ_0-separated Γ_0-orbits in Y form a net with the density constant $\varepsilon_0 \leq C_0\Delta_0$ for a constant $C_0 = C_0(Y, \Gamma)$. Then for every Δ there exists a locally constant Δ-separated Γ-net $Z \subset Y \times Y$ with $\varepsilon(Z) \leq C_1\Delta$ for some constant $C_1 = C_1(Y, \Gamma)$.*

Proof: Since Y is finite dimensional (as well as locally compact and metrizable) one can choose the covering $Y = \bigcup Y_i$ of finite multiplicity. The above condition on Γ_0 gives us Γ_i-equivariant Δ-separated nets $Z(i)$, $i \in I$, with $\varepsilon\big(Z(i)\big) \leq C_0\Delta$ which then can be slightly (thanks to the bounded multiplicity) rarefied in order to satisfy (6.1). □

A typical (hyperbolic) example where the Γ_0-assumption of the lemma is *not* satisfied is where $Y = [-1, 1] \times \mathbb{R}$ with Γ consisting of the transformations $(t_1, t_2) \mapsto (\pm t_1, t_2 + k)$, $k \in \mathbb{Z}$. However, if we pass from Y to Y^+ considered in II.6.F″(b), then this difficulty disappears as the points in $Y^+ = Y_{\text{in}} \subset Y_{\text{co}} = Y$ lying close to ∂Y are "strongly moved" by all $\gamma \in \Gamma$. This effect can be achieved in the general case even if we start with a space Y without any boundary by first multiplying Y by $[0, 1]$ and then by applying the blow-up +construction to $Y \times [0, 1]$. Thus we obtain the following improvment of the above lemma.

6.G″. *Suppose that for every Γ_0 as in II.6.G′ there exist numbers $R > 0$ and $d > 0$, such that every ball in Y of radius R contains a point y, such that $\operatorname{dist}\big(y, \gamma(y)\big) \geq d$ for all $\gamma \in \Gamma_0 - \{\text{id}\}$. Then for every $\Delta > 0$ the* *space* $Y' = \left(Y \times [0,1]\right)^+$ *admits a closed locally constant* Δ*-separated* Γ*-net* $Z' \subset Y' \times Y'$ *with* $\varepsilon(Z') \leq C'\Delta$.

Remark: Notice that the assumption in II.6.G″ is satisfied in many natural cases, for example if the action of Γ on Y is cocompact. In particular, one always has the desired Γ-nets in the context of the word hyperbolic groups.

6.H Geodesic Γ-dia-contraction of nets. Here we assume Y is hyperbolic and every two points can be joined by a geodesic segment (which follows from our definition of hyperbolicity if Y is complete). We recall that for each positive $\lambda < 1$ and $y \in Y$ there is a natural set-valued geodesic similarity map denoted $f_{\lambda,y} : Y \to Y$ (see II.6.A) which sends each $y' \in Y$ to the union of the convex combinations $(1-\lambda)y + \lambda y'$ over all geodesic segments $[y, y'] \subset Y$ between y and y'. Notice that if Y is δ-hyperbolic, then (by an easy argument) this union, call it $\{(1-\lambda)y + \lambda y'\}$, has diameter $\leq 2\delta$.

6.H′ LEMMA. *Let the space* Y *satisfy the assumptions of* II.6.G′ *and let* $\mu \leq 1$ *be an arbitrary positive constant. Then there exists a closed* Γ*-net* $Z \subset Y \times Y$ *containing the diagonal of* $Y \times Y$ *and a continuous map* $F_0 : Z \to Y \times Y$ *with the following six properties* (*compare* II.3.A).

(i) *The map* F_0 *fixed the diagonal of* $Y \times Y$.

(ii) *The map* F_0 *commutes with the diagonal action of* Γ *on* $Y \times Y$ *and on* $Z \subset Y \times Y$.

(iii) *For every* $y \in Y$ *the map* F_0 *sends* $Z_y = Z \cap (y \times Y)$ *into* $y \times Y$ *and the resulting map, call it* $f_y : Z_y \to Y = y \times Y$, *is* μ*-Lipschitz for each* $y \in Y$.

(iv) *For every two points* y *and* y', *the image* $f_y(y')$ *lies* 3δ*-close to some segment* $[y, y'] \subset Y$ *between* y *and* y'.

(v) *The density constant* $\varepsilon(Z)$ *satisfies the inequality*

$$\varepsilon(Z) \leq C\Delta$$

for a constant $C = C(Y, \Gamma)$.

(vi) *The maps* f_y *are uniformly proper* (*see* II.3.A). *In fact*, $\operatorname{dist}\left(y, f_y(y')\right) \geq C\mu \operatorname{dist}(y, y')$.

Proof: First we construct Z as earlier of the form

$$Z = \bigcup_i Y_i \times Z(i) ,$$

such that the separation constant Δ of Z is large compared to the hyperbolicity constant δ of Y. Then we take a point y_i in each subset $Y_i \subset Y$, such that the istropy subgroup Γ_{y_i} equals the subgroup Γ_i mapping Y_i into itself and we join every point in $Z(i) \subset Y$ with y_i by a geodesic segment, say $[y_i, y']$, $y' \in Z(i)$, such that

$$[y_i, \gamma(y')] = \gamma([y_i, y'])$$

for all $\gamma \in \Gamma_i$. Then we take $\lambda = \mu/2$ and set

$$f_y(y') = (1-\lambda)y_i + \lambda y'$$

for all $y' \in Z(i)$, where the convex combination refers to the above chosen segment between y' and y_i. This gives us a map F_i of $Y_i \times Z(i) \subset Z$ to $Y \times Y$ and as all of Z is the union over $i \in I$ of such products, which can be assumed mutally disjoint by the proof of II.6.G$'$, we obtain our map $F : Z \to Y \times Y$. The only point which needs verification is the μ-Lipschitz, which follows from the hyperbolicity essentially the same way as earlier in II.6.D.

6.E Lipschitz cohomology of hyperbolic groups. Now we are ready to prove the following

THEOREM . *Let a group Γ admit an isometric discrete (see* II.6.F$''$*) action on a* ULB *hyperbolic metric space, e.g. Γ is a subgroup in a word hyperbolic group. Then the cohomology of Γ is properly Lipschitz.*

Proof: We already now (see II.6.F$''$) that Γ may act on a complete Riemannian hyperbolic manifold Y which is ULB and ULC and such that (due to the uniform discreteness assumption and thanks to II.6.G$''$) the action satisfies the assumptions of II.6.G$'$. Therefore, there exists a Γ-net $Z \subset Y \times Y$ and a map $F_0 : Z \to Y \times Y$ as claimed by II.6.H$'$. Then by the Lipschitz interpolation discussion in II.5.F and II.5.G this map extends to a Γ-dia-selfcontraction F of Y (see II.3.A), where the properties (i)-(iii) of F in II.3.A follow from the corresponding properties of F_0 (see II.6.H$'$) and (v) in II.6.H$'$ needed to insure the contracting property of F (according to II.5.F$'$). Then the existence of a uniformly proper homotopy F_t (see (iv) in II.3.A) trivially follows from (iv) and (vi) in II.6.H$'$.

Finally, the existence of a Γ-dia-selfcontraction implies (see II.4.B) that the cohomology of Γ is Lipschitz. $\square$

Recall that the most important corollary (for us) reads:

Γ satisfies the Novikov conjecture.

Remark: The uniform discreteness assumption is rather unpleasant as it rules out, for example, the actions of Γ on the hyperbolic space H^n with parabolic elements. It is easy to remove this restriction with an infinite dimensional version of the Lipschitz cohomology, appropriate for the Novikov conjecture. On the other hand, removing the ULB-condition requires more effort. This will be discussed further in another paper.

III. Lipschitz Cocyles and Secondary Classes

In this section we shall show that the hypothesis of *properness* of the action of the discrete group Γ on the space $\widetilde{P}$, in the construction of Lipschitz classes, is unnecessary. Thus, both the construction of the group cocycle c and the Novikov conjecture for a cohomology class of c will remain valid for the extended notion of Lipschitz classes. Besides being more natural (Lipschitz cohomology becomes functorial for any group homomorphism), the extended theory now covers the group cocycles on diffeomorphism groups coming from Gelfand-Fuchs cohomology.

1.A Families with a fixed target. This means a continuous map $\alpha : P \longrightarrow \mathbb{R}^N$, where P is an oriented smooth manifold on which Γ acts by orientation preserving diffeomorphisms. We not longer assume that the action of Γ on P is proper. The assumptions on α are:

(I) *Displacement bound*:

$$\operatorname{dist}_{\mathbb{R}^N}\big(\alpha(p), \alpha(\gamma p)\big) \leq \|\gamma\| , \qquad \forall p \in P , \ \gamma \in \Gamma .$$

(II) *Properness*: $\alpha : P \longrightarrow \mathbb{R}^N$ is proper.

The construction done in I.8 above, of the group cocycle $\alpha_\cap[P] \in H^k(\Gamma)$, works without any change. One has:

$$c(\gamma_0, \ldots, \gamma_k) = \int_{\Delta \times P} \alpha_\Delta^*(u) ,$$

where $\alpha_\Delta : \Delta \times P \longrightarrow T = P \times \mathbb{R}^N$, and $u = 1 \times \beta$ is the (Γ-invariant) Thom class of the trivial bundle $T = P \times \mathbb{R}^N$ over P (β is the generator of $H^N_{\text{comp}}(\mathbb{R}^N)$). For any Γ-invariant smooth form $\omega \in H^j(\widetilde{P} : \Gamma)$ the same construction works and yields the following cocycle of dimension $j + (N - \dim \widetilde{P}) = q$

$$c(\gamma_0, \ldots, \gamma_q) = \int_{\Delta \times P} \alpha_\Delta^*(\beta \times \omega) .$$

One can easily remove the orientation hypothesis on P and work with twisted Γ-invariant forms on P. We thus get a map $\alpha_\cap : H_i(P : \Gamma) \to H^{N-i}(\Gamma)$.

1.B Families with a variable target. Here the Γ-manifold P comes along with an Euclidean, oriented Γ-bundle $T \xrightarrow{\pi} P$. This means, as above, that T is a vector bundle over P which is Γ-equivariant, the Γ-action preserving both the metric and the orientation. The map α is now a continuous section

$$\alpha : P \to T$$

of the bundle T on P, which satisfies the following two conditions:

(I)* *Displacement bound:*

$$\operatorname{dist}\big(\gamma\alpha(p), \alpha(\gamma p)\big) \leq \|\gamma\| \,, \qquad \forall \gamma \in \Gamma \,,\ p \in P \,.$$

(II)* *Properness*: The function $p \to \big\|\alpha(p)\big\|$ is proper on P.

