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Geometry from the Spectral Point of View

This paper is dedicated to the memory of J. Schwinger

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Abstract. In this Letter, we develop geometry from a spectral point of view, the geometric data being encoded by a triple $(\mathcal{A}, \mathcal{H}, D)$ of an algebra $\mathcal{A}$ represented in a Hilbert space $\mathcal{H}$ with selfadjoint operator D. This point of view is dictated by the general framework of noncommutative geometry and allows us to use geometric ideas in many situations beyond Riemannian geometry.

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0. Introduction

The notions of manifold and of Riemannian metric play a basic role in our usual formulation of geometry. The obtained notion of geometric space is flexible enough to encompass not only the Euclidean and non-Euclidean geometries, but also the space like hypersurfaces in general relativity. The tools of the differential and integral calculus allow us to develop the general theory of Riemannian manifolds. These tools are replaced in noncommutative geometry by the quantized calculus (cf. below).

In mathematics, one meets many natural spaces such as the space of Penrose tilings of the plane, spaces of leaves of foliations, spaces of irreducible representations of discrete group, fractal spaces, etc., which are not Riemannian manifolds but to which one would like to apply geometric ideas. Such spaces give rise in a natural manner to an associative algebra $\mathcal{A}$ that plays the role of the algebra of functions $f: X \to \mathbb{C}$ with the product:

$$f_1 f_2(p) = f_1(p) f_2(p), \quad \forall p \in X \tag{1}$$

and involution $*$ given by

$$f^*(p) = \overline{f(p)}, \quad \forall p \in X. \tag{2}$$

In general, the algebra $\mathcal{A}$ associated to the above spaces is not commutative, this accounts for the difficulty in identifying the notion of *point* in the above spaces. In simple examples such as manifolds or fractals, the algebra $\mathcal{A}$ is commutative and is the algebra of functions on X, but allowing noncommutative algebras is essential in order to deal with more elaborate examples such as quotients of manifolds by a pseudogroup of transformations. It is the relations between different points that generate the noncommutativity. For instance, if one considers a set Y consisting of two points $\{1,2\}$ and the relation which identifies 1 and 2, then $\mathcal{A}(Y, \mathrm{rel})$ is the algebra $M_2(\mathbb{C})$ of 2×2 complex matrices with the product

$$(f_1 f_2)(i,j) = \sum f_1(i,k) f_2(k,j), \quad i,j,k \in \{1,2\}, \tag{3}$$

i.e. the usual product of matrices.

In this simple example, the ordinary space $\{1,2\}$, given by the two points without any relation, is described by the subalgebra of diagonal matrices. It is the 'off-diagonal' matrices, such as

$$e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which describe the relation. This type of construction of an algebra $\mathcal{A}$ is rather general. It extends to a pseudogroup of transformations of a manifold and also to the holonomy pseudogroup of a foliation (see [1]). The resulting noncommutative algebra encodes the structure of the 'space with relations'. It also applies to a smooth manifold together with its full diffeomorphism group.

As another simple example, we can consider the case of a single point divided by a discrete group Γ. Then the corresponding algebra $\mathcal{A}$ is the group ring attached to Γ, whose elements f are functions (with finite support) on Γ,

$$g \to f_g \in \mathbb{C}, \tag{4}$$

with the product given by linearization of the group law $g_1, g_2 \to g_1 g_2$ in Γ:

$$(f_1 f_2)_g = \sum_{g_1 g_2 = g} f_{1,g_1} f_{2,g_2}. \tag{5}$$

So far, in describing the algebra $\mathcal{A}$ associated to an ordinary space X we have ignored the degree of regularity of the elements $f \in \mathcal{A}$ as functions of $p \in X$. To various degrees of regularity correspond various branches of the general theory of noncommutative associative algebras. The latter are assumed to be algebras over $\mathbb{C}$ which, moreover, are involutive, i.e. endowed with an antilinear involution

$$f \to f^*, \quad (f_1 f_2)^* = f_2^* f_1^*. \tag{6}$$

The two kinds of regularity assumptions for which the corresponding algebraic theory is satisfactory are

measurability, which corresponds to the theory of von Neumann algebras;
continuity, which corresponds to the theory of C^*-algebras.

The Hilbert space plays a key role in both theories. Indeed, both types of algebras are faithfully representable as algebras of operators in Hilbert space with a suitable closure hypothesis. One can trace the role of Hilbert space to the simple fact that *positive* complex numbers are those of the form

$$\lambda = z^* z. \tag{7}$$

In any of the above algebras, functional analysis provides the existence, via Hahn–Banach arguments, of sufficiently many linear functionals L which are positive

$$L(f^* f) \geqslant 0. \tag{8}$$

From such an L, one easily constructs a Hilbert space together with a representation, by left multiplication, of the original algebra.

Next, many of the tools of *differential topology*, such as the de Rham theory of differential forms and currents, the Chern character, etc., are well captured (see [1]) by cyclic cohomology applied to *pre-C^*-algebras*, i.e. to dense subalgebras of C^*-algebras which are stable under the holomorphic functional calculus

$$f \to h(f) = \frac{1}{2i\pi} \int \frac{h(z)}{f - z} \, \mathrm{d}z, \tag{9}$$

where h is holomorphic in a neighbourhood of Spec(f). The prototype of such an algebra is the algebra $C^\infty(X)$ of smooth functions on a manifold X. The cyclic cohomology construction then recovers the ordinary differential forms, the de Rham complex of currents, and so on. More significantly, this construction also applies to the highly noncommutative example of group rings, in which case the group cocycles give rise to cyclic cocycles with direct application to the Novikov conjecture on the homotopy invariance of the higher signatures of nonsimply connected manifolds with given fundamental group. (For a more thorough discussion, see [1].)

If one wants to go beyond differential topology and reach the geometric structure itself, including the metric and the real analytic aspects, it turns out that the most fruitful point of view is that of *spectral geometry*. More precisely, while our measure theoretic understanding of the space X was encoded by a (von Neumann) algebra of operators $\mathcal{A}$ acting in the Hilbert space $\mathcal{H}$, the *geometric* understanding of the space X will be encoded, not by a suitable subalgebra of $\mathcal{A}$, but by an operator in Hilbert space

$$D = D^*, \quad \text{selfadjoint unbounded operator in } \mathcal{H}. \tag{10}$$

In the compact case, i.e. X compact, the operator D will have discrete spectrum, with (real) eigenvalues λ_n, $|\lambda_n| \to \infty$, when $n \to \infty$.

TABLE I.

Classical	Quantum
Complex variable	Bounded operator in Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order $\alpha > 0$	Compact operator whose characteristic values μ_n satisfy $\mu_n = O(n^{-\alpha})$
Differential	$dT = [F, T] = FT - TF$
Integral of infinitesimal of order 1	Dixmier trace $\mathrm{Tr}_\omega(T)$

Formulating the precise conditions to which the triples $(\mathcal{A}, \mathcal{H}, D)$ should be subjected is tantamount to devising the axioms of noncommutative geometry. If we let F and $|D|$ be the elements of the polar decomposition of D,

$$D = F|D|, \qquad |D|^2 = D^2, \quad F = \mathrm{Sign}\, D, \tag{11}$$

then the operators F and $|D|$ play a similar role to the measurements of angles and, respectively, of length in Hilbert's axioms of geometry. In particular, the operator $F = \mathrm{Sign}\, D$ captures the conformal aspect, while D describes the full geometric situation.

Considering F alone, the quantized calculus was developed (cf. [1]) based on the dictionary produced in Table I.

We refer to [1] for a thorough treatment of the Dixmier trace. For a host of applications of the quantized calculus, including Julia sets, the quantum Hall effect and the analysis of group rings, the reader is referred to [1]. A further application, namely the construction of a four-dimensional conformal invariant analogue of the two-dimensional Polyakov action, is discussed in [3].

Our goal in this Letter is to use the quantized calculus to develop geometry from a spectral point of view. In more precise terms, our initial datum (called spectral triple) will be a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$ and D is a selfadjoint operator in $\mathcal{H}$ with compact resolvent, which almost commutes with any $a \in \mathcal{A}$, to the extent that $[D, a]$ is bounded for any $a \in \mathcal{A}$.

The basic example of such a triple is provided by the Dirac operator on a closed Riemannian (Spin) manifold. In that case, $\mathcal{H}$ is the Hilbert space of L^2 spinors on the manifold M, $\mathcal{A}$ is the algebra of (smooth) functions acting in $\mathcal{H}$ by multiplication operators and D is the (selfadjoint) Dirac operator. One can easily check that no information has been lost in trading the geometric space M for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Indeed (see [1]), one recovers

(i) the space M, as the spectrum $\mathrm{Spec}(\mathcal{A})$, of the norm closure of the algebra $\mathcal{A}$ of operators in $\mathcal{H}$;

(ii) the geodesic distance d on M, from the formula:

$$d(p,q) = \operatorname{Sup}\{|f(p) - f(q)|; \|[D,f]\| \leqslant 1\}, \quad \forall p,q \in M.$$

The right-hand side of the above formula continues to make sense in general and the simplest non-Riemannian example, where it applies, is the 0-dimensional situation in which the geometric space is finite. In that case, both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ are finite-dimensional, so that D is a selfadjoint matrix. For instance, for a two-point space, one lets $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ act in the two-dimensional Hilbert space $\mathcal{H}$ by

$$f \in \mathcal{A} \to \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix},$$

and one takes

$$d = \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}.$$

The above formula gives $d(a,b) = 1/\mu$.