Note that we *do not* impose the condition that would normally follow from I.7.B, i.e. that $\alpha(p)$ is fixed by the isotropy subgroup $\{\gamma \in \Gamma \ ,\ \gamma p = p\}$. It turns out to be unnecessary both for the construction of $\alpha_\cap$ and for the Novikov conjecture.

To construct $\alpha_\cap$ let us consider the classifying space $B\Gamma$, together with the universal Γ-principal bundle:

$$E\Gamma \to B\Gamma$$

with $E\Gamma$ a contractible Γ-space on which Γ acts properly and freely. Let then $P_\Gamma = P \times_\Gamma E\Gamma$, $T_\Gamma = T \times_\Gamma E\Gamma$ be the corresponding induced bundles over $B\Gamma$, and $\pi_\Gamma : T_\Gamma \to P_\Gamma$ be the corresponding projection. The fibers of $p : P_\Gamma \to B\Gamma$, $t : T_\Gamma \to E\Gamma$ are naturally isomorphic to P and T respectively and the bundle $T_\Gamma \xrightarrow{\pi_\Gamma} P_\Gamma$ is an oriented Euclidean vector bundle. In particular it has a Thom class which we can view as a cohomology class:

$$u \in H^N_{\text{proper}}(\widetilde{T}_\Gamma) \,,$$

where the projection π_Γ is *proper* on the support of u; in fact we can assume that the support of u is contained in the unit ball bundle of T_Γ.

Next, using the local triviality of the Γ-principal bundle $E\Gamma \to B\Gamma$, we get sections $\alpha_i : p^{-1}(U_i) \to T_\Gamma \mid p^{-1}(U_i)$ for open sets $U_i \subset B\Gamma$, and we can, using a partition of unity $\{\chi_i\}$, combine them into a section $s(q) =$

$\sum \chi_i(p(q))\alpha_i(q)$ of $T_\Gamma \xrightarrow{\pi_\Gamma} P_\Gamma$. The displacement bound shows that the norm, $\|s(q)\|$ of this section is a proper function on each fiber of the fibration $P_\Gamma \xrightarrow{p} B_\Gamma$.

It thus follows that the pull-back of the Thom class u is a cohomology class

$$s^*(u) \in H^N_{\text{proper}}(P_\Gamma)$$

in the cohomology of P_Γ with *proper support* for the projection $p : P_\Gamma \to B\Gamma$.

If P is oriented and Γ preserves this orientation we can now define the class $\alpha_\cap[P] \in H^{N-d}(B\Gamma) = H^{N-d}(\Gamma)$, where $d = \dim P$, by integrating $s^*(u)$ along the fibers of the fibration $P_\Gamma \xrightarrow{p} B\Gamma$. More specifically, when evaluated on an $N-d$ dimensional singular simplex $f : \Delta \to B\Gamma$, the cocycle $c = p!s^*(u)$ is given by $\int_{P_\Delta} f^*s^*(u)$, where P_Δ is the pull-back to Δ by f of the bundle $P_\Gamma \xrightarrow{p} B\Gamma$, while $f : P_\Delta \to P_\Gamma$ is the corresponding map.

In general, given a Γ invariant current of order 0, i.e., a current given locally by a differential form with measures as coefficients, $\omega \in H^j(P : \Gamma)$, we define $\alpha_\cap(\omega \cdot [P]) \in H^{N-d+j}(B\Gamma) = H^{N-d+j}(\Gamma)$ as $p!\big((\omega \times_\Gamma 1)s^*(u)\big)$. Here $\omega \times_\Gamma 1$ is the extension of ω as a cohomology class on $P_\Gamma = P \times_\Gamma E\Gamma$.

As above, we thus get a well defined map

$$H_k(P : \Gamma) \xrightarrow{\alpha_\cap} H^{N-K}(\Gamma)\ .$$

1.C DEFINITION. *A cohomology class c in $H^*(\Gamma)$ with real coefficients is called Lipschitz if there exists $P, T \to P$ and α as above, such that $c = \alpha_\cap(b)$ for some $b \in H_*(P : \Gamma)$.*

The main result of this section will be that any Lipschitz cohomology class satisfies the Novikov conjecture. Before embarking on the proof, which will rely on cyclic cohomology, we shall exhibit interesting examples of Lipschitz classes for which the Γ-action is not proper.

2. Examples.

The first two examples will be trivial but not the third one whose generalization will show that Gelfand-Fuchs classes on Diff are Lipschitz.

EXAMPLE 1: *Any 0-dimensional class is Lipschitz, with P = the one-point Γ-space and $N = 0$.* Note that if we insisted that the Γ-action on P be proper it would not be clear at all that such a class is Lipschitz.

EXAMPLE 2: *Any* 1-*dimensional class* $h \in H^1(\Gamma, \mathbb{R})$ *is Lipschitz.* Indeed, it comes from a group homomorphism $h : \Gamma \to \mathbb{R}$, being a group 1-cocycle. We then let $P = \mathbb{R}$ be endowed with the action of Γ by translation: $\gamma \cdot p = h(\gamma) + p$, $\forall \gamma \in \Gamma$, $p \in \mathbb{R}$. We let α be the identity map from $\mathbb{R}$ to $\mathbb{R}^N$, $N = 1$ and let $b \in H_0(P : \Gamma)$ be the homology class of dimension 0 given by Lebesgue measure $\mu = dp$, i.e., with the notation of 1.A by the translation invariant 1-form $\omega = dp$. For $\gamma_0, \gamma_1 \in \Gamma$ one has:

$$c(\gamma_0, \gamma_1) = \int_{\Delta \times P} \alpha_\Gamma^*(\omega \times \beta)$$

where $\alpha_\Delta\big(\sum_0^1 \lambda_j \gamma_j, p\big) = \big(p, \sum_0^1 \lambda_j\big(h(\gamma_j) + p\big)\big)$; thus, with $\beta(p) = f(p)\, dp$, $\int f\, dp = 1$, and $\lambda_0 = 1 - \lambda$, $\lambda_1 = \lambda$ we get

$$\alpha_\Delta^*(\omega \times \beta) = (dp \wedge d\lambda)\big(h(\gamma_1) - h(\gamma_0)\big) f\Big(\sum_0^1 \lambda_j\big(h(\gamma_j) + p\big)\Big) \ .$$

Hence, integration in dp eliminates f and integration in λ gives $h(\gamma_1) - h(\gamma_0)$ as expected.

EXAMPLE 3: We consider the group $\mathrm{Diff}^+(S^1) = \Gamma$ of orientation preserving diffeomorphism of the circle (or any countable subgroup) together with the Godbillon-Vey class, viewed as an element of $H^2(\Gamma, \mathbb{R})$ thanks to the following formula of Bott and Thurston:

$$c(g^1, g^2) = \int_{S^1} g_2^*\big(d\ell(g_1)\big)\ell(g_2) \ .$$

Here c is a group 2-cocycle, i.e. $c(g^1, g^2) = c'(1, g^1, g^1 g^2)$ with c' a left invariant straight cocycle; also $\ell(g)$, for $g \in \Gamma$, is the logarithm of the Jacobian:

$$\ell(g) = \mathrm{Log}\left(\frac{dg(x)}{dx}\right) \ .$$

We shall now show that this 2-cocycle on Γ is Lipschitz. In fact the same proof will work for Gelfand-Fuchs cohomology classes.

Let us carefully construct the triple (P, α, ω). The Γ-space P is the 3-dimensional manifold $J_2^+ = P$ of 2-jets of diffeomorphisms of a neighborhood of $0 \in \mathbb{R}$ with an open set of S^1. With $S^1 = \mathbb{R}/\mathbb{Z}$ any $j \in J_2^+$ can be written $j(t) = y + ty_1 + t^2 y_2$, where $y \in S^1$, $y_1 > 0$ (this is the meaning of the $+$ in J_2^+) and $y_2 \in \mathbb{R}$. The group $\Gamma = \mathrm{Diff}^+(S^1)$ acts on J_2^+ by composition, thus, with $\gamma \in \Gamma$ one has

$$(\gamma \circ j)(t) = \gamma(y) + ty_1\gamma'(y) + t^2\big(\gamma'(y)y_2 + \frac{\gamma''(y)}{2} y_1\big) \ .$$

This action commutes with the natural structure of G-principal bundle on J_2^+, where the Lie group G is the group of 2-jets of orientation preserving diffeomorphisms of $\mathbb{R}^2$ fixing the origin, i.e. the 2-dimensional group of upper triangular unimodular matrices.

Obviously the action of $\Gamma = \mathrm{Diff}^+(S^1)$ on P is not proper. The proper map $\alpha : P \to \mathbb{R}^2$ is constructed as follows: the principal G-bundle J_2^+ is trivial (since G is contractible) as is obvious from the choice of section $j_0(y) = y + t$, i.e. $y_1 = 1$, $y_2 = 0$. This gives us a map $\alpha_0 : P \to G$ such that $\alpha_0(j \cdot a) = \alpha_0(j)a$, $\forall\, a \in G$. Now we endow G with a right invariant Riemannian metric of negative curvature (and G is then isometric to the Poincaré disk). We then take:

$$\alpha(j) = \exp_1^{-1}\big(\alpha_0(j)\big) = \log_1\big(\alpha_0(j)\big) \in \mathbb{R}^2 \ ,$$

where $\log_1$ is the inverse of the Riemannian exponential map at $1 \in G$, $\exp_1 : \mathbb{R}^2 = T_1(G) \to G$.

The map α is proper by construction, and we shall check that it satisfies the displacement bound (with some constant c) for each finitely generated subgroup of $\mathrm{Diff}^+(S^1)$. Since $\log_1$ is a contraction, we just have to check the bound:

$$\mathrm{dist}_G\big(\alpha_0(\gamma j), \alpha_0(j)\big) \leq c\ell(\gamma) \ , \qquad \forall\, j \in P \ , \ \gamma \in \Gamma_0 \ ,$$

where $\ell(\gamma)$ is the word length of $\gamma \in \Gamma_0$ with respect to a finite set of generators. But such a bound follows from the finiteness of $\operatorname{Sup}_{j \in p}\big(\mathrm{dist}_p(\gamma j, j)\big)$, for any $\gamma \in \Gamma$, where we use on P any Riemannian metric which is invariant under the right action of G.

Since P is oriented and Γ preserves the orientation, to get an element b of $H_0(P : \Gamma)$ we look for a Γ-invariant differential form ω of degree 3 on P. We take $\omega = 2y_1^{-3}\, dy \wedge dy_1 \wedge dy_2$.

LEMMA. *The 2-cocycle $\alpha_\cap(\omega \cdot [P]) \in H^2(\mathrm{Diff}^+(S^2), \mathbb{R})$ is the Godbillon-Vey cocycle.*

Proof: Since our main concern is to show that $\alpha_\cap(\omega \cdot [P])$ is non-zero, we shall handle only the case when $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}^+(S^1)$ is the fundamental group of a compact Riemann surface of genus > 1. The general case follows along the same lines but is more cumbersome. Thus we identify $E\Gamma_0$ with the Poincaré disk U and let Γ_0 act on $U = E\Gamma_0$ by isometries with quotient $M = B\Gamma_0$. As above in III.1.B, we let $P_{\Gamma_0} \to M = B\Gamma_0$ be the induced bundle over M from the action of Γ_0 on $P = J_2^+$. This bundle is a principal G-bundle over $S^1_{\Gamma_0} = S^1 \times_{\Gamma_0} M$. Since G is contractible, it is trivial and admits G-equivariant smooth maps

$$\alpha_1 : P_{\Gamma_0} \to G$$

whose composition with $\log_1$ yields the map s,

$$s : P_{\Gamma_0} \to \mathbb{R}^2 \ , \qquad s = \log_1 \circ \alpha_1 \ ,$$

which is used in III.1.B. It follows that the cohomology class $s^*(u) \in H^2_{\text{proper}}(P_{\Gamma_0})$ is the pull-back by s of the fundamental class of $\mathbb{R}^2$ in cohomology with compact support, and is hence *Poincaré dual* to any smooth section of $P_{\Gamma_0} \xrightarrow{\rho} S^1_{\Gamma_0}$.

We want to compute $\alpha_\cap(\omega \cdot [P])$ evaluated on the homology fundamental class $[M] \in H_2(M,\mathbb{R})$. Thus, by III.1.B, we just need to compute

$$\int_{P_{\Gamma_0}} (\omega \times_{\Gamma_0} 1) s^*(u) \ ,$$

where the 3-form ω has been extended to $P_{\Gamma_0} = P \times_{\Gamma_0} E\Gamma_0$ using its Γ_0-invariance. But since $s^*(u)$ is Poincaré dual to (any) section $\sigma : S^1_{\Gamma_0} \to P_{\Gamma_0}$ of ρ we get:

$$\int_{S^1_{\Gamma_0}} \sigma^*(\omega \times_{\Gamma_0}) \ .$$

We shall now check that this is the Godbillon-Vey invariant of the foliation of $S^1_{\Gamma_0} = S^1 \times_{\Gamma_0} E\Gamma_0$ which is the horizontal foliation of this flat bundle with fiber S^1 over M. For this, it is enough to check that the pull-back $\rho^*(GV)$ of the Goodbillon-Vey class of this foliation is given by the 3-form $\omega \times_{\Gamma_0} 1$. By naturality of the Godbillon-Vey class we just need to check that $\omega \times_{\Gamma_0} 1$ is the Godbillon Vey class of the pulled back foliation. However, this latter foliation is defined by the closed non-vanishing 1-form $\theta \times_{\Gamma_0} 1$, where θ is the Diff^+-invariant 1-form $\theta = \frac{dy}{y_1}$ on J_2^+. Thus, the result follows from the following classical equalities between Diff^+ invariant forms on J_2^+:

$$d\theta = \theta_1 \wedge \theta \ , \qquad \omega = d\theta_1 \wedge \theta_1$$

where $\theta_1 = 2\frac{y_2}{y_1^2}dy - \frac{dy_1}{y_1}$.

We shall now extend the construction of example 3 to higher dimensional Gelfand-Fuchs classes.

3. Higher dimensional Gelfand Fuchs classes are Lipschitz.

Let M be a smooth oriented, compact, n-dimensional manifold and let $\Gamma = \mathrm{Diff}^+(M)$ be the group of orientation preserving diffeomorphisms of M. Let $k \in \mathbb{N}$ and $J_k^+(M)$ be the positive higher frame bundle over M (cf. [H], I.8); an element of $J_k^+(M)$ is the k-jet at $0 \in \mathbb{R}^n$ of a germ of orientation preserving local diffeomorphism of a neighborhood of 0 in $\mathbb{R}^n$ with an open subset of M. As above, the manifold $J_k^+(M)$ is a principal G_k bundle over M, where G_k is the Lie group of k-jets of orientation preserving local diffeomorphisms of $\mathbb{R}^n$ fixing 0. Moreover $\Gamma = \mathrm{Diff}^+(M)$ acts naturally on $J_k^+(M)$ and its action commutes with the action of G_k. The group $\mathrm{SO}(n)$ sits in G_k as a maximal compact subgroup, so that we can consider the quotient $M_k = J_k^+(M)/\mathrm{SO}(n)$ as a Γ-manifold. Now let θ_k be a natural map of the complex $\mathrm{WO}(n)$ (cf. [H] loc. cit.) to the complex of Γ-invariant differential forms on $M_\infty = \varprojlim M_k$. For any $\omega \in H^q(\mathrm{WO}(n))$ let k be large enough so that $\theta_k(\omega)$ is well defined on M_k, then one obtains a group cocycle on Γ, $\rho(\omega) \in H^{q-n}(\Gamma, \mathbb{R})$ as follows.

The Γ-manifold M_k is the total space of a Γ-equivariant bundle over M whose fibers F_k are isomorphic to the quotient $G_k/\mathrm{SO}(n)$ of G_k by its maximal compact subgroup. Let $N = \dim F_k$, and fix an orientation of F_k. The corresponding orientation of M_k is then Γ-invariant. One can then consider the following fibration with fiber F_k:

$$M_k \times_\Gamma E\Gamma \xrightarrow{p} M \times_\Gamma E\Gamma \ .$$

The Thom class u of this bundle with contractible fibers can be viewed as an element of $H^N_{\mathrm{proper}}(M_{k,\Gamma})$, with $M_{k,\Gamma} = M_k \times_\Gamma E\Gamma$ and "proper" meaning that p restricted to supports is a proper map. Thus it makes sense to use the homology class $\theta_k(\omega) \cdot [M_k]$ of dimension $n + N - q$ to integrate u over the fibers of the fibration $M_{k,\Gamma} \to B\Gamma$ and obtain a group cocycle $\rho(\omega) \in H^*(B\Gamma)$ of dimension $N - (n + N - q) = q - n$.

This is exactly what we did in example 3 above. In general we shall prove:

3.A THEOREM. *Any $c \in \rho H^*(\mathrm{WO}(n)) \subset H^*(\Gamma, \mathbb{R})$ is Lipschitz.*

With the above notation, we take as a Γ-manifold the space $P = M_k$. The problem is to realize the above Thom class u (which was on P_Γ) from the suitable section α of a Γ-Euclidean vector bundle T on P in such a way that α ***satisfies the displacement bound*** for $\Gamma = \mathrm{Diff}^+(M)$ acting on M_k. This would be easy if one could endow the homogeneous space $F = G_k/\mathrm{SO}(n)$ of the Lie group G_k, with a left invariant Riemannian metric of non-positive sectional curvature. Indeed, one would then take T as the tangent bundle along the fibers of $P \xrightarrow{p} M$ and $\alpha(j) = \exp_j^{-1}(sp(j))$ where s is a fixed smooth section of this bundle with contractible fibers. A left invariant Riemannian metric of non-positive sectional curvature on $F = G_k/\mathrm{SO}(n)$ exists for $k = 1$ or for $k = 2$, $n = 1$, which was the situation of example 3 above. However, it does not exist in general. We shall overcome this difficulty by a technique of inductive construction of proper maps satisfying the displacement bound which will apply to many other situations.

Since we want to deal with groups Γ which are not necessarily finitely generated we shall reformulate the displacement bound as follows:

3.B DEFINITION. *Let Γ be a discrete group, P a Γ-space, T a Γ-equivariant Euclidean vector bundle over P and $\alpha : P \to T$ a section of T. Then α satisfies the displacement bound if*

$$\forall g \in \Gamma\,, \qquad \operatorname*{Sup}_{p \in P} \big\|\alpha(gp) - g\alpha(p)\big\| < \infty\,.$$

When Γ is finitely generated it follows immediately that for λ small enough $\lambda\alpha$ satisfies the previous displacement bound.

Our key technical tool is the following lemma.

3.C LEMMA. *Let Γ be a discrete group, P a Γ-space, T_1 and T_2 two Euclidean Γ-equivariant vector bundles on P, α_1 a continuous section of T_1 satisfying the displacement bound and α_2 a continuous section of T_2 such that:*

1) $\forall g \in G\,,\ \exists C_g < \infty$ *with* $\big\|\alpha_2(gp) - g\alpha_2(p)\big\| \le C_g e^{\|\alpha_1(p)\|} \qquad \forall p \in P\,.$

Then α_2 is homotopic among sections satisfying 2) to a continuous section α_2' of T_2 which satisfies the displacement bound and:

2) $$\big\|\alpha_1(p)\big\| + \big\|\alpha_2'(p)\big\| \ge \operatorname{Log}\big\|\alpha_2(p)\big\| \qquad \forall p \in P\,.$$

Proof: We look for a continuous function $f(x,y) > 0$ of two real variables $x \geq 1$, $y \geq 1$, such that:

$$\begin{array}{llll} \alpha) & xf(x,y) \leq 1 \quad \forall x,y; & \beta) & yf(x,y) \geq \operatorname{Log} y - \operatorname{Log} x \quad \forall x,y; \\ \gamma) & \left|xy\,\partial_x f(x,y)\right| \leq 1 \quad \forall x,y; & \delta) & \left|xy\,\partial_y f(x,y)\right| \leq 1 \quad \forall x,y. \end{array}$$

Let us check that the following function f fulfills these conditions:

$$f(x,y) = \frac{1 + \operatorname{Log} y - \operatorname{Log} x}{y} \quad \text{for} \quad y \geq x$$
$$f(x,y) = \frac{1}{x} \quad \text{for} \quad y \leq x \ .$$

It is clearly continuous since $x \geq 1$, $y \geq 1$ and the two definitions agree on the diagonal.

α) It is clear from $y \leq x$, otherwise $x\,f(x,y) = \frac{1+\operatorname{Log} t}{t}$ where $t = y/x \geq 1$, which is all right.

β) This is clear: for $y < x$, $\operatorname{Log} y - \operatorname{Log} x$ is negative.

γ) One has, for $y \geq x$, $\partial_x f(x,y) = -\frac{1}{xy}$, while for $y \leq x$, $\partial_x f(x,y) = -\frac{1}{x^2}$. This shows that $\partial_x f(x,y)$ is a continuous function and also that $\left|xy\,d_y f(x,y)\right| \leq 1$.