As a slightly more involved 0-dimensional example, one can consider the algebraic structure provided by the elementary Fermions, i.e. the three families of quarks (and leptons). Thus, one hets $\mathcal{H}$ be the finite-dimensional Hilbert space with orthonormal basis labelled by the left-handed and right-handed elementary quarks such as $u^r_L, u^b_R, \ldots$ (and similarly for leptons). The algebra $\mathcal{A}$ in $\mathbb{C} \oplus \mathbb{H}$, where the complex number λ in $(\lambda, q) \in \mathcal{A}$ acts on the right-handed part by λ on 'up' particles and λ on 'down' particles. The isodoublet structure of the left-handed (up, down) pairs allows the quaternion q to act on them by the matrix

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad q = \alpha + \beta j; \quad \alpha, \beta \in \mathbb{C}.$$

Then the Yukawa coupling matrix of the standard model provides the selfadjoint matrix D.

In [4], the theory of matter fields was developed within the above framework, under the finite-dimensionality hypothesis that the characteristic values of D^{-1} are $O(n^{-1/d})$, for some finite d.

This allows us to define the action functional of quantum electrodynamics at the same level of generality (cf. [1]). The striking fact there is that if one replaces the usual picture of spacetime by its product by the above 0-dimensional example, the QED action functional gives the Glashow–Weinberg–Salam standard model Lagrangian with its Higgs fields and symmetry-breaking mechanism. This shows that our geometric framework of spectral triples is flexible enough to encompass the 'effective geometry' of spacetime at the energy levels that we can probe at present.

Of course, no one believes that the standard model is the ultimate answer, but the fact is that we can interpret the three additional terms to the QED Lagrangian, provided by the weak forces and Higgs mechanism, as coming from a simple modification of the texture of spacetime at short distances (cf. [1]). The latter, instead of being purely the continuum, becomes a mixture of continuum and finite discrete, the geometric space being the product of the ordinary continuum by a finite space (cf. [1]). In the development of this theory, the tools of the quantized calculus, in particular the Dixmier trace as the substitute for the Lebesgue integral, played an essential role.

The matter field Lagrangian which we have just discussed involves the metric $g_{\mu\nu}$ but does not involve any derivative of $g_{\mu\nu}$. This indicates that the difficulty involved in developing the analogue of gravity in the above context is of a different scale. In order to overcome it, one needs both a good list of examples of spectrally defined spaces and a difficult mathematical problem to solve. By a spectrally defined space, we mean a triple $(\mathcal{A}, \mathcal{H}, D)$ as above; the involution algebra $\mathcal{A}$ is not necessarily commutative. We shall also refer to them as *spectral triples*.

Let us give a list of examples

1. *Riemannian manifolds* (with some variations allowing for Finsler metrics and also for the replacement of $|D|$ by $|D|^\alpha, \alpha \in]0, 1]$).
2. *Manifolds with singularities*. For this, the work of J. Cheeger on conical singularities is very relevant. In fact, the spectral triples are stable under the operation of 'coning', which is easy to formulate algebraically.
3. *Discrete spaces and their product with manifolds* (as in the discussion in [1] of the standard model). The spectral triples are, of course, stable under products.
4. *Cantor sets*. Their importance lies in the fact that they provide examples of dimension spectra which contain complex numbers (cf. Section 2).
5. *Nilpotent discrete groups*. The algebra $\mathcal{A}$ is the group ring of the discrete group Γ, and the nilpotency of Γ is required to ensure the finite-summability condition $D^{-1} \in \mathcal{L}^{(p,\infty)}$. We refer to [1] for the construction of the triple for subgroups of Lie groups.
6. *Transverse structure for foliations*. This example, or rather the intimately related example of the *Diff*-equivariant structure of a manifold is treated in detail in [6].

Let us now state the mathematical problem which will be the guide to develop geometric concepts:

> *compute by a local formula the cyclic cohomology Chern character of $(\mathcal{A}, \mathcal{H}, D)$.*

More specifically, the representation of $\mathcal{A}$ in $\mathcal{H}$ together with the operator D allows us to set up an index problem $\mathrm{Ind}_D \colon K_j(\mathcal{A}) \to \mathbb{Z}$, where $j = 0$ in the $\mathbb{Z}/2$-graded (or even case) and $j = 1$ otherwise. The index map turns out to be polynomial and given, in the above generality, by the pairing of $K_j(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, a^1, \ldots, a^n) = \mathrm{Trace}(a^0[F, a^1] \ldots [F, a^n]), \quad \forall a^j \in \mathcal{A},$$

where n has the same parity as j and $n > d - 1$. In the even case, one replaces the trace by the supertrace, i.e. one uses the $\mathbb{Z}/2$-grading γ of $\mathcal{H}$ to write

$$\tau(a^0, a^1, \ldots, a^n) = \mathrm{Trace}(\gamma a^0[F, a^1] \ldots [F, a^n]), \quad \forall a^j \in \mathcal{A}.$$

The class of τ in the cyclic cohomology $\mathrm{HC}^n(\mathcal{A})$ is the Chern character of $(\mathcal{A}, \mathcal{H}, D)$. We refer to [1] for more details as well as for the appropriate normalizations.

The general problem is to compute the class of τ by a *local formula*. A partial answer to this problem was already obtained in [1], by means of a general local formula for the *Hochschild class* of τ as the Hochschild n-cocycle:

$$\varphi(a^0, \ldots, a^n) = \mathrm{Tr}_\omega(a^0[D, a^1] \ldots [D, a^n]|D|^{-n}), \quad \forall a^j \in \mathcal{A}, \tag{12}$$

where n is as above and, in the even case, with γ inserted in front of a^0.

In the above formula, Tr_ω is the Dixmier trace, which when evaluated on a given operator T only depends upon the asymptotic behavior of its eigenvalues. More precisely, for $T \geqslant 0$, with $\mu_n(T)$ the nth eigenvalue of T in decreasing order, one has (cf. [1]);

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_0^N \mu_n(T);$$

this is insensitive to the perturbation of μ_n by any sequence $\varepsilon_n = o(1/n)$, i.e. such that $n\varepsilon_n \to 0, n \to \infty$.

For a classical pseudodifferential operator P with distributional kernel $k(x, y)$, the Dixmier trace is given by the Wodzicki residue $\mathrm{Tr}_\omega(T) = \int a(x)$, where $k(x, y)$ has an asymptotic expansion near the diagonal of the form

$$k(x, y) = a(x) \log(d(x, y)) + b(x, y),$$

with b bounded [10].

In particular, when one evaluates Tr_ω on a product $T_1 \ldots T_n$ of such operators, the result is expressed as an integral *in a single variable* x of a local quantity. This is in sharp contrast with what happens for the ordinary trace, which when evaluated on $T_1 \ldots T_n$ involves a multiple integral, of the form

$$\int k_1(x_1, x_2)k_2(x_2, x_3) \ldots k_n(x_n, x_1).$$

where the x_j's vary arbitrarily in the manifold.

While the expression (12) of the Hochschild cocycle φ is local in full generality, it only accounts for the Hochschild class of the Chern character of $(\mathcal{A}, \mathcal{H}, D)$, which is not sufficient to recover the index map. In the manifold case, for instance, it only gives the index of D with coefficients in the Bott K-theory class supported by an arbitrarily small disk.

In Section 3 of this Letter, we shall describe a general local formula for all the components of the cyclic cocycle τ. This will be achieved by adapting the Wodzicki residue, the unique extension of the Dixmier trace to pseudodifferential operators of arbitrary order, to all our examples. For spectrally defined spaces $(\mathcal{A}, \mathcal{H}, D)$, we shall see that the usual notion of dimension is replaced by a *dimension spectrum* (Section 2), which is a subset of $\mathbb{C}$. Under the assumption of simple discrete dimension spectrum, the Wodzicki residue makes sense and defines a trace on the algebra of the pseudodifferential operators of $(\mathcal{A}, \mathcal{H}, D)$. The latter algebra is obtained by analyzing the one-parameter group $\sigma_t = |D|^{it} \cdot |D|^{-it}$ in a manner very similar to Tomita's analysis of the modular automorphism group of von Neumann algebras (Section 1). When the dimension spectrum is discrete but not simple, the analogue of the Wodzicki residue is no longer a trace; it satisfies, however, cohomological identities which relate it to higher residues (cf. Section 2).

Under the sole hypothesis of discreteness of the dimension spectrum, we shall obtain (Section 3) a *universal local formula for the Chern character* of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, expressing the components of the Chern character in terms of finite linear combinations, with *rational coefficients*, of higher residues applied to products of iterated commutators of D^2 with $[D, a^j], a^j \in \mathcal{A}$. A noteworthy feature of the proof is the use of renormalization group techniques to remove the transcendental coefficients which arise when the dimension spectrum has multiplicity (Section 4). In the manifold case, this formula reduces, of course, to the classical local index formula. In general, however, it is necessarily more intricate, in several respects, because of its large domain of applicability, which encompasses for instance the diffeomorphisms-equivariant situation described in [6]. Finally, in the last section, we shall describe the analogue within our framework of the usual geometric notions of cotangent space, geodesic flow, and Levi-Civita connection. The justification for these notions is that they play an implicit role in the elaboration of proof of Theorem 2 and are not just mere analogues of the classical notions.