δ) One has, for $y \geq x$, $\partial_y f(x,y) = -\frac{1}{y^2} \operatorname{Log}(y/x)$ and for $y \leq x$, $\partial_y f(x,y) = 0$. This shows that $\partial_y f$ is continuous and $xy\,\partial_y f(x,y) = -\frac{x}{y} \operatorname{Log}(y/x)$ for $y \geq x$ is of the form $-\frac{1}{t} \operatorname{Log} t$, $t \geq 1$, which is bounded by 1 in norm.

We shall now prove that conditions α), γ), δ) are sufficient to get the displacement bound for $f\left(e^{\|\alpha_1\|}, \|\alpha_2\|\right)$, $\alpha_2 = \alpha_2'$. It is clear that condition β) insures that α_2' is homotopic to α_2 among sections of T_2 satisfying 2).

Let $g \in \Gamma$, there are constants $C_1, C_2 < \infty$ such that

$$\left\|\alpha_1(gp) - g\alpha_1(p)\right\| \leq C_1 \ , \qquad \forall p \in P$$
$$\left\|\alpha_2(gp) - g\alpha_2(p)\right\| \leq C_2 \exp\left(\left\|\alpha_1(p)\right\|\right) \ , \qquad \forall p \in P \ .$$

Of course both C_1 and C_2 depend on $g \in \Gamma$. But we just need to show that $\operatorname*{Sup}_{p \in P} \left\|\alpha_2'(gp) - g\alpha_2'(p)\right\| < \infty$.

We shall prove the following inequality:

$$\left\|\alpha_2'(gp) - g\alpha_2'(p)\right\| \leq (2C_2 + 1)e^{C_1} \ , \qquad \forall p \in P \ .$$

To do this, we let $x = \exp\left\|\alpha_1(p)\right\|$, $y = \left\|\alpha_2(p)\right\|$, $x' = \exp\left\|\alpha_1(gp)\right\|$, $y' = \left\|\alpha_2(gp)\right\|$.

First, the diplacement bound for α_1 shows that $\|\alpha_2(gp) - g\alpha(p)\| \leq C_1$ and hence that:

a) $|\operatorname{Log} x - \operatorname{Log} x'| \leq C_1$.

Next, the inequality 2) of the lemma shows that with $\xi = \alpha_2(p)$, $\xi' = g^{-1}\alpha_2(gp)$ one has:

$$\|\xi - \xi'\| \leq C_2 x$$

which using a) we can replace by the symmetric condition:

b) $\|\xi - \xi'\| \leq \inf(x, x')C_2 e^{C_1}$.

It follows then that $y = \|\xi\|$ and $y' = \|\xi'\|$ satisfy:

c) $|y - y'| \leq \inf(x, x')C_2 e^{C_1}$.

We have to estimate $\|f(x, y)\xi - f(x', y')\xi'\|$. Since conditions a),b),c) are symmetric under the exchange of (x, y, ξ) with (x', y', ξ'), we may assume that $y \leq y'$. One then has

$$f(x', y')\xi' - f(x, y)\xi =$$
$$= f(x', y')(\xi' - \xi) + \big(f(x', y') - f(x', y)\big)\xi + \big(f(x', y) - f(x, y)\big)\xi \ .$$

The first term $f(x', y')(\xi' - \xi)$ is bounded in norm by

$$(x')^{-1}\|\xi' - \xi\| \ ,$$

using condition α) for f, and hence by $C_2 e^{C_1}$, using the inequality b).

The second term $\big(f(x', y') - f(x', y)\big)\xi$ is bounded in norm by

$$\big|f(x', y') - f(x', y)\big| y \leq y \operatorname*{Sup}_{[y,y']} \big|\partial_t f(x', t)\big| |y' - y| \ .$$

But the inequality δ) on $\partial_y f$ gives the bound $(x')^{-1}$ for $y \operatorname{Sup}_{[y,y']} \big|\partial_t f(x', t)\big|$, and hence the bound $C_2 e^{C_1}$ for the second term, using inequality c).

The third term $\big(f(x', y) - f(x, y)\big)\xi$ is bounded in norm by

$$\big|f(x', y) - f(x, y)\big| y \leq y \operatorname*{Sup}_{[x,x']} \big|\partial_u f(u, y)\big| |x' - x| \ .$$

Again, the inequality γ) on $\partial_x f$ gives the bound

$$\big(\operatorname{Inf}(x, x')\big)^{-1} \quad \text{for} \quad y \operatorname*{Sup}_{[x,x']} \big|\partial_u f(u, y)\big|$$

and hence the bound e^{C_1} for the third term using inequality a).

3.D COROLLARY. *Let $\Gamma, P, T_1, T_2, \alpha_1$ and α_2 be as in Lemma 3.C and assume that $\|\alpha_1(p)\| + \|\alpha_2(p)\|$ is a proper function on P. Then the section $\alpha(p) = \big(\alpha_1(p), \alpha_2'(p)\big)$ for $T = T_1 \oplus T_2$ satisfies the displacement bound and $p \to \|\alpha(p)\|$ is a proper function on P.*

Proof: One has $\|\alpha(p)\| = \|\alpha_1(p)\| + \|\alpha_2'(p)\|| \geq \mathrm{Sup}\,(\|\alpha_1(p)\|, \mathrm{Log}\,\|\alpha_2(p)\|)$ by condition 2) of the lemma.

This corollary gives us a tool to construct inductively proper maps satisfying the displacement bound. The precise estimate in $\mathrm{Log}\,\|\alpha_2\|$ of Lemma 3.C is important. Simple examples such as the following show that this Log bound cannot be improved:

3.E EXAMPLE: Let P be the connected solvable 2-dimensional Lie group of transformations $t \to at+n$, $a > 0$ of $\mathbb{R}$. Thus $(a,n)(a',n') = (aa', an'+n)$. Let $\Gamma = P$ act on P by left multiplication and T_1, T_2 be the bundles on P with constant fiber $\mathbb{R}$. Let $\alpha_1(a,n) = \mathrm{Log}\, a, \alpha_2(a,n) = a^{-1}n$. Then clearly α_1 being a group homomorphism satisfies the displacement bound while, with $\gamma = (x,y) \in \Gamma$,

$$\alpha_2(\gamma p) - \alpha_2(p) = (xa)^{-1}(xn+y) - a^{-1}n = a^{-1}x^{-1}y$$

so that $\|\alpha(\gamma p) - \alpha_2(p)\| \leq C_\gamma \exp \|\alpha_1(p)\|$. Thus the conditions of Lemma 3.C are fulfilled, but it is impossible to improve the growth of α_2' since for a left invariant metric on $G = P$, its restriction to unipotent subgroups $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $n \in \mathbb{R}$, is typically the metric $d(n,n') = \mathrm{Log}\,(1 + |n - n'|)$.

3.F COROLLARY. *Let G be a linear algebraic group (over $\mathbb{R}$) and K a maximal compact subgroup. Then there exists a smooth section α of the tangent bundle T of G/K, such that (for a fixed left G-invariant Euclidean metric on T):*

1) *$\|\alpha\|$ is a proper function on G/K;*
2) *$\mathrm{Sup}_{G/K} \|\alpha(gp) - g\alpha(p)\| < \infty$ for any $g \in G$;*
3) *the pull-back by α of the Thom class of T is the fundamental class in $H^*_{\mathrm{comp}}(G/K)$.*

Proof: We shall construct α by induction using Lemma 3.C. Recall that (cf. [M1]) $G = N \rtimes H$ (semidirect product) with N nilpotent simply connected and H reductive. When $N = \{0\}$, the homogeneous space G/K admits an invariant metric of non-positive curvature so that the answer follows from section I.9. Moreover the resulting section α of $T(G/K)$ satisfies the following further condition:

$$\|\pi(g)\| \leq \exp\,(\lambda_\pi \|\alpha(gK)\|)\,, \qquad \forall g \in G\,,$$

where the left hand side is the norm of the *matrix* $\pi(g)$ in an arbitrary finite dimensional linear representation of G whose restriction on K is orthogonal.

In general, we shall assume that we have proved the lemma for any N of dimension $< d$ and with the following further condition fulfilled by the constructed section α:

4) For any representation π of H in a finite dimensional space with $\pi|K$ orthogonal, and for any polynomial P on the Lie algebra of N there exists $\lambda = \lambda(\pi, P) < \infty$ such that

$$\|\pi(h)\|\,|P(n)| \leq \exp\lambda\|\alpha(g)\| , \qquad \forall\, g = (n,h) \in G .$$

We have already seen how to fulfill this condition for $N = \{0\}$. Let us now construct α for $G = N \rtimes H$, N of dimension d. Let Z be the center of N, it is by construction a normal subgroup of $N \rtimes H = G$ and we let G_1 be the quotient group G/Z, ρ the canonical homomorphism form G to G_1. One has $G_1 = (N/Z) \rtimes H$ so that G_1 has the same form as G with $\dim N_1 < d$. The restriction of ρ to $H \subset G$ is injective and we let $K_1 = \rho(K)$ be the corresponding maximal compact subgroup of G_1.

The induction hypothesis thus provides us with a smooth section α_1 of $T(G_1/K_1)$ which satisfies 1),2),3),4).

Let us consider the G-equivariant fibration

$$G/K \xrightarrow{p} G_1/K_1$$

with $p(gK) = \rho(g)K_1$ for any $g \in G$.

Let us first understand the fibers $p^{-1}\{x\}$, $x \in G_1/K_1$. The center Z of N is a vector space and the fibers of

$$\rho : G \to G_1$$

are, in a left G-invariant manner, affine spaces over Z (which acts on the right). When one divides by $K \simeq K_1$, this affine structure is lost but not the Euclidean metric coming from any K invariant fixed Euclidean metric on the vector space Z. With this choice we get a G-invariant metric on the vertical bundle $T_2 = \operatorname{Ker} p_* \subset T(G/K)$. We shall construct a section α_2 of T_2 using a section $s : G_1/K_1 \to G/K$ of the map p by the formula

$$\alpha_2(x) = \overrightarrow{xs\big(p(x)\big)} \in T_{2,x}$$

(or, in a more fancy notation, the inverse of the exponential map $\exp_x$ along the fibers, applied to the point $s(p(x))$). The lack of G-invariance of α_2, i.e. $\|\alpha_2(gx) - g\alpha_2(x)\|$ is governed by the lack of G-invariance of s, i.e. by $\|(s(gx))^{-1}gs(x)\|$, where $s(gx)^{-1}gs(x) \in Z$.

To construct s, we use the linear section $\mathrm{Lie}\, N_1 \xrightarrow{s_0} \mathrm{Lie}\, N$ and the respective group exponential maps. We then extend it to $N_1 \rtimes H \to N \rtimes H$ by $s(n_1, h) = (s_1(n_1), h)$ $\forall n_1 \in N_1$, $h \in H$.