1. Pseudodifferential Calculus for Spectral Triples

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. For each $s \in \mathbb{R}$, we let $\mathcal{H}^s = \text{Domain}\,(|D|^s)$ and

$$\mathcal{H}^\infty = \bigcap_{s \geqslant 0} \mathcal{H}^s, \qquad \mathcal{H}^{-\infty} = \text{dual of } \mathcal{H}^\infty.$$

In this way we obtain a scale of Hilbert spaces, and for each r we define op^r to be the linear space of operators in $\mathcal{H}^\infty$ which are continuous for every s:

$$\mathrm{op}^r : \mathcal{H}^s \to \mathcal{H}^{s-r}.$$

We shall use the following smoothness condition on $\mathcal{A}$: $\forall a \in \mathcal{A}$, both a and $[D, a]$ are in the domain of all powers of the derivation $\delta = [|D|, \cdot]$.

LEMMA 1. *Then* $a, [D, a]$ *are in* op^0 *and*

$$b - |D| b |D|^{-1} \in \mathrm{op}^{-1} \quad (b = a \textit{ or } [D, a]).$$

Proof. Let us first check that $|D|^n b |D|^{-n}$ is bounded for $n \geqslant 0$. With $\sigma(\cdot) = |D| \cdot |D|^{-1}$, one has

$$\sigma = \mathrm{id} + \varepsilon, \quad \varepsilon(b) = \delta(b)|D|^{-1}.$$

Since $\varepsilon^k(b)$ is bounded, equal to $\delta^k(b)|D|^{-k}$, we get the result using $\sigma^n = \sum C_n^k \varepsilon^k$.

Moreover, $\sigma^{-1}(b) = |D|^{-1} b |D| = b - |D|^{-1}\delta(b)$ and the same argument shows that $\sigma^n(b)$ is bounded for $n < 0$. Then one uses interpolation.

For the second part, one applies the above argument to $\delta(b)$; thus,

$$\delta(b) \in \mathrm{op}^0, \quad \delta(b)|D|^{-1} \in \mathrm{op}^{-1}.$$

□

It is important to note that the above smoothness hypothesis can be replaced by

$$a \text{ and } [D, a] \in \cap \operatorname{Dom} L^k R^q, \quad L(b) = |D|^{-1}[D^2, b], \quad R(b) = [D^2, b]|D|^{-1}.$$

Indeed, assuming the above, one has

$$L(b) = |D|^{-1}(|D|\delta(b) + \delta(b)|D|) \in \mathrm{op}^0, \quad R(b) \in \mathrm{op}^0$$

and the same applies to $L^k R^q(b)$.

COROLLARY 1. *Under the above hypothesis, one has*

$$[D^2, [D^2, \dots [D^2, b\underbrace{]] \cdots]}_{n} \in \mathrm{op}^n, \quad \forall b \in \mathcal{A} \textit{ or } [D, \mathcal{A}].$$

Let us now show that if $b \in \cap \operatorname{Dom} L^k R^q$, then $b \in \operatorname{Dom} \delta$. The proof is more subtle than one would expect, because the obvious argument, using

$$|D| = \pi^{-1} \int_0^\infty \frac{D^2}{D^2 + \mu} \mu^{-1/2}\, \mathrm{d}\mu,$$

requires some care. Indeed, one gets from the above

$$[|D|, b] = \pi^{-1} \int_0^\infty (D^2 + \mu)^{-1}[D^2, b](D^2 + \mu)^{-1}\mu^{1/2}\, d\mu.$$

We can replace $[D^2, b]$ by $|D|$, which has the same size, and get

$$\int_0^\infty (D^2 + \mu)^{-2}|D|\mu^{1/2}\, d\mu = \int_0^\infty (1 + t)^{-2} t^{1/2}\, dt.$$

For this to work, we need to move $[D^2, b]$ in front of the above integral, i.e. use the finiteness of the norm of

$$\int_0^\infty \underbrace{[(D^2 + \mu)^{-1}, [D^2, b]]}_{-(D^2+\mu)^{-1}[D^2,[D^2,b]](D^2+\mu)^{-1}} (D^2 + \mu)^{-1}\mu^{1/2}\, d\mu.$$

This finiteness follows from

(1) $(D^2 + \mu)^{-1}[D^2, [D^2, b]]$ bounded since $b \in \text{Dom } L^2$,
(2) $\int_0^\infty \|(D^2 + \mu)^{-2}\|\mu^{1/2}\, d\mu \leqslant C \int_0^1 \mu^{1/2}\, d\mu + \int_1^\infty \mu^{-3/2}\, d\mu < \infty$.

Once $[D^2, b]$ is moved in front the above calculation applies.

It follows that $b \in \text{Dom}\, \delta$ and applying the same proof to $\delta(b), \ldots$ we get $b \in \cap \text{ Dom } \delta^k$. We thus obtain the following lemma.

LEMMA 2. $\cap_{k,q} \text{ Dom } L^k R^q = \cap_n \text{ Dom } \delta^n$.

We shall define the order of operators by the following filtration:

$$P \in \text{OP}^\alpha \quad \text{iff } |D|^{-\alpha} P \in \cap \text{ Dom } \delta^n.$$

Thus, $\text{OP}^0 = \cap \text{ Dom } \delta^n$ and we have

$$\text{OP}^\alpha \subset \text{op}^\alpha \quad \forall \alpha.$$

Let us now describe the general pseudodifferential calculus.

We let ∇ be the derivation: $\nabla(T) = [D^2, T]$ and consider the algebra generated by the $\nabla^n(T)$, $T \in \mathcal{A}$ or $[D, \mathcal{A}]$.

We view this algebra $\mathcal{D}$ as an analogue of the algebra of differential operators. In fact, by Corollary 1, we have a natural filtration of $\mathcal{D}$ by the total power of ∇ applied, and moreover

$$\mathcal{D}^n \subset \text{OP}^n. \tag{1}$$

We want to develop a calculus for operators of the form:

$$A|D|^z \quad z \in \mathbb{C},\ A \in \mathcal{D}. \tag{2}$$

We shall use the notation $\Delta = D^2$ and begin by understanding the action of $\mathbb{C}$ given by

$$\sigma^{2z} = \Delta^z \cdot \Delta^{-z}. \tag{3}$$

By construction $\mathcal{D}$ is stable under the derivation ∇ and

$$\nabla(\mathcal{D}^n) \subset \mathcal{D}^{n+1}. \tag{4}$$

Also for $A \in \mathcal{D}^n$ and $z \in \mathbb{C}$, one has

$$A|D|^z \in \mathrm{OP}^{n+\mathrm{Re}(z)}. \tag{5}$$

We shall use the group σ^{2z} to understand how to multiply operators of complex order modulo OP^{-k} for any k. One has $\sigma^2 = 1 + \mathcal{E}$,

$$\mathcal{E}(T) = \nabla(T)\Delta^{-1}. \tag{6}$$

LEMMA 3. *Let $T \in \mathcal{D}^q$ then $\mathcal{E}^k(T) \in \mathrm{OP}^{q-k}$*, $\forall k \geqslant 0$.

Proof.

$$\mathcal{E}^k(T) = \nabla^k(T)\Delta^{-k} \in \mathrm{OP}^{q+k}\Delta^{-k} \subset \mathrm{OP}^{q-k}.$$ □

We just wish to justify the formal expansion:

$$\sigma^{2z}(T) = \left(1 + z\mathcal{E} + \frac{z(z-1)}{2!}\mathcal{E}^2 + \cdots\right)(T).$$

It should give a control of $\sigma^{2z}(T)$ modulo OP^{q-k-1} if we stop at $\mathcal{E}^k(T)$.

To do this, we need to control the remainder in the Taylor formula:

$$\begin{aligned}
&(1+\mathcal{E})^{n+1-\alpha} \\
&\quad = 1 + (n+1-\alpha)\mathcal{E} + \frac{(n+1-\alpha)(n-\alpha)}{2!}\mathcal{E}^2 + \cdots \\
&\qquad + (n+1-\alpha)\cdots(n+1-k-\alpha)\frac{\mathcal{E}^{k+1}}{(k+1)!} + \cdots \\
&\qquad + (n+1-\alpha)\cdots(2-\alpha)\frac{\mathcal{E}^n}{n!} + \\
&\qquad + \mathcal{E}^{n+1}\int_0^1 (n+1-\alpha)\cdots(1-\alpha)(1+t\mathcal{E})^{-\alpha}\frac{(1-t)^n}{n!}\,\mathrm{d}t.
\end{aligned} \tag{7}$$

The main lemma is the following:

LEMMA 4. *Let* $\alpha \in \mathbb{C}, 0 < \operatorname{Re} \alpha < 1$ *and* $\beta > 0$, $\beta < a = \operatorname{Re} \alpha$. *Then the following operator preserves the space* OP^{α} *for any* α:

$$\Psi = \sigma^{2\beta} \int_0^1 (1 + t\mathcal{E})^{-\alpha}(1 - t)^{n} \, \mathrm{d}t.$$

Proof. This will be done by expressing Ψ as an integral of the form

$$\Psi = \int \sigma^{2is} \, \mathrm{d}\mu(s), \quad \|\mu\| < \infty. \tag{8}$$

One writes

$$(1 + t\mathcal{E})^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{1}{1 + t\mathcal{E} + \mu} \mu^{-\alpha} \, \mathrm{d}\mu \tag{9}$$

using the standard formula

$$x^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{1}{x + \mu} \mu^{-\alpha} \, \mathrm{d}\mu. \tag{10}$$