Let $g = (n, h) \in G$, $x = (n_1, h_1 K_1) \in G_1/K_1$ and let us compute $(s(gx))^{-1}gs(x) \in Z$ in terms of $s_1 : N_1 \to N$. One gets:

$$h_1^{-1}h^{-1}\left(\left(s_1(nh(n_1))\right)^{-1}nh(s_1(n_1))\right) \in Z \ .$$

Here $g = (n, h)$ is fixed while $x = (n_1, h_1 K_1)$ is the variable and we have to estimate the size of the above expression in the vector space Z. But h_1 only appears once and through its action by the representation π of H in Z, we can thus bound it by $\|\pi(h_1^{-1})\|$. Let us show that the remaining term, i.e. $s_1(nh(n_1))^{-1}nh(s_1(n_1))$ is bounded by a polynomial in $\exp^{-1}(n_1) \in$ Lie algebra of N_1. This is clear since for fixed h and n the above expression applied to $n_1 = \exp a_1$, $a_1 \in \mathrm{Lie}\, N_1$ is a polynomial map of $\mathrm{Lie}\, N_1$ to Z. This shows that we can bound the norm of the above element of Z by $\|\pi(h_1^{-1})\|\ \|P(\exp^{-1}(n_1))\|$.

Moreover, the dependence of P on $g = (n, h)$ can be absorbed in an overall multiplicative constant C_g. Thus, the section α_2 of T_2 satisfies the bound

$$\|\alpha_2(gy) - g\alpha_2(y)\| \leq C_g \exp \lambda \|\alpha_1(p(y))\| \qquad \forall y \in G/K \ ,$$

for a suitable λ provided by the induction hypothesis.

Let us then fix a G-invariant Euclidean metric on the tangent bundle $T = T(G/K)$, whose restriction to T_2 is the previously chosen metric. We have a natural G-equivariant isomorphism of $T_1 \subset T$, the orthogonal of $T_2 \subset T$, with the pull-back $p^*(T(G_1/K_1))$ of the tangent bundle of G_1/K_1. Let $\widetilde{\alpha}_1$ be $\lambda p^* \alpha_1$, the pull-back of the section α_1 of $T(G_1/K_1)$, then the section α_2 satisfies the bound

$$\|\alpha_2(gy) - g\alpha_2(y)\| \leq C_g \exp \|\widetilde{\alpha}_1(y)\| \qquad \forall y \in G/K$$

and we can thus apply Lemma 3.C to $P = G/K$ with the pair $(\widetilde{\alpha}_1, \alpha_2)$. Let then α_2' be as in Lemma 3.C; the section $\alpha(y) = (\widetilde{\alpha}_1(y), \alpha_2'(y))$ of $T(P)$ satisfies the displacement bound and $\|\alpha(y)\|$ is a proper function on $P = G/K$.

In fact it clearly satisfies the stronger condition 4) since any polynomial is bounded by a function $\exp(\lambda \operatorname{Log} t)$ for large t. Finally, α verifies 3) by construction. We have thus shown how to proceed by induction, which proves Corollary 3.F.

Proof of Theorem III.3.A: As above, we let $J_k^+(M)$ be the positive k-frame bundle over M, and $\Gamma = \mathrm{Diff}^+(M)$. Consider the Γ-equivariant fibration $Y = J_k^+/\mathrm{SO}(n) \xrightarrow{\pi} M$. It is induced from the G_k-principal bundle $J_k^+ \to M$ and the left action of G_k on $P = G_k/\mathrm{SO}(n)$, i.e. one has $Y = J_k^+ \times_{G_k} P$.

Let V be the tangent bundle along the fibers of π; it is a Γ-equivariant bundle on Y and is induced from the action of G_k on the tangent bundle of P. Thus we can endow it with the Γ-invariant Euclidean metric coming from a G_k The local triviality of the G_k-principal bundle J_k^+ and the compactness of M give us a finite covering $\{U_i\}$ of M by open sets, and isomorphisms $\pi^{-1}(U_i) \overset{\phi_i}{\simeq} U_i \times P$ such that on $U_i \cap U_j$, $\phi_i\phi_j^{-1}(x,p) = \big(x, g_{ij}(x)p\big)$, where (g_{ij}) is a 1-cocycle with values in G_k. Now since G_k satisfies the hypothesis of Corollary 3.F we let α be a smooth section of $T(P)$ fulfilling the conditions of the corollary, and using a partition of unity we get a corresponding section $\tilde{\alpha} = \sum(\chi_i \circ p)(\alpha \circ \phi_i)$ of V. Since Γ acts by gauge transformations on J_k^+, the compactness of M and the displacement bound for α show that $\tilde{\alpha}$ satisfies the displacement bound relative to the action of Γ.

Finally, condition 3) of Corollary 3.F shows that $\tilde{\alpha}$ has the right proper homotopy class.

Remark: Using Lemma 4.B below, one can actually extend Corollary 3.F to almost connected Lie groups.

3.G THEOREM. *Let G be an almost connected locally compact group, Γ a finitely generated discrete group and $\iota : \Gamma \to G$ a homomorphism. Then any cohomology class in $\iota^*\big(H^*_{\mathrm{cont}}(G, \mathbb{R})\big)$ is Lipschitz.*

Proof: By a well-known theorem, G has a normal compact subgroup such that the quotient G_{Lie} is a Lie group. By another well-known result (see e.g. [M2]),

$$H^*_{\mathrm{cont}}(G, \mathbb{R}) = H^*_{\mathrm{cont}}(G_{\mathrm{Lie}}, \mathbb{R}) \ .$$

Thus, the statement follows from the above remark.

4. Lipschitz cocycles and Chern classes of Γ-equivariant bundles.

Let Γ be a discrete group, P a Γ-manifold and α a proper section, satisfying the displacement bound, of a Γ-equivariant Euclidean vector bundle

T over P. In constructing the map $\alpha_\cap$ from $H_*(P:\Gamma)$ to group cohomology we defined the homology $H_*(P:\Gamma)$ from invariant closed currents of order 0 on P. Our aim in this section is to show that the class of Lipschitz cocycles is stable under multplication by arbitrary polynomials in the Chern classes of Γ-equivariant complex vector bundles E over P, without any hypothesis of existence of a Γ-invariant metric on E.

4.A THEOREM. *Let (P,T,α,C) be a geometric group cocycle, E a Γ-equivariant complex vector bundle over P and E_Γ the associated complex vector bundle over $P_\Gamma = P \times_\Gamma E\Gamma$. Let $Q = Q(c_1,\ldots,c_n)$ be an arbitrary polynomial in the Chern classes of E_Γ. Then the cohomology class $\rho = \langle C \times_\Gamma Q\alpha^*(u_T)\rangle \in H^*(\Gamma)$ is Lipschitz.*

The first difficulty we meet in proving this theorem is the absence of a Γ-invariant Hermitian metric on the vector bundle E. To deal with it, we shall use the following two lemmas.

4.B LEMMA. *Let P be a locally compact Γ-space, f a positive continuous function on P such that $f(p)\to\infty$ when $p\to\infty$. There exists a continuous positive function $h(p)\le f(p)$ such that $h(p)\to\infty$ when $p\to\infty$ and satisfying the displacement bound:*

$$\forall\,\gamma\in\Gamma\,, \qquad \operatorname*{Sup}_p \big|h(\gamma p)-h(p)\big| < \infty\,.$$

Proof: Let $\beta>0$ be such that $\sum\limits_{g\in\Gamma} e^{-\beta\ell(g)} = C_\beta < \infty$ where $\ell(g)$ is the word length relative to a finite set of generators $Z\subset\Gamma$. Let then

$$\theta(p) = \sum_{g\in\Gamma} e^{-\beta\ell(g)} e^{-f(gp)}\,.$$

One has by construction $0<\theta(p)\le C_\beta<\infty$. Moreover the inequality $\ell(g_1^{-1}g)\ge\ell(g)-\ell(g_1)$, $\forall\,g,g_1\in\Gamma$ shows that $\theta(g_1,p)\le e^{\beta\ell(g_1)}\theta(p)$, $\forall\,g_1\in\Gamma$, $p\in P$.

It follows that the function $\operatorname{Log}\theta(p)$ satisfies the displacement bound, as well as $-\operatorname{Log}\theta(p)$. Now the inequality $\theta(p)\ge e^{-f(p)}$ shows that $-\operatorname{Log}\big(\theta(p)\big)\le f(p)$. Since $f(gp)\to\infty$ when $p\to\infty$, one has by the Lebesgue dominated convergence theorem that $\theta(p)\to 0$ when $p\to\infty$, so that $-\operatorname{Log}\theta(p)\to\infty$ when $p\to\infty$. Thus we can take for h the positive part $h(p) = \big(-\operatorname{Log}\theta(p)\big)^+ = -\operatorname{Log}^-\theta(p)$.

4.C LEMMA. *Let (P, T, α, C) be a geometric cocycle and E a Γ-equivariant complex vector bundle on P. There exists a Γ-equivariant fibration $P_1 \xrightarrow{p} P$ and an equivalent cocycle $(P_1, T_1, \alpha_1, C_1)$ such that the bundle $p^*(E) = E_1$ admits a Γ-invariant Hermition metric.*

Proof: The frame bundle of E is a Γ-equivariant G-principal bundle over P, where $G = \mathrm{GL}(n, \mathbb{C})$, $n = \dim_{\mathbb{C}} E$.

Let $P_1 \xrightarrow{p} P$ be the bundle associated with the action of $\mathrm{GL}(n, \mathbb{C})$ on the space $H = \mathrm{GL}(n, \mathbb{C})/U(n)$ of positive matrices. We endow H with its canonical Riemannian metric which is G-invariant and of non-positive sectional curvature. This yields a Γ-invariant Euclidean metric on the vertical bundle V of the fibration $P_1 \xrightarrow{p} P$.

By construction, the fiber $p^{-1}(x)$ over a point $x \in P$ is the space of all Hermitian metrics on the fiber E_x of E, thus $p^*(E)$ has a canonical "tautological" metric which is Γ-invariant.

We let $T_1 = p^*T \oplus V$; it is a Γ-equivariant Euclidean oriented bundle. We already have the section $p^*\alpha$ of the bundle p^*T, we need a section α_2 of V which satisfies the displacement bound, is *fiberwise* proper, and pulls back the Thom class of V to the Thom class of $P_1 \xrightarrow{p} P$, an oriented bundle with contractible fibers.

We shall get α_2 as follows. First we take an arbitrary Hermitian metric on E, i.e. a smooth section s of the bundle $P_1 \xrightarrow{p} P$. To this section s we associate the following *bounded* section β_1 of the vertical bundle V on P_1

$$\beta_1(y) = \left(1 + \left\|\beta(y)\right\|\right)^{-1} \beta(y)$$

where $\beta(y) = \overrightarrow{y, sp(y)} \in V_y$.