Let us then consider the resolvent of $-\sigma^2$, namely

$$R(\lambda) = (\lambda + \sigma^2)^{-1}.$$

One has, with $\beta \in]0, 1[$ as above,

$$R(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \sigma^{-2(\beta+is)} \lambda^{\beta+is-1} \frac{\mathrm{d}s}{\sin \pi(\beta + is)} \tag{11}$$

which follows from

$$\frac{1}{1 + y} = \frac{1}{2} \int_{-\infty}^{\infty} y^{-(\beta+is)} \frac{\mathrm{d}s}{\sin \pi(\beta + is)}. \tag{12}$$

(With $y = \mathrm{e}^u$, this means that $\mathrm{e}^{\beta u}/(1 + \mathrm{e}^u)$ is the Fourier transform of $1/(\sin \pi(\beta + is))$, which also follows from (10) written as

$$\frac{\pi}{\sin(\pi(1 - \beta - is))} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{(1-\alpha)u}}{1 + \mathrm{e}^u} \, \mathrm{d}u, \quad \alpha = 1 - \beta - is.)$$

Thus, from (11), we get

$$\sigma^{2\beta} R(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \sigma^{-2is} \lambda^{\beta-1} \frac{\lambda^{is} \, \mathrm{d}s}{\sin \pi(\beta + is)} \tag{13}$$

where the measure $\lambda^{is}\,\mathrm{d}s/(\sin\pi(\beta+is))$ is well controlled by $\mathrm{e}^{-|s|}\,\mathrm{d}s$. By (9) we have

$$(1+t\mathcal{E})^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{1}{t}R\left(\frac{\mu+1}{t}-1\right)\mu^{-\alpha}\,\mathrm{d}\mu,$$

$$\sigma^{2\beta}(1+t\mathcal{E})^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\,\frac{1}{2}\int_0^\infty \frac{1}{t}\int_{-\infty}^{\infty}\sigma^{-2is}\left(\frac{\mu+1}{t}-1\right)^{\beta-1}\mu^{-\alpha}\,\frac{\lambda^{is}\,\mathrm{d}s}{\sin\pi(\beta+is)}\,\mathrm{d}\mu,$$

with

$$\lambda = \left(\frac{\mu+1}{t}-1\right).$$

For fixed s, we are thus dealing with the size

$$\frac{1}{t}\int_0^\infty\left(\frac{\mu+1}{t}-1\right)^{\beta-1}\mu^{-\alpha}\,\mathrm{d}\mu = I.$$

One has $0<t<1$ so that the behavior at $\mu=0$ is fine, also the integral converges for $\mu\to\infty$ as $\mu^{(\beta-\alpha)-1}$, since $\beta<\alpha$.

We get

$$\begin{aligned} I &= \frac{1}{t}\int_0^\infty\left(u+\frac{1}{t}-1\right)^{\beta-1}t^{-\alpha}u^{-\alpha}t\,\mathrm{d}u \\ &= \frac{1}{t}\int_0^\infty\left(\frac{1}{t}-1\right)^{\beta-1}(v+1)^{\beta-1}t^{-\alpha}\left(\frac{1}{t}-1\right)^{-\alpha}v^{-\alpha}t\left(\frac{1}{t}-1\right)\mathrm{d}v \\ &= t^{-\alpha}\left(\frac{1}{t}-1\right)^{\beta-\alpha}\int_0^\infty(v+1)^{\beta-1}v^{-\alpha}\,\mathrm{d}v \\ &= (1-t)^{\beta-\alpha}t^{-\beta}c(\alpha,\beta). \end{aligned}$$

Finally, we get an equality

$$\int_0^1\sigma^{2\beta}(1+t\mathcal{E})^{-\alpha}\frac{(1-t)^n}{n!}\,\mathrm{d}t = \int_{-\infty}^{\infty}\sigma^{2is}\,\mathrm{d}\nu(s),$$

where the total mass of the measure ν is finite.

Let us check this in another way, by looking directly at the L^1-norm of the Fourier transform of the function

$$u\to\int_0^1\mathrm{e}^{\beta u}(1+t(\mathrm{e}^u-1))^{-\alpha}\frac{(1-t)^n}{n!}\,\mathrm{d}t.$$

Thus, it is enough to check that the following function of u is in the Schwartz space $\mathcal{S}(\mathbb{R})$:

$$\begin{aligned}
&\varphi_n(u)\\
&= (e^u - 1)^{-(n+1)} e^{\beta u} \Big(e^{(n+1-\alpha)u} - 1 - (n+1-\alpha)(e^u - 1) -\\
&\quad - \frac{(n+1-\alpha)(n-\alpha)}{2!}(e^u-1)^2\\
&\quad - \cdots - \frac{(n+1-\alpha)(n-\alpha)\cdots(2-\alpha)}{n!}(e^u - 1)^n \Big).
\end{aligned}$$

First, for $u \to \infty$, the size is

$$\sim e^{-(n+1)u} e^{\beta u} e^{(n+1-\alpha)u} = e^{(\beta-\alpha)u} \to 0.$$

For $u \to -\infty$, it behaves like $e^{\beta u} \to 0$. We need to know that it is smooth at $u = 0$ but this follows from the Taylor expansion. The same argument applies to all derivatives. Thus, this gives another proof of the lemma. □

We are now ready to prove the following theorem.

THEOREM 1. *Let $T \in \mathcal{D}^q$ and $n \in \mathbb{N}$. Then, for any $z \in \mathbb{C}$*

$$\begin{aligned}
&\sigma^{2z}(T) - \Big(T + z\mathcal{E}(T) + \frac{z(z-1)}{2!}\mathcal{E}^2(T)\\
&\qquad + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}\mathcal{E}^n(T) \Big) \in \mathrm{OP}^{q-(n+1)}.
\end{aligned}$$

Proof. First for any $z \in \mathbb{C}$ and $k \in \mathbb{N}$ one has

$$\mathcal{E}^k(\sigma^{2z}(T)) \in \mathrm{OP}^{q-k}. \tag{14}$$

Indeed, by (8) we know that σ^{2z} leaves any OP^{α} invariant; as $\mathcal{E}^k \circ \sigma^{2z} = \sigma^{2z} \circ \mathcal{E}^k$, we just use Lemma 3.

One has, for $0 < \mathrm{Re}\,\alpha < 1$, β as above and $z = (n+1) - \alpha$:

$$\begin{aligned}
&\sigma^{2(\beta+(n+1)-\alpha)}(T)\\
&\quad - \Big(\sigma^{2\beta}(T) + z\sigma^{2\beta}\mathcal{E}(T) + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}\sigma^{2\beta}\mathcal{E}^n(T) \Big)\\
&= \lambda\Psi(\mathcal{E}^{n+1}(T)).
\end{aligned} \tag{15}$$

If we apply this equality to $\sigma^{2s}(T)$ and use (14) and Lemma 4, we see that for any s, there exists a polynomial $P_s(\alpha)$ of degree n in α such that

$$\sigma^{2(s-\alpha)}(T) - P_s(\alpha) \in \mathrm{OP}^{q-(n+1)}, \qquad \beta < \operatorname{Re}\alpha < 1.$$

The polynomials $P_s(\alpha + s)$ have to agree modulo $\mathrm{OP}^{q-(n+1)}$ on the overlap of the bands $\beta < \operatorname{Re}\alpha < 1$ and, thus, the differences between the two will belong to $\mathrm{OP}^{q-(n+1)}$ for all z. It follows then that there is $P(z)$ which works for all z. To obtain its coefficients, one takes the integral values $z = 0, 1, \ldots, n$ which yields the formula of Theorem 1. □

2. Dimension Spectrum

In this Letter, we shall describe a general local index formula in terms of the Dixmier trace, extended to operators of arbitrary order, for our spectral triples:

$$(\mathcal{A}, \mathcal{H}, D). \tag{1}$$

Contrary to the standard practice, we shall focus on the odd case, the point being that in the even case, there is a natural obstruction to express the (cyclic cocycle) character (cf. [1]) of the triple (1) in terms of a residue or Dixmier trace. Indeed, the latter vanishes on any finite rank operator and, thus, will give the result 0 whenever $\mathcal{H}$ is finite-dimensional. Since it is easy to construct finite-dimensional (i.e. $\dim \mathcal{H} < \infty$) even triples with $\operatorname{Ind}(D) \neq 0$, one cannot expect to cover this case as well. However, one can convert any even triple into an odd one by crossing it with S^1, i.e. with the triple

$$\left(C^\infty(S^1), L^2(S^1), D = \frac{1}{i}\frac{\partial}{\partial\theta}\right). \tag{2}$$

Thus, there is no real loss of generality in treating the odd case only. The next point is that the usual notion of dimension (cf. [1]) for spectral triples, provided by the degree of summability

$$D^{-1} \in \mathcal{L}^{(p,\infty)}, \tag{3}$$

gives only an upper bound on the dimension and cannot detect the dimensions of the various pieces of a space constructed as a union of pieces of different dimensions $(\mathcal{A}_k, \mathcal{H}_k, D_k)$, $k = 1, \ldots, N$,

$$\mathcal{A} = \oplus \mathcal{A}_k, \qquad \mathcal{H} = \oplus \mathcal{H}_k, \qquad D = \oplus D_k. \tag{4}$$

In [1], we gave a formula for the p-dimensional Hochschild cohomology class of the character, namely:

$$\tau(a^0, \ldots, a^p) = \operatorname{Tr}_\omega(a^0[D, a^1] \cdots [D, a^p]|D|^{-p}). \tag{5}$$

Clearly, this Hochschild cocycle cannot account for lower-dimensional pieces in a union such as (4).

As it turns out, the correct notion of dimension is given not by a single real number p but by a subset

$$\mathrm{Sd} \subset \mathbb{C} \tag{6}$$

which shall be called the *dimension spectrum* of the given triple. We shall assume that Sd is a discrete subset of $\mathbb{C}$, a condition which can be incorporated into the definition of Sd, as follows:

DEFINITION 1. A spectral triple (1) has discrete dimension spectrum Sd, if $\mathrm{Sd} \subset \mathbb{C}$ is discrete and for any element of the algebra $\mathcal{B}$ generated by the $\delta^n(a)$, $a \in \mathcal{A}$, the function

$$\zeta_b(z) = \mathrm{Trace}(b|D|^{-z})$$

extends holomorphically to $\mathbb{C}\backslash\mathrm{Sd}$.

Here δ denotes the derivation $\delta(T) = [|D|, T]$ and we assume that $\mathcal{A} \subset \cap_{n>0}\mathrm{Dom}\,\delta^n$ (see also Section 1). The operator $b|D|^{-z}$ of Definition 1 is then of trace class for $\mathrm{Re}\, z > p$, with p as in (3). On the technical side, we shall assume that the analytic continuation of ζ_b is such that $\Gamma(z)\zeta_b(z)$ is of rapid decay on vertical lines $z = s + it$, for any s with $\mathrm{Re}\, s > 0$. It is not difficult to check that Sd has the correct behavior with respect to the operations of sum and product for spectral triples:

$$\mathrm{Sd}\,(\text{Sum of two spaces}) = \cup\mathrm{Sd}(\text{Spaces}). \tag{7}$$

$$\mathrm{Sd}\,(\text{Product of two spaces}) = \mathrm{Sd}(\text{Space}_1) + \mathrm{Sd}(\text{Space}_2); \tag{8}$$

more precisely, (8) holds with the exception of $\mathrm{Sd} \cap -\mathbb{N}$.

It is easy to give many examples of spectral triples with discrete dimension spectrum, all examples listed in the introduction do, but we shall now concentrate on the general theory of such spaces.

Our first task will be to extend the Wodzicki residue to this general framework, or equivalently, to extend the Dixmier trace to operators $P|D|^{-z}$ of arbitrary order, where P is an element of $\mathcal{B}$. In fact, it is more convenient (cf. Section 1) to introduce the algebra $\Psi^*(\mathcal{A})$ of operators which have an expansion:

$$P \simeq b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots, \quad b_q \in \mathcal{B}, \tag{9}$$

where the equality with $\sum_{-N<n\leqslant q} b_n|D|^n$ holds modulo OP^{-N}.

To see that it is an algebra, one uses Theorem 1 of Section 1, which gives an identity of the form

$$|D|^\alpha b \simeq \sum_0^\infty c_{\alpha,k}\delta^k(b)|D|^{\alpha-k}. \tag{10}$$

where $c_{\alpha,k}$ is the coefficient of ε^k in the expansion of

$$(1+\varepsilon)^{\alpha} = \sum_0^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\varepsilon^k, \tag{11}$$

with $\varepsilon(b) = \delta(b)|D|^{-1}$.

We shall say that the dimension spectrum Sd is *simple*, when the singularities of the functions $\zeta_b(z)$ of Definition 1 at $z \in$ Sd are at most simple poles. Similarly, we say that Sd has finite multiplicity k when ζ_b has, at most, a pole of order k. We shall assume for simplicity that Sd has finite multiplicity in this section.

PROPOSITION 1. *Let $p < \infty$ be the degree of summability of D.*

(a) *For $P \in \Psi^*(\mathcal{A})$ the function $h(z) =$ Trace$(P|D|^{-2z})$ is holomorphic for* Re $z > \frac{1}{2}$(Order $P + p$) *and extends to a holomorphic function on the complement of a discrete subset of $\mathbb{C}$.*

(b) *Let $\tau_k(P)$ be the residue at 0 of $z^k h(z)$, $k \geqslant 0$; then*

$$\tau_k(P_1P_2 - P_2P_1) = \sum_{n>0} \frac{(-1)^{n-1}}{n!}\tau_{k+n}(P_1L^n(P_2)),$$

where L is the derivation $L = 2\log(1+\varepsilon)$.

Proof. (a) The statement follows immediately from Definition 1 for any finite sum of operators $b_\alpha|D|^\alpha$. Furthermore, if P is of order less than $-N$, then $h(z)$ is holomorphic if Re $z > \frac{1}{2}(p - N)$ and, for any given z, this is achieved for N large enough.

(b) First, the derivation $L = 2\log(1+\varepsilon)$ makes sense as a power series in ε and can be viewed at the formal level as implemented by $\log|D|^2$.

For any P, one has an expansion near 0

$$\text{Trace}(P|D|^{-2z}) = \sum_{k\geqslant 0} \tau_k(P)z^{-(k+1)} + O(1). \tag{12}$$

We can then write

$$\text{Trace}(P_2P_1|D|^{-2z}) = \text{Trace}(P_1(1+\varepsilon)^{-2z}(P_2)|D|^{-2z}) \tag{13}$$

and, since

$$(1+\varepsilon)^{-2z} = \exp(-zL), \tag{14}$$

we get

$$\text{Trace}(P_2P_1|D|^{-2z}) = \sum \frac{(-z)^n}{n!}\text{Trace}(P_1L^n(P_2)|D|^{-2z}). \tag{15}$$

By (13), we can expand

$$\mathrm{Trace}(P_1 L^n(P_2)|D|^{-2z}) = \sum \tau_q(P_1 L^n(P_2)) z^{-(q+1)} + \mathrm{O}(1)$$

and, when multiplied by z^n, we see that we get the exponent $z^{-(k+1)}$ for $n - q = -k$. Thus, the coefficient of $z^{-(k+1)}$ in the expansion (16) is

$$\sum_{n=q-k} \frac{(-1)^n}{n!} \tau_q(P_1 L^n(P_2)) = \sum \frac{(-1)^n}{n!} \tau_{n+k}(P_1 L^n(P_2)).$$

Therefore, we obtain

$$\tau_k(P_2 P_1) - \tau_k(P_1 P_2) = \sum_{n>0} \frac{(-1)^n}{n!} \tau_{n+k}(P_1 L^n(P_2)). \tag{16}$$

□

It follows, of course, that if q is the multiplicity of Sd, i.e. the highest order of poles, then τ_q is a trace.

In the case of a simple spectrum, the trace $\tau = \tau_0$ is an extension of the Dixmier trace, the latter being defined only when the operator $P \in \Psi^*(\mathcal{A})$ belongs to OP^{-p}.

3. Local Formula for the Chern Character

Before giving the general local formula for the Chern character of a triple $(\mathcal{A}, \mathcal{H}, D)$ with discrete dimension spectrum, we need to recall a few basic definitions from [1].

First, the cyclic cohomology $\mathrm{HC}^n(\mathcal{A})$ is defined as the cohomology of the complex of cyclic cochains, i.e. those satisfying

$$\varphi(a^1, \ldots, a^n, a^0) = (-1)^n \varphi(a^0, \ldots, a^n), \quad \forall a^j \in \mathcal{A}, \tag{1}$$

under the coboundary operation b given by:

$$\begin{aligned}(b\varphi)&(a^0, \ldots, a^{n+1}) \\ &= \sum_0^n (-1)^j \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) \\ &\quad + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n), \quad \forall a^j \in \mathcal{A}.\end{aligned} \tag{2}$$

Equivalently, $\mathrm{HC}^n(\mathcal{A})$ can be described in terms of the second filtration of the (b, B) bicomplex of arbitrary (noncyclic) cochains on $\mathcal{A}$, where $B: C^m \to C^{m-1}$ is given by

$$
\begin{aligned}
&(B_0\varphi)(a^0, \ldots, a^{m-1}) \\
&\quad = \varphi(1, a^0, \ldots, a^{m-1}) - (-1)^m \varphi(a^0, \ldots, a^{m-1}, 1), \\
&B = AB_0, \quad (A\psi)(a^0, \ldots, a^{m-1}) = \sum (-1)^{(m-1)j} \psi(a^j, \ldots, a^{j-1}).
\end{aligned} \tag{3}
$$

To an n-dimensional cyclic cocycle ψ, one associates the (b, B) cocycle $\varphi \in Z^p(F^q C)$, $n = p - 2q$ given by

$$(-1)^{[n/2]}(n!)^{-1}\psi = \varphi_{p,q} \tag{4}$$

where $\varphi_{p,q}$ is the only nonzero component of φ.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $D^{-1} \in \mathcal{L}^{(p,\infty)}$, its Chern character in cyclic cohomology is obtained from the following cyclic cocycle τ_n, $n \geqslant p$, n odd,

$$\tau_n(a^0, \ldots, a^n) = \lambda_n \operatorname{Tr}'(a^0[F, a^1] \ldots [F, a^n]), \quad \forall a^j \in \mathcal{A}, \tag{5}$$

where

$$F = \operatorname{Sign} D, \qquad \lambda_n = \sqrt{2i}(-1)^{n(n-1)/2}\Gamma\left(\frac{n}{2} + 1\right)$$

and

$$\operatorname{Tr}'(T) = \tfrac{1}{2}\operatorname{Trace}(F(FT + TF)). \tag{6}$$

In [1], we obtained the following general formula for the Hochschild cohomology class of τ_n in terms of the Dixmier trace:

$$\varphi_n(a^0, \ldots, a^n) = \lambda_n \operatorname{Tr}_\omega(a^0[D, a^1] \ldots [D, a^n]|D|^{-n}), \quad \forall a^j \in \mathcal{A}. \tag{7}$$