The non-positive curvature of the fibers shows that

$$\left\|\beta^g(y) - \beta(y)\right\| \le d\big(s^g(x), s(x)\big) \ . \qquad \forall\, y \in p^{-1}(x),$$

where one uses the action of $g \in \Gamma$ on both β and s, and the Riemannian distance in the fibers of P_1.

It follows that for any compact $K \subset P$ one has

$$\left\|\beta_1^g(y) - \beta_1(y)\right\| \to 0 \quad \text{when} \quad y \to \infty \quad \text{in} \quad p^{-1}(K) \ .$$

Let then $Z \subset \Gamma$ be a finite set of generators and

$$f(y) = \Big(\sum_{g \in Z} \left\|\beta_1^g(y) - \beta_1(y)\right\|\Big)^{-1} \ .$$

Let $h(y)$ be given by the formula of Lemma 4.B, it satifies the displacement bound and $h(y) \leq f(y)$, also

$$h(y) \to \infty \text{ in } p^{-1}(K) \text{ for any } K \text{ compact of } P \ .$$

Then $\alpha_2(y) = h(y)\beta_1(y)$ is fiberwise homotopic among proper sections of V to the original section β. The norm of α_2 is fiberwise proper. Finally, let us check that α_2 satisfies the displacement bound. One has, for $g \in Z$,

$$\left\|\alpha_2^g(y) - \alpha_2(y)\right\| \leq \left|h(gy) - h(y)\right| \left\|\beta_1(gy)\right\| + h(y)\left\|\beta_1^g(y) - \beta(y)\right\| \ ,$$

which is bounded since $h(y)\left\|\beta_1^g(y) - \beta_1(y)\right\| \leq 1$.

We now let $\alpha_1 = p^*\alpha \oplus \alpha_2$; it is a proper section of T_1. We let $C_1 = p^*C$; since $\alpha_2^*(u_V)$ is the Thom class of the bundle $P_1 \xrightarrow{p} P$ associated to the section s, we get the required equivalence $(P_1, T_1, \alpha_1, C_1) \sim (P, T, \alpha, C)$.

4.D LEMMA. *Let (P, T, α, C) be a geometric Γ-cocycle.*

1) *Let E be an oriented Γ-equivariant Euclidean vector bundle over P. Then there exists a Γ-equivariant fibration $P_1 \xrightarrow{p} P$ and a geometric Γ-cocycle $(P_1, T_1, \alpha_1, C_1)$ with associated group cocycle $\langle C_{X_\Gamma}, e(E_\Gamma)\widetilde{\alpha}^*(u_T)\rangle$, where $e(E_\Gamma)$ is the Euler class of the bundle associated to E on P_Γ.*
2) *Let E be a Hermitian Γ-equivariant vector bundle over P. Then there exist a Γ equivariant fibration $P_1 \xrightarrow{p} P$ and an equivalent geometric Γ-cocycle $(P_1, T_1, \alpha_1, C_1)$ such that the pull-back $p_1^*(E)$ admits a Γ-equivariant subbundle of rank one.*

Proof: 1) We let P_1 be the total space of E, we take $T_1 = p^*(T \oplus E)$ and $\alpha_1 = p^*\alpha \oplus \alpha_2$, where $\alpha_2(y) = \overrightarrow{ys(p(y))}$, s being the Γ-invariant zero section of E. Now instead of taking $C_1 = p^*C$, which would yield an equivalent cocycle, we take $C_1 = s_*C$. It is a Γ-invariant current, and the dimension of the associated group cocycle is now increased by $\dim E$. The equality $e(E) = s^*(u_E)$, where u_E is the Thom class of E, yields the desired answer.

2) Let first $P_0 \xrightarrow{p_0} P$ be the Γ-equivaraint fibration with fiber over $x \in P$ the projective space $P(E_x)$ of the fiber E_x.

Let L be the canonical complex line bundle over P_0. The construction of 1) applies to the real oriented bundle $E_0 = L \oplus \ldots \oplus L$, direct sum of $n-1$ copies of L, $n = \dim_{\mathbb{C}} E$, and yields a triple (P_1, T_1, α_1), where P_1 is the total space of E_0, T_1 is $p_1^* \circ p_0^* T \oplus p_1^* E_0$ and α_1 is as above. We let $C_1 = s_* p_0^* C$. Since the pushforward p_{0*} of the Euler class of E_0 is a constant $\neq 0$ we get the conclusion.

Proof of Theorem III.4.A: Using Lemmas 4.C and 4.D we can assume that the bundle E is a direct sum of Γ-equivariant Hermitian line bundles E_i. We are thus dealing with a polynomial in the first Chern classes $c_1(E_i)$, i.e. in the Euler classes $e(E_i)$. The answer follows from the multiplicativity of the Euler class combined with 4.D.1).

Let us now state and prove two propositions which are corollaries of the above lemmas.

Let us recall from [C2] the following

4.E DEFINITION: An action of the discrete group Γ on a manifold P is *almost isometric* if there exists a Γ-equivariant reduction of the structure group of P to a group of block triangular matrices with orthogonal diagonal blocks.

If we apply the proof of Lemma 4.C in the *real* context, i.e. with $\mathrm{GL}(n,\mathbb{R})$ instead of $\mathrm{GL}(n,\mathbb{C})$, to the tangent bundle of P we get, using [C2]:

4.F PROPOSITION. *Let (P,T,α,C) be a geometric Γ-cocycle, then there exists a Γ-equivariant fibration $P_1 \xrightarrow{p} P$ and an equivalent geometric Γ-cocycle (P_1,T_1,α_1,C_1) such that the action of Γ on P_1 is almost isometric.*

This reduction to the almost isometric case is crucial for the use of cyclic cohomology and the proof of the Novikov conjecture for Lipschitz cocycles.

The next proposition relaxes a bit the axioms in the definition of a Lipschitz class.

4.G PROPOSITION. *Let P be a locally compact Γ-space with a Γ-equivariant Euclidean vector bundle T. Let α_1 be a continuous section of T such that*

a) $\|\alpha_1(y)\| \geq \varepsilon > 0$ *outside a compact subset of P,*
b) $\|\alpha_1(y)\| \leq 1, \forall\, y \in P$,
c) $\|\alpha_1^g(y) - \alpha_1(y)\| \to 0$ *when $y \to \infty$, for any $g \in \Gamma$.*

Then there exists a proper, displacement bounded section α of T on P of the form $\alpha(y) = h(y)\alpha_1(y)$, $h(y) \to \infty$, $y \to \infty$.

Proof: Let $f(y)$ be a positive proper continuous function on P such that $f(y) \leq \Big(\sum_{g\in Z} \|\alpha_1^g(y) - \alpha_1(y)\|\Big)^{-1}$ where $Z \subset \Gamma$ is a finite set of generators of Γ, and with $f(y) \to \infty$ when $y \to \infty$. (This is possible by c).) Let $h(y) \leq f(y)$ be given by Lemma 4.B. Then $\alpha(y) - h(y)\alpha_1(y)$ is the required answer.

5. Lipschitz cohomology and the Novikov Conjecture.

In this section we shall show how to remove the hypothesis that the action of Γ on P is proper in the proof of chapter I of the Novikov conjecture for Lipschitz classes.

It is cyclic cohomology which will play a crucial role in this proof. This is not surprising in view of Example 3 in III.2. Indeed, in that example, the Γ-invariant form ω of degree 3 on J_2^+ (the space of higher 2-frames on S^1) gives an *invariant measure* on the space J_2^+ and hence a *trace* ϕ on the crossed product C^* algebra $A = C_0(J_2^+)\rtimes\Gamma$. Such a trace ϕ gives a natural map from $K_0(A)$ to $\mathbb{C}$ and it is this map which replaces in this example the Chern character ch : $K_c^*(P/\Gamma) \to H^*(P:\Gamma)$ which we used in the proper case. Traces are 0-dimensional cyclic cocycles and in general a Γ-invariant differential form ω of degree q on the Γ-manifold P provides us with a cyclic cocycle ϕ of dimension $d = \dim P - q$ on a dense subalgebra $\mathfrak{A}$ of the C^*-algebra $A = C_0(P)\rtimes\Gamma$. The very delicate problem of extension of the corresponding map $K_d(\mathfrak{A}) \to \mathbb{C}$ to a map $K_d(A) \to \mathbb{C}$ has been solved in [C2] and we shall use this result to conclude.

5.A THEOREM. *Let Γ be a discrete group. Every Lipschitz cohomology class $c \in H^k(\Gamma, \mathbb{R})$ satisfies the Novikov conjecture.*

We shall use as a tool in the proof of this theorem the natural "assembly map" μ from the geometric K-group of the pair (P, Γ) where P is a Γ manifold, to the (analytic) K-theory of the C^*-algebra $A = C_0(P)\rtimes\Gamma$, crossed product of $C_0(P)$ by Γ. This map (see [BC]) extends to group actions the usual assembly map: $K_*(B\Gamma) \to K(C^*(\Gamma))$ (cf. [K]). The geometric group $K^*(P,\Gamma)$ of the Γ space P is by definition the K-homology of the pair $(B\tau, S\tau)$ of the unit ball, unit sphere bundle of the vector bundle τ on $P_\Gamma = P \times_\Gamma E\Gamma$ which is associated to the tangent bundle of P. (A more refined version is necessary to take care of the torsion of Γ, but we shall not need it here).

We shall now recall briefly the properties of the generalized assembly map $\mu : K^*(P,\Gamma) \to K_*(C_0(P)\rtimes\Gamma)$ which we shall need for the proof of the theorem.

Using the Baum-Douglas description of K homology ([BD]) (i.e. as a quotient of Spin_c cobordism) one gets that every element $x \in K^*(P,\Gamma)$ can be obtained from a quadruple (N, q, F, h) where N is a manifold, $q : \widetilde{N} \to N$ is a Γ-principal bundle over N, F is an element of K-theory *with compact support* $F \in K_c(N)$, and $h : \widetilde{N} \to P$ is a Γ equivariant K-*oriented* map. The

K-orientation of h is given by the choice of a Γ-invariant Spin_c-structure on the bundle $T\widetilde{N} \oplus h^*TP$ on $\widetilde{N}$. This notion is well defined because the action of Γ on $\widetilde{N}$ is proper.