Our local formula for the *cyclic cohomology* Chern characer, i.e. for a cyclic cocycle cohomologous to (5), will be expressed in terms of the (b, B) bicomplex. Bearing this in mind, we see that if we want to regard the cochain φ_n of (7) as a cochain of the (b, B) bicomplex, we should use, instead of λ_n, the normalization constant

$$\mu_n = (-1)^{[n/2]}(n!)^{-1}\lambda_n = \sqrt{2i}\,\frac{\Gamma(\frac{n}{2} + 1)}{n!} \quad (\text{for } n \text{ odd}). \tag{8}$$

Let us now state the result. We let $\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete dimension spectrum and $D^{-1} \in \mathcal{L}^{(p,\infty)}$. We shall use the following notations:

$$\mathrm{d}a = [D, a], \quad \forall a \text{ operator in } \mathcal{H}. \tag{9}$$

$$\nabla(a) = [D^2, a]; \qquad a^{(k)} = \nabla^k(a), \quad \forall a \text{ operator in } \mathcal{H}. \tag{10}$$

THEOREM 2 [6]. (a) *The following formula defines a cocycle in the* (b, B) *bicomplex of* $\mathcal{A}$:

$$\begin{aligned} &\varphi_n(a^0, \ldots, a^n) \\ &\quad = \sqrt{2i} \sum_{q \geqslant 0, k_j \geqslant 0} c_{n,k,q} \tau_q(a^0 (\mathrm{d}a^1)^{(k_1)} \ldots (\mathrm{d}a^n)^{(k_n)} |D|^{-(n+2\Sigma k_j)}), \end{aligned}$$

where

$$\begin{aligned} c_{n,k,q} &= (-1)^{k_1+\cdots+k_n}(k_1! \ldots k_n!)^{-1} \Gamma^{(q)}\left(k_1 + \cdots + k_n + \frac{n}{2}\right) \times \\ &\quad \times \frac{1}{q!}((k_1+1)(k_1+k_2+2)\ldots(k_1+\cdots+k_n+n))^{-1}, \end{aligned}$$

with $\Gamma^{(q)}$ *the* qth *derivative of the* Γ *function.*

(b) *The cohomology class of the cocyle* (φ_n), n *odd, in* $\mathrm{HC}^{\mathrm{odd}}(\mathcal{A})$ *coincides with the cyclic cohomology Chern character* $\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)$.

Let us note that the term $\tau_q(T_{n,k})$ with coefficient $c_{n,k,q}$ in the above sum vanishes when

$$n + \sum k_j > p, \tag{11}$$

since the operator $T_{n,k}$ is in $\mathcal{L}_0^{(1,\infty)}$ when (11) holds. This implies that, for fixed n, the sum involved contains only finitely many terms. (We assume that Sd has finite multiplicity so that only finitely many q's are involved.) It also implies that

$$\varphi_n = 0, \quad \text{if } n > p. \tag{12}$$

Assertion (b) is the cyclic cohomology analogue of Theorem IV.2.8 of [1].

Note also that all the operators $T_{n,k}$ involved in the above formula are *homogeneous of degree* 0 in D, i.e. they are unaffected by the scaling

$$D \to \lambda D, \quad \lambda \in \mathbb{R}_+^*. \tag{13}$$

Finally, let us remark that assertion (a), i.e. the equality

$$b\varphi_n + B\varphi_{n+2} = 0, \quad \forall n, \tag{14}$$

is actually a consequence of our proof of (b). However, it is an instructive exercise to check it directly. This is done [6] (in the case of a simple spectrum) by making use of the following properties (with $\tau = \tau_0$):

$$D\,\mathrm{d}a + \mathrm{d}a\, D = \nabla(a), \forall a \text{ operator in } \mathcal{H} \tag{15}$$

$$\tau((\mathrm{d}a)^{(k)}|D|^{-q}) = 0, \quad \forall a \in \mathcal{A}, \forall k \geqslant 0, \forall q \tag{16}$$

$$\tau(\nabla(T)|D|^{-q}) = 0, \quad \forall T, \forall q \tag{17}$$

$$D_{(k)}b = \sum_{0}^{\infty} \frac{(-1)^{\ell}}{\ell!} b^{(\ell)} D_{(k+\ell)}, \quad \forall b \in B, \tag{18}$$

where, by definition, $D_{(k)} = \Gamma(k)|D|^{-2k}$.

The meaning of (15) is that, if we view the graded commutator with D as a graded derivation in the appropriate way, then $d^2 = \nabla$. The meaning of (16) and (17) is that integration by parts is possible, since both d and ∇ are derivations. Finally, (18) follows from Theorem 1 of Section 1.

4. Renormalization

There is one unpleasant feature of the formula of Theorem 2(a) for the cyclic cocycle φ, namely the occurrence of the transcendental numbers which enters in the Taylor expansion of the Γ function at the points $\Gamma(\frac{1}{2} + q), q \in \mathbb{N}$. Also the sum

$$\sum \frac{\Gamma(|k| + \frac{n}{2})^{(q)}}{q!} \operatorname*{Res}_{s=0}(s^q \zeta(s)) \tag{1}$$

is an infinite sum when ζ is not meromorphic at $s = 0$. We can, of course, rewrite it as

$$\operatorname*{Res}_{s=0} \Gamma\left(|k| + \frac{n}{2} + s\right) \zeta(s). \tag{2}$$

We shall, however, proceed to show how to obtain a modified cyclic cocycle φ', giving the same result Theorem 2(b) as φ, but involving a *finite* linear combination *with rational coefficients* of the terms

$$\sqrt{2i}\,\Gamma(\tfrac{1}{2})\tau_q(a^0(\mathrm{d}a^1)^{(k_1)} \ldots (\mathrm{d}a^n)^{(k_n)}|D|^{-2\Sigma k_j - n}). \tag{3}$$

To achieve this, we shall exploit the freedom of replacing the operator D by $\mu^{-1}D, \mu \in \mathbb{R}_+^*$ without affecting Theorem 2(b). The effect of this transformation on the functionals τ_q is

$$\tau_q^{\mu} = \sum \frac{(\log \mu)^m}{m!} \tau_{q+m}. \tag{4}$$

This implies that for any integer $m \geqslant 1$, the following formula defines the components of a cyclic cocycle which pairs trivially with cyclic homology

$$\begin{aligned} &\varphi_n^{(m)}(a^0, \ldots, a^n) \\ &\quad = \sum c_{n,k,q} \tau_{q+m}(a^0(\mathrm{d}a^1)^{(k_1)} \ldots (\mathrm{d}a^n)^{(k_n)}|D|^{-(n+2|k|)}). \end{aligned} \tag{5}$$

What we shall do now is add a suitable linear combination of counterterms $\varphi^{(m)}$ in order to cancel all the transcendental coefficients occurring in the Taylor expansion of $\Gamma(1/2)^{-1}\Gamma(s)$ at half integers. Even though we could right away write down the list of the coefficients β_m needed in front of $\varphi^{(m)}$, we shall rather explain carefully how they are obtained. To begin with, there is no problem at all if Sd is *simple*, i.e. if one has, at most, a simple pole. In that case, one simply writes

$$\Gamma(\tfrac{1}{2}+q)=\tfrac{1}{2}\tfrac{3}{2}\cdots(\tfrac{1}{2}+q-1)\Gamma(\tfrac{1}{2}) \tag{6}$$

and, since all τ_q's with $q \geqslant 1$ vanish, one gets the desired answer.

Let us see what happens when Sd has multiplicity two, i.e. when we have, at most, a double pole. In that case by Proposition 1 we know that τ_1 is a trace, while $\tau_q = 0$ for $q \geqslant 2$. This means that the formula for φ_n involves the combination

$$\Gamma\left(|k|+\frac{n}{2}\right)\tau_0(A)+\Gamma'\left(|k|+\frac{n}{2}\right)\tau_1(A), \tag{7}$$

where A is some operator. Now since the Hochschild coboundary $b\,\tau_0$ is given rationally in terms of τ_1 (cf. Proposition 1), we do not expect to need the transcendental coefficient

$$\frac{\Gamma'(\frac{1}{2}+m)}{\Gamma(\frac{1}{2}+m)} \tag{8}$$

in order to compensate for the lack of trace property of τ_0. If we replace the term $\Gamma'(|k|+(n/2))\,\tau_1(A)$ by $\Gamma(|k|+(n/2))\,\tau_1(A)$, then we get exactly the components of $\varphi_n^{(1)}$ which we can subtract from φ without affecting Theorem 2(b). Thus, we shall look for a coefficient λ such that

$$\Gamma'(\tfrac{1}{2}+m)=\lambda\Gamma(\tfrac{1}{2}+m)+c_m\Gamma(\tfrac{1}{2}+m), \quad m\in\mathbb{N}, \tag{9}$$

where the c_m are rational numbers.

To obtain (9), one just uses the equality

$$\frac{\Gamma'(\frac{1}{2}+m+\varepsilon)}{\Gamma(\frac{1}{2}+m+\varepsilon)}=\sum_{a=0}^{m-1}\frac{1}{\frac{1}{2}+a+\varepsilon}+\frac{\Gamma'(\frac{1}{2}+\varepsilon)}{\Gamma(\frac{1}{2}+\varepsilon)}, \tag{10}$$

which we write with ε for later use.

Thus, the constant λ is

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}=-(\gamma_E+2\log 2),$$

where γ_E is Euler's constant.

If we replace φ by $\varphi - \lambda\varphi^{(1)}$, then using (9), we find that in the formula giving φ the terms $\Gamma'(|k| + (n/2))\,\tau_1(A)$ should be replaced simply by

$$c_{|k|+\frac{n-1}{2}}\Gamma(|k| + \tfrac{n}{2})\,\tau_1(A),$$

where

$$c_i = \sum_{a=0}^{i-1} \frac{1}{\frac{1}{2} + a}. \tag{11}$$

Let us consider the next case, when Sd has multiplicity 3, i.e. when we have triple poles. This time we shall get the combination:

$$\Gamma\left(|k| + \frac{n}{2}\right)\tau_0(A) + \Gamma'\left(|k| + \frac{n}{2}\right)\tau_1(A) + \frac{\Gamma''}{2!}\left(|k| + \frac{n}{2}\right)\tau_2(A). \tag{12}$$

We want to use a further subtraction, say of $\lambda_2\varphi^{(2)}$, to get coefficients for $\tau_1(A)$ and $\tau_2(A)$ of the form $\Gamma(|k| + (n/2)) \times \mathbb{Q}$. From (10), we get the formula

$$\Gamma'(\tfrac{1}{2} + m + \varepsilon) = R_m(\varepsilon)\Gamma(\tfrac{1}{2} + m + \varepsilon) + f(\varepsilon)\Gamma(\tfrac{1}{2} + m + \varepsilon), \tag{13}$$

where R_m is the rational fraction

$$R_m(\varepsilon) = \sum_{a=0}^{m-1} \frac{1}{\frac{1}{2} + a + \varepsilon} \tag{14}$$

and where the function f is given by

$$f(\varepsilon) = \frac{\Gamma'(\frac{1}{2} + \varepsilon)}{\Gamma(\frac{1}{2} + \varepsilon)}. \tag{15}$$

If we differentiate (13), we get

$$\begin{aligned}\Gamma''&(\tfrac{1}{2} + m + \varepsilon)\\ &= R'_m(\varepsilon)\Gamma(\tfrac{1}{2} + m + \varepsilon) + R_m(\varepsilon)\Gamma'(\tfrac{1}{2} + m + \varepsilon)\\ &\quad + f'(\varepsilon)\Gamma(\tfrac{1}{2} + m + \varepsilon) + f(\varepsilon)\Gamma'(\tfrac{1}{2} + m + \varepsilon).\end{aligned} \tag{16}$$

We have to transform the term $R_m(\varepsilon)\Gamma'(\frac{1}{2} + m + \varepsilon)$ because, at the same time, it involves a function of m, $R_m(\varepsilon)$ and a derivative of Γ. To do this, we replace $\Gamma'(\frac{1}{2} + m + \varepsilon)$ by its value (13) which yields

$$R_m(\varepsilon)^2\Gamma(\tfrac{1}{2} + m + \varepsilon) + R_m(\varepsilon)f(\varepsilon)\Gamma(\tfrac{1}{2} + m + \varepsilon) \tag{17}$$

and we use again (13) to replace the second term of the formula by

$$f(\varepsilon)\Gamma'(\tfrac{1}{2}+m+\varepsilon)-f(\varepsilon)^2\Gamma(\tfrac{1}{2}+m+\varepsilon). \tag{18}$$

Coming back to all the terms of (16), we have thus proved

$$\begin{aligned}\Gamma''(\tfrac{1}{2}+m+\varepsilon)\\ &=(R'_m(\varepsilon)+R_m(\varepsilon)^2)\Gamma(\tfrac{1}{2}+m+\varepsilon)\\ &\quad+(f'(\varepsilon)-f(\varepsilon)^2)\Gamma(\tfrac{1}{2}+m+\varepsilon)+2f(\varepsilon)\Gamma'(\tfrac{1}{2}+m+\varepsilon).\end{aligned} \tag{19}$$

This shows that, if we replace φ by $\varphi-\lambda_1\varphi^{(1)}-\lambda_2\varphi^{(2)}$, where $\lambda_1=\lambda=f(0)$ as above, and

$$\lambda_2=\frac{1}{2!}f_2(0),\quad f_2(\varepsilon)=f'(\varepsilon)-f(\varepsilon)^2, \tag{20}$$

then the combination (12) gets replaced by

$$\begin{aligned}&\Gamma\left(|k|+\frac{n}{2}\right)\tau_0(A)+c_{|k|+\frac{n-1}{2}}\\ &\quad\cdot\Gamma\left(|k|+\frac{n}{2}\right)\tau_1(A)+c'_{|k|+\frac{n-1}{2}}\Gamma\left(|k|+\frac{n}{2}\right)\tau_2(A),\end{aligned} \tag{21}$$

where the rational number c'_m is $(1/2!)R_m^{(1)}(0)$, with

$$R_m^{(1)}(\varepsilon)=R'_m(\varepsilon)+R_m(\varepsilon)^2. \tag{22}$$

We can now proceed by induction to the general case. One proves by induction on ℓ, the following formula on the ℓth derivative $\Gamma^{(\ell)}$ of the Γ function:

$$\begin{aligned}&\Gamma^{(\ell)}(\tfrac{1}{2}+m+\varepsilon)\\ &\quad=R_m^{(\ell-1)}(\varepsilon)\Gamma(\tfrac{1}{2}+m+\varepsilon)+\sum_{j=1}^{\ell}C_\ell^j f_j(\varepsilon)\Gamma^{(\ell-j)}(\tfrac{1}{2}+m+\varepsilon),\end{aligned} \tag{23}$$

where $R_m^{(\ell)}$ and f_j are defined, inductively, by

$$R_m^{(\ell+1)}(\varepsilon)=R_m^{(\ell)}(\varepsilon)'+R_m(\varepsilon)R_m^{(\ell)}(\varepsilon), \tag{24}$$

$$f_{j+1}(\varepsilon)=f'_j(\varepsilon)-f(\varepsilon)f_j(\varepsilon). \tag{25}$$

The proof uses the same pattern as above and is straightforward. It shows that if we replace φ by

$$\varphi' = \varphi - \lambda_1\varphi^{(1)} - \lambda_2\varphi^{(2)} - \cdots - \lambda_\ell\varphi^{(\ell)} \cdots \tag{26}$$

where $\lambda_\ell = \frac{1}{\ell!} f_\ell(0)$, then the combination of terms

$$\sum \frac{1}{q!} \Gamma\left(|k| + \frac{n}{2}\right)^{(q)} \tau_q(A) \tag{27}$$

in the expression of the cocycle φ, gets replaced by

$$\Gamma\left(|k| + \frac{n}{2}\right) \sum \frac{1}{q!} R_m^{(q-1)}(0)\tau_q(A), \quad m = |k| + \frac{n-1}{2}. \tag{28}$$

Now the functions $E_m^{(\ell)}(\varepsilon)$ are easy to compute, since

$$R_m(\varepsilon) = \sum_{a=0}^{m-1} T_a(\varepsilon) \quad \text{with } T_a'(\varepsilon) = -T_a(\varepsilon)^2, \quad T_a(\varepsilon) = \frac{1}{\frac{1}{2} + a + \varepsilon}.$$

One gets that $R_m^{(\ell)}(\varepsilon)$ is the $(\ell + 1)$th symmetric function of the m terms $1/(\frac{1}{2} + a + \varepsilon)$:

$$\prod_{a=0}^{m-1} \left(1 + \frac{z}{\frac{1}{2} + a + \varepsilon}\right) = \sum R_m^\ell(\varepsilon) z^{\ell+1}. \tag{29}$$

We can then easily compute the product $\Gamma(\frac{1}{2} + m)\, R_m^{(q-1)}(0)$ which appears in (28) and get

$$\Gamma(\tfrac{1}{2} + m) R_m^{(q-1)}(0) = \Gamma(\tfrac{1}{2})\sigma_{m-q}(m), \tag{30}$$

where $\sigma_j(m)$ is the jth symmetric function of the first m odd 1/2 integers:

$$\prod_{\ell=0}^{m-1} \left(z + \frac{(2\ell + 1)}{2}\right) = \sum z^j \sigma_{(m-j)}(m). \tag{31}$$

What is remarkable now is that these coefficients vanish if q is larger than m so that not only have we transformed them to elements of $\Gamma(\frac{1}{2})\mathbb{Q}$, but we have also eliminated all but a finite number.

We note that the function $f_j(\varepsilon)$ is not difficult to compute, and by induction we get the formula

$$f_j(\varepsilon) = -\Gamma(\tfrac{1}{2} + \varepsilon)\left(\frac{1}{\Gamma(\frac{1}{2} + \varepsilon)}\right)^{(j)}. \tag{32}$$

as can be seen by interpreting the transformation

$$T(h) = h' - fh \tag{33}$$

as $T(h) = \Gamma(h/\Gamma)'$, and using (25).

It follows then that

$$\lambda_\ell = -\Gamma(\tfrac{1}{2})\frac{1}{\ell!}\left(\frac{1}{\Gamma(\frac{1}{2}+\varepsilon)}\right)^{(\ell)_{\varepsilon=0}} \tag{34}$$

and that the above operation of subtraction has a very simple interpretation, namely the following. In the proof of Theorem 2(a), we are applying the linear form $\mathrm{Res}_{s=0}$ to meromorphic functions of the form

$$\Gamma\left(|k| + \frac{n}{2} + s\right)\mathrm{Trace}(A|D|^{-2s}) = \zeta(s). \tag{35}$$

where A is an operator. But any other linear form such as

$$\zeta \to \mathrm{Res}_{s=0}\, g(s)\zeta(s), \tag{36}$$

with g holomorphic at 0, would have worked equally well. What the subtraction of $\sum\lambda_\ell\varphi^{(\ell)}$ is doing is exactly to take

$$g(s) = \frac{\Gamma(1/2)}{\Gamma(1/2+s)}.$$

If we combine this with

$$\frac{\Gamma(\frac{1}{2}+m+s)}{\Gamma(\frac{1}{2}+s)} = \prod_0^{m-1}(\tfrac{1}{2}+a+s), \tag{37}$$

we can summarize the above discussion by the following variant of Theorem 2 [6].