Each such quadruple (N, q, F, h) defines an element of $K^*(P,\Gamma) = K_*(B\tau, S\tau)$ whose image by μ can be explicitly described (cf. [C2]). We shall just need

5.B LEMMA [C2]. *If $h : \widetilde{N} \to P$ is a submersion, one has $\mu(N,q,F,h) = F \oplus h! \in K(C_0(P) \rtimes \Gamma)$, where $h! \in KK_\Gamma(C_0(\widetilde{N}), C_0(P))$ is the Γ-equivariant family of Dirac operators along the fibers of h.*

We have used here the natural map of $KK_\Gamma(A,B)$ to $KK(A \rtimes \Gamma, B \rtimes \Gamma)$ and the Morita equivalence $C_0(\widetilde{N}) \rtimes \Gamma \simeq C_0(N)$. We refer to [C2] for more details.

Let us assume that P is Γ-invariantly oriented. Then the Chern character in K-homology:

$$\mathrm{ch}_* : K_*(B\tau, S\tau) \to H_*(B\tau, S\tau)$$

can be composed with the Thom isomorphism

$$\Phi : H_*(B\tau, S\tau) \simeq H_{*-n}(P_\Gamma) \ , \quad \text{with} \quad P_\Gamma = P \times_\Gamma E\Gamma$$

and an explicit computation of $\Phi\,\mathrm{ch}_*(x)$ for $x = (N, q, F, h)$ gives the following

5.C LEMMA [C2]. *Let $x = (N, q, F, h) \in K^*(P, \Gamma \simeq K_*(B\tau, S\tau)$. Then $\Phi \circ \mathrm{ch}_*(x) = \widetilde{h}_*(\mathrm{ch}F)Td(TN \oplus h^*\tau) \cap [N])$.*

Here N is oriented since h is K oriented and τ is oriented. Moreover, $\widetilde{h}$ is a map from M to P_Γ associated to $h_\Gamma : \widetilde{N}_\Gamma \to P_\Gamma$, since $\widetilde{N}_\Gamma = \widetilde{N} \times_\Gamma E\Gamma$ is naturally homotopic to N.

We can now state the main result of [C2] which will be the key fact used in the proof of Theorem 5.A.

5.D LEMMA. *Let Γ be a discrete group acting by orientation preserving diffeomorphisms of the (not necessarily compact) manifold P. Let $\mathcal{R}$ be the $\mathbb{C}$ subalgebra of $H^*(P_\Gamma, \mathbb{C})$ generated by the cohomology classes of Γ-invariant closed currents of order 0 on P and by the Chern classes of Γ-equivariant bundles on P. Then for any $y \in \mathcal{R}$ there exists an additive map ϕ of the K theory group $K(A)$ of the C^* algebra $A = C_0(P) \rtimes \Gamma$, to $\mathbb{C}$, such that*

$$\phi(\mu(x)) = \langle \Phi \circ \mathrm{ch}_*(x), y \rangle \ , \qquad \forall\, x \in K^*(V, \Gamma) \ .$$

Proof: Since the result is not stated like this in [C2; Thm. 6.8] (it does not involve Γ-invariant differential forms), we need to explain how to prove it using the technique of [C2]. One first uses Proposition 4.F to reduce to the *almost isometric* case, i.e. the case where the action of Γ on P preserves a G-structure on the manifold P, where G is a group of triangular block matrices with orthogonal diagonal blocks.

Next, given a Γ-invariant closed current of order 0, ω on P, one shows exactly as in Lemma 4.4 of [C2] that the following cyclic cocycle on the algebra $\mathfrak{A} = C_c^\infty(P \rtimes \Gamma)$ defines an m-trace of the Banach algebra completion B of $\mathfrak{A}$ given in [C2; Thm. 3.7]:

$$\tau(f^0, \dots, f^m) = \sum_{g^0 \dots g^m = 1} \int_P f^0_{g_0} g_0 d(f^1_{g_1}) \wedge g_0 g_1 d(f^2_{g_2}) \wedge \dots \wedge g_0 \dots g_{m-1} df^m_{g_m} \wedge \omega$$

where $m = \dim P - \deg \omega$, and $f^i \in C_c^\infty(P \rtimes \Gamma)$ is viewed as a family f_g, $g \in \Gamma$ of smooth functions on P.

The rest of the proof is then exactly the same as the proof of Thm. 6.8 of [C2].

Proof of Theorem 5.A: Let P be an oriented smooth Γ-manifold, ω a Γ invariant closed current of order 0 on P and α a section of a Γ-equivariant Euclidean oriented vector bundle T on P fulfilling conditions (I)* and (II)* of III.1.B. We want to prove that the corresponding group cocycle $c = \alpha_\cap(\omega \cdot [P])$ on Γ satisfies the Novikov conjecture.

As a first step, let us show that we can assume that the Γ-bundle T is endowed with a Γ-invariant structure of complex vector bundle. For this we use the suspension (I.7.C) with the bundle T. Thus the new space P' is the total space of T and the new bundle T' is the pull back to T of the complexification $T_{\mathbb{C}} = T \oplus iT$ of T. The new differential form ω' is the pull-back of ω to T which does not change its degree $k = \deg \omega$. In particular the difference $\dim T - (\dim P - \deg \omega)$ is equal to $\dim T' - (\dim P' - \deg \omega)$. Finally the section α' is $\alpha'(p, \xi) = \alpha'(p) + i\xi$ for $p \in P$, $\xi \in T_p$. As before (I.8) one has $\alpha'_\cap(\omega'[P']) = \alpha_\cap(\omega[P])$ and the new bundle T' is now complex.

As a second step we use, assuming that T is complex, the bundle $S = \wedge T$, exterior algebra of the complex space T, and the natural representation of the Clifford algebra $\mathrm{Cliff}(T_{\mathbb{R}})$ of the underlying real vector bundle, to construct as in I.10, an element $K(\alpha) \in KK_\Gamma(\mathbb{C}, C_0(P))$. This is done as follows. The C^* module $\mathcal{E}$ over $C_0(P)$ is the space of continuous sections vanishing at ∞ of S and the remaining formulas are identical with those of

I.10. Thus the endomorphism F of $\mathcal{E}$ is given by $(F\xi)_p = F_p\xi_p$, $\forall \xi \in \mathcal{E} = C_0(P,S)$ where $F_p = \gamma(\alpha_1(p))$ is the Clifford multiplication by

$$\alpha_1(p) = \left(1 + \|\alpha(p)\|\right)^{-1}\alpha(p) \ .$$

As before the displacement bound and the properness of $\|\alpha\|$ show that $(\mathcal{E},F)$ defines an element $K(\alpha)$ of $KK_\Gamma(\mathbb{C},C_0(P))$.

Using $K(\alpha) \in KK_\Gamma(\mathbb{C},C_0(P))$, or rather its image $K'(\alpha)$ in $KK(C^*(\Gamma),C_0(P)\rtimes\Gamma)$, we obtain a natural map $\psi_\alpha : K(C^*(\Gamma)) \to K(C_0(P)\rtimes\Gamma)$, $\psi_\alpha(y) = y \otimes K'(\alpha)$.

As a third step we shall compute the composition $\psi_\alpha \otimes \mu$ of ψ_α with the assembly map $\mu : K_*(B\Gamma) \to K(C^*(\Gamma))$. Let $x \in K_*(B\Gamma)$ be given by a geometric cycle, i.e. a compact Spin$_c$-manifold V, a Γ-principal bundle $\widetilde{V} \xrightarrow{\rho} V$ on V and a K theory class $E \in K^*(V)$. Then Lemma 5.B shows that $\mu(x) \oplus K'(\alpha) = \psi_\alpha\mu(x)$ is given by the following geometric cycle: $(N,Q,F,h) = z$, i.e. $\psi_\alpha\mu(x) = \mu(z)$. Here N is the manifold $\widetilde{V} \times_\Gamma P$, q is the Γ-principal bundle over N given by $\widetilde{N} = \widetilde{V} \times P \xrightarrow{q} \widetilde{V} \times_\Gamma P$, F is the class in $K_c^*(N)$, the K-theory with compact support of N given by the formula

$$F = \pi^*(E)\widetilde{\alpha}^*(t)$$

where π is the projection $N \to V$ associated to $\widetilde{N} \to \widetilde{V}$ the projection on the first factor, t is the Thom class in K-theory of the complex vector bundle $\widetilde{T}$ on N associated to the Γ-equivariant bundle T on P, and $\widetilde{\alpha}$ is a section of $\widetilde{T}$ on N associated, using the local triviality of the bundle $N \xrightarrow{\pi} V$ with fiber P, to the section α of T over P.

Finally the map $h : \widetilde{N} \to P$ is the second projection.

Let us compute the Chern character $\Phi \circ \mathrm{ch}_*(z) \in H_*(P_\Gamma)$, where $P_\Gamma = P \times_\Gamma E\Gamma$, of the element z.

The K-orientation of $h : \widetilde{N} \to P$ is given by the Spin$_c$-structure of the fiber $\widetilde{V}$. The Chern character $\mathrm{ch}(F)$ is equal, by the above formula, to $\pi^*(\mathrm{ch}(E))\widetilde{\alpha}^*(\mathrm{ch}t)$ and, as is well known in the original proof [AS] of the index theorem, $\mathrm{ch}\,t$ is the product of the Thom class u of $\widetilde{T}$ by the inverse of the Todd genus of this complex bundle, a characteristic class $\nu(\widetilde{T}) =$ 1+ higher degree. We thus have

$$\Phi \circ \mathrm{ch}_*(x) = \widetilde{h}_*\left(\nu(\widetilde{T})\widetilde{\alpha}^*(u)\pi^*\left(\hat{A}(V)\mathrm{ch}(E)\right)[N]\right)$$

where $\widetilde{h} : N \to P_\Gamma$ is obtained from $h_\Gamma : N_\Gamma \to P_\Gamma$ and the homotopy equivalence $N_\Gamma = \widetilde{N} \times_\Gamma E\Gamma \simeq N$.

If we express this in terms of $\mathrm{ch}_*(x) = \widetilde{\rho}_*\big(\widehat{A}(V)\mathrm{ch}E[V]\big)$ where $\widetilde{\rho} : V \to B\Gamma$ is the classifying map of the Γ principal bundle $\widetilde{V}$ over V, we get the following formula

$$\Phi \circ \mathrm{ch}_*(z) = \nu(\widetilde{\widetilde{T}})\widetilde{\widetilde{\alpha}}^*(u)\big(\mathrm{ch}_*(x) \times [P]\big)$$

where $\widetilde{\widetilde{T}}$ is the complex vector bundle on P_Γ associated to T on P, $\widetilde{\widetilde{\alpha}}$ is a section of $\widetilde{\widetilde{T}}$ associated to the section α, and where the orientation of the fibers P of the fibration $P_\Gamma \to B\Gamma$, is used to define the product homology class $\mathrm{ch}_*(x) \times [P]$. It follows from this formula together with Lemma 5.D that for any Γ-invariant closed current of order 0, $\omega \in H^j(P : \Gamma)$, there exists a linear map $L : K\big(C^*(\Gamma)\big) \to \mathbb{C}$ such that

$$L\big(\mu(x)\big) = \big\langle \mathrm{ch}_*(x), \alpha_\cap\big(\omega \cdot [P]\big)\big\rangle \ , \quad \forall\, x \in K_*(B\Gamma) \ .$$

It follows that the Lipschitz class $\alpha_\cap\big(\omega\cdot[P]\big)$ satisfies the Novikov conjecture.

Epilogue. Lipschitz-Poincaré Dual of a Group

The construction of Lipschitz cohomology classes for a finitely generated discrete group Γ can be systematized by means of a "dual object" to $B\Gamma$, consisting of a Γ-space equipped with a "Poincaré duality" map from its Γ-invariant homology (with infinite chains) to $H_L^*(\Gamma)$. We devote the concluding section to this notion of "Lipschitz-Poincaré dual" of $B\Gamma$, which we believe to be of independent interest.

Fix a word-length function $|\gamma|$, $\gamma \in \Gamma$ and let $d(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|$, $\gamma_1, \gamma_2 \in \Gamma$ be the associated left-invariant distance on Γ. For $N \in \mathbb{N}$, let

$$L_N\Gamma = \big\{\phi : \Gamma \to \mathbb{R}^N \ : \ \big\|\phi(\gamma_1) - \phi(\gamma_2)\big\| \le d(\gamma_1, \gamma_2) \ , \ \forall\, \gamma_1, \gamma_2 \in \Gamma\big\} \ .$$

Endowed with the product topology, $L_N\Gamma$ is a *locally compact space*. Indeed, $L_N\Gamma$ is a closed subspace of $\mathbb{R}^N \times \prod_{\gamma \neq e} B_\gamma$, where $B_\gamma \subset \mathbb{R}^N$ is a ball centered at the origin of radius $|\gamma|$. Γ acts on $L_N\Gamma$ by left translations. On the other hand, $\mathrm{SO}(N)$ acts on $L_N\Gamma$ in the obvious way: $u\phi = u \circ \phi$, $\forall\, u \in \mathrm{SO}(N)$ and $\phi \in L_N\Gamma$. The two actions commute.

Consider now the subspace $F_N\Gamma$ of $L_N\Gamma$ consisting of all Lipschitz contractions $\phi : \Gamma \to \mathbb{R}^N$ such that $\mathbb{R}^N$ = linear span of $\phi(\Gamma)$. It is easy to see that $F_N\Gamma$ is precisely the union of all $\mathrm{SO}(N)$-orbits with trivial isotropy. As such, it is an open subset of $L_N\Gamma$ and therefore a *locally compact space* itself. Moreover, the action of $\mathrm{SO}(N)$ on $F_N\Gamma$ is *free*.

Let $P_N\Gamma$ denote the quotient space of $F_N\Gamma$ under the action of $\mathrm{SO}(N)$ and form the vector bundle

$$T_N\Gamma = F_N\Gamma \times_{\mathrm{SO}(N)} \mathbb{R}^N \to P_N\Gamma \ ,$$

associated to the principal $\mathrm{SO}(N)$-bundle $\pi : F_N\Gamma \to P_N\Gamma$ and to the standard representation of $\mathrm{SO}(N)$. Evaluation at identity gives rise to a canonical cross-section $\alpha_N : P_N\Gamma \to T_N\Gamma$ as follows:

$$\alpha_N\big(\pi(\phi)\big) = \phi \times_{\mathrm{SO}(N)} \phi(e) \ , \qquad \forall \phi \in F_N\Gamma \ .$$

E.1 Lemma. *$T_N\Gamma \to P_N\Gamma$ is an Euclidean, oriented Γ-bundle and the continuous section $\alpha_N : P_N\Gamma \to T_N\Gamma$ satisfies the following conditions:*

(I*) $\big\|\gamma\alpha_N(\gamma^{-1}p) - \alpha_N(p)\big\| \le |\gamma| \ , \qquad \forall \gamma \in \Gamma \ , \ p \in P_N\Gamma;$

(II*) *the function* $\|\alpha_N\|(p) = \big\|\alpha_N(p)\big\|$, $p \in P_N\Gamma$, *is proper;*

(III*) $(T_N\Gamma)_p =$ *linear span of* $\big\{\gamma\alpha_N(\gamma^{-1}p) \ ; \ \gamma \in \Gamma\big\}$, $\forall p \in P_N\Gamma$.

Proof: If $p = \pi(\phi)$, then $\gamma\alpha_N(\gamma^{-1}p) = \phi \times_{\mathrm{SO}(N)} \phi(\gamma)$. Thus, (I*) and (III*) follow just from the fact that $\phi \in F_N\Gamma$. Likewise, (II*) is a consequence of the definition of $F_N\Gamma$ as a topological space. □

Note that (I*) and (II*) are the displacement bound and, respectively, the *properness* conditions required in the definition of a family with variable target (cf. I.7.B, III.1.B). On the other hand, (III*) is a *non-degeneracy* condition which could have been added to that definition without altering any of the subsequent developments. A family satisfying the extra assumption (III*) will be called *non-degenerate*.

The point of the above construction is that it provides a *universal family with variable target*.

E.2 Proposition. *Let (P,T,α) be a non-degenerate family with variable target. There exists a proper, Γ-equivariant map $\kappa : P \to P_N\Gamma$, where $N = \mathrm{rank}(T)$, such that $(P,T,\alpha) \cong \kappa^*(P_N\Gamma, T_N\Gamma, \alpha_N)$, i.e. $T \cong \kappa^*(T_N\Gamma)$ and $\kappa_T \circ \alpha = \alpha_N \circ \kappa$, where $\kappa_T : T \to T_N\Gamma$ is the canonical lift of κ.*

Proof: Let $F \to P$ be the orthonormal frame bundle associted to $T \to P$. An element of F is a pair (p, f) with $p \in P$ and $f : T_p \to \mathbf{R}^N$ as orientation preserving linear isometry. Define $\phi = \kappa_F(p, f) : \Gamma \to \mathbf{R}^N$ by the formula:

$$\phi(\gamma) = f\big(\gamma\alpha(\gamma^{-1}p)\big) , \qquad \forall \gamma \in \Gamma .$$

Since (P, T, α) satisfies (I*) and (III*), it follows that $\phi \in F_N\Gamma$. The map $\kappa_F : F \to F_N\Gamma$ thus defined is obviously $\mathrm{SO}(N)$-equivariant and, in view of the fact that (P, T, α) fulfills the properness axiom (II*), κ_F is proper. Therefore, it induces proper maps $\kappa : P \to P_N\Gamma$ and $\kappa_T : T \cong F \times_{\mathrm{SO}(N)} \mathbf{R}^N \to T_N\Gamma$. Moreover, κ_F provides a canonical identification of F with $\kappa_F^*(F_N\Gamma)$ and, consequently, of T with $\kappa^*(T_N\Gamma)$. With these identifications made, it is easily seen that $\kappa_T \circ \alpha = \alpha_N \circ \kappa$.

Finally, κ is Γ-equivariant because κ_F is; indeed, if $g \in \Gamma$, $p' = gp$, $f' = g \cdot f$ and $\phi' = \kappa_F(p', f')$, then

$$\phi'(\gamma) = f'\big(\gamma\alpha(\gamma^{-1}ap)\big) = f\big(g^{-1}\gamma\alpha(\gamma^{-1}gp)\big) = \phi(g^{-1}\gamma) = (g \cdot \phi)(\gamma) . \quad \square$$

Since $(P_N\Gamma, T_N\Gamma, \alpha_N)$ is itself a family with variable target, one can define, as in section I.8, a *duality map* $\alpha_N\cap : H_*(P_N\Gamma : \Gamma) \to H^{N-*}(\Gamma)$. Namely, let $\Delta = \Delta(\gamma_0, \dots, \gamma_k) = \big\{ \sum_{0 \le j \le k} t_j\gamma_j \, ; \, \sum_{0 \le j \le k} t_j = 1 \, , \, t_j \ge 0 \big\}$ be the k-simplex spanned by $\gamma_0, \dots, \gamma_k \in \Gamma$ and let $\alpha_\Delta : \Delta \times P_N\Gamma \to T_N\Gamma$ be the map defined by $\alpha_\Delta\big(\sum_{0 \le j \le k} t_j\gamma_j, p\big) = \sum_{0 \le j \le k} t_j\gamma_j^{-1}\alpha(\gamma_j p)$. Let U_N be the (Γ-invariant) Thom class of the bundle $T_N\Gamma \to P_N\Gamma$. Since $U_N \in H^N_{cv}(T_N\Gamma, \mathbf{R})$ (= compact vertical cohomology of $T_N\Gamma$) and α_Δ is proper, $\alpha_\Delta^*(U_N) \in H^N_c(\Delta \times P_N\Gamma, \mathbf{R})$. One can therefore evaluate $\alpha_\Delta^*(U_N)$ on a cycle of the form $\Delta \times C$, where $C \in H_{N-k}(P_N\Gamma : \Gamma)$. The resulting number, $c(\gamma_0, \dots, \gamma_k)$, satisfies the invariance property $c(\gamma\gamma_0, \dots, \gamma\gamma_k) = c(\gamma_0, \dots, \gamma_k)$, $\forall \gamma \in \Gamma$. Moreover, the assignment $(\gamma_0, \dots, \gamma_k) \to c(\gamma_0, \dots, \gamma_k)$ defines a group cocycle; its class is, by definition, $\alpha_N\cap(C) \in H^*(\gamma)$.

E.3 Corollary. *Let (P, T, α) be a non-degenerate family with variable target and let $\kappa : P \to P_N\Gamma$ be its classifying map. The following diagram is commutative:*

$$\begin{array}{ccc} H_*(P\Gamma : \Gamma) & \xrightarrow{\kappa_*} & H_*(P_N\Gamma : \Gamma) \\ & {\scriptstyle \alpha\cap} \searrow \quad \swarrow {\scriptstyle \alpha_N\cap} & \\ & H^{N-*}(\Gamma) . & \end{array}$$

Thus, a natural definition for the *Lipschitz cohomology of* Γ is the following:

$$H_L^*(\Gamma) = \bigcup_{N\geq 1} \text{Image}\left(\alpha_{N\cap} : H_{N-*}(P_N\Gamma : \Gamma) \to H^*(\Gamma)\right) \subseteq H^*(\Gamma) .$$

Note, however, that in the body of the paper we have been using a "smoothed" version of this cohomology, consisting only of Lipschitz classes which admit a smooth realization, i.e. arise from geometric cycles (P, T, α, C) with P a smooth manifold.

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