THEOREM 3. *The statements of Theorem 2 are true for the cocycle φ'_n given, for n-odd, $n \leqslant p$, by the formula*

$$\varphi'_n(a^0, \ldots, a^n)$$
$$= \sqrt{2\pi i}\sum_{k,q}\frac{(-1)^{|k|}}{k_1!\ldots k_n!}\alpha_k\frac{1}{q!}\sigma_{m-q}(m)\tau_q$$
$$\times(a^0(\mathrm{d}a^1)^{(k_1)}\ldots(\mathrm{d}a^n)^{(k_n)}|D|^{-(2|k|+n)}),$$

with

$$m = |k| + \frac{n-1}{2}, \qquad \alpha_k^{-1} = (k_1+1)(k_1+k_2+2)\dots(k_1+\dots+k_n+n)$$

and σ defined in (31).

Not only the coefficients are all rational multiples of the overall factor $\sqrt{2\pi i}$, but also the total number of terms in the formula is now finite and bounded in terms of p alone and not the order of the poles. Indeed,

$$q \leqslant |k| + \frac{n-1}{2} \quad \text{and} \quad |k| + n \leqslant p,$$

so that the formula does not involve more than p terms in the Laurent expansion.

Let us see what this formula looks like for small values of p.

$p = 1$. Then only φ_1' is nonzero; we have $k = 0$ and $q = 0$, also

$$\varphi_1'(a^0, a^1) = \sqrt{2\pi i}\,\tau_0(a^0\,\mathrm{d}a^1|D|^{-1}). \tag{38}$$

This shows that, even if we had poles of arbitrary order for the function $\zeta(s) = \mathrm{Trace}(a^0\,\mathrm{d}a^1|D|^{-1-2s})$ at $s = 0$, they do not contribute to φ_1' except for the residue of ζ at $s = 0$.

If we had used the formula of Theorem 2, we would be taking the residue of $\Gamma(\frac{1}{2}+s)\zeta(s)$ at $s = 0$ which involves infinitely many of the functionals τ_k. Note also that here τ_0 is a trace.

$p = 2$. Again, only φ_1' is nonzero, but now we can have $k_1 = 1$ and also $q = 1$ if $k_1 = 1$. Thus, we get three terms

$$\begin{aligned}\varphi_1'&(a_0, a_1)\\ &= \sqrt{2\pi i}(\tau_0(a^0\,\mathrm{d}a^1|D|^{-1}) - \tfrac{1}{2}\tau_0(a^0(\mathrm{d}a^1)^{(1)}|D|^{-3}) -\\ &\qquad - \tfrac{1}{2}\tau_1(a^0(\mathrm{d}a^1)^{(1)}|D|^{-3})).\end{aligned} \tag{39}$$

This time τ_0 is no longer a trace, as one can see using Proposition 1, and the formula involves τ_1, i.e. the coefficient of s^{-2} in some ζ-function. However, no higher-order coefficient is involved, unlike the formula for φ in Theorem 2(a).

$p = 3$. Let us look at φ_3' in this case. Here, we must have $k = 0$ but since $q \leqslant |k| + \frac{1}{2}(n-1)$, we can have $q = 1$. Thus, we get two terms for φ_3':

$$\begin{aligned}\varphi_3'&(a_0, a_1, a_2, a_3)\\ &= \sqrt{2\pi i}(\tau_0(a^0\,\mathrm{d}a^1\,\mathrm{d}a^2\,\mathrm{d}a^3|D|^{-3}) + \tau_1(a^0\,\mathrm{d}a^1\,\mathrm{d}a^2\,\mathrm{d}a^3|D|^{-3})).\end{aligned} \tag{40}$$

This shows that, even for φ_3', the coefficient of s^{-2} in the expansion of the ζ-function is playing a role, i.e. that τ_1 enters into play.

5. Local Index Formula

To get more insight into the content of Theorems 2 and 3, we shall now write down a corollary whose statement does not involve cyclic cohomology or noncommutative geometry but computes a Fredholm index (called *spectral flow*) as a sum of residues of ζ-functions attached to the problem.

To formulate the problem, we just need a pair (D, U) of operators in Hilbert space, where D is selfadjoint with discrete spectrum, while U is unitary. The main assumption that we need is that $[D, U]$ is bounded, which implies immediately that the compression PUP of U of the *positive* part of D, ($P = \frac{1}{2}(1 + F)$, $F = \text{Sign}\, D$) is a Fredholm operator. The index

$$\text{Index}\, PUP = \dim \text{Ker}\, PUP - \dim \text{Ker}\, PU^*P \tag{1}$$

can be interpreted as spectral flow, i.e. as the number of eigenvalues which cross the origin in the natural homotopy between D and $UDU^* = D + U[D, U^*]$. In any case, it is an integer, and we shall compute it as a sum of residues.

We make the following hypotheses:

(2a) If S is the spectrum of D (with multiplicity), then

$$\sum_{\lambda \in S} |\lambda|^{-s} < \infty \quad \text{for some finite } s.$$

(We call p the lower bound of such s.)

(2b) The operators U and $[D, U]$ are in the domain of δ^k, $\delta = [|D|, \cdot]$ for $1 \leqslant k \leqslant N$, $N \gg 0$.

(2c) The following functions, holomorphic for $\text{Re}\, s \gg 0$, are meromorphic, with finitely many poles for $\text{Re}\, s > -\varepsilon$,

$$\zeta_{(k,n)}(s) = \text{Trace}(U^{-1}[D, U]^{(k_1)}[D, U^{-1}]^{(k_2)} \ldots [D, U]^{(k_n)}|D|^{-2|k|-n-s}),$$

where we use the notation $X^{(k)} = \nabla^k(X)$, $\nabla(X) = [D^2, X]$.

In (2c) only *finitely many* functions are involved because of the inequality $|k| + n \leqslant p$. At the technical level, we need to assume that $\Gamma(s)\zeta(s)$ restricted to vertical lines is of rapid decay.

COROLLARY 2. *Let D and U be as above. Then*

$$\text{Index } PUP$$
$$= \sum_{n \leqslant p} (-1)^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! \times$$
$$\times \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \dots k_n!} \alpha_k \frac{1}{q!} \sigma_{m-q}(m) \operatorname{Res}_{s=0} s^q \zeta_{(k,n)}(s),$$

with $m = |k| + \frac{1}{2}(n-1)$.

Proof. We just apply Theorem 3 to the special case when $\mathcal{A} = C^\infty(S^1)$, acting on $\mathcal{H}$ by the unique representation which sends the function $f(e^{i\theta})$ to $f(U)$. We use the formula for the pairing between K^1-theory and odd cyclic cohomology, together with the index formula (cf. [1]),

$$\langle \varphi', U \rangle = \frac{1}{\sqrt{2\pi i}} \sum_{n \text{ odd}} (-1)^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! \varphi'_n(U^{-1}, U, \dots, U^{-1}, U). \tag{3}$$

The proof of Theorem 2(b) shows that the hypothesis (2) is sufficient to conclude. □

At this point, we should stress the considerable freedom that one has in applying Corollary 2. The data is a discrete subset (perhaps with multiplicity) of $\mathbb{R}$,

$$S = \text{Spectrum } D, \tag{4}$$

together with a unitary matrix, $u(\lambda, \lambda')_{\lambda, \lambda' \in S}$ which signifies a 'unitary correspondence' on the list S. The main hypothesis is that when D is shifted by this correspondence (i.e. UDU^* is considered), it stays at bounded distance from D. Then one writes down a finite number of ζ-functions, the $\zeta_{(k,n)}$ above, which can be expressed as Dirichlet series of the form

$$\sum a_n^\pm |\lambda_n^\pm|^{-s} \tag{5}$$

when one computes the trace in the basis of eigenvectors

$$D e_n^\pm = \lambda_n^\pm e_n^\pm \tag{6}$$

for the operator D.

The statement is that a certain rational combination of residues of these functions gives the index of PUP or spectral flow. In particular, one has the following corollary.

COROLLARY 3. *If* Index $PUP \neq 0$, *at least one of the functions* $\zeta_{(k,n)}(s)$ *has a nontrivial pole at* $s = 0$.

6. Geodesic Flow and Covariant Differentiation

In this section, we shall explain the geometric content of the ingredients of the local formula of Theorem 2. This will yield the analogue within our framework of the unit sphere of the cotangent bundle, of the geodesic flow and of covariant differentiation.

First of all, the computations involved in Theorem 2 all take place within the algebra $\Psi^*(\mathcal{A})$ of pseudodifferential operators (cf. Section 2 (10)). Let us consider first only 'scalar' pseudodifferential operators, i.e. those which have an expansion of the form (Section 2 (10))

$$P \simeq b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots, \quad b_q \in \mathcal{B} \tag{1}$$

where $\mathcal{B}$ is the algebra generated by the $\delta^n(a)$, $a \in \mathcal{A}$, $n \in \mathbb{N}$. We say that P is of order α when $P \in \mathrm{OP}^\alpha$ as defined in Section 1, i.e.

$$P \in \mathrm{OP}^\alpha \quad \text{iff } P|D|^{-\alpha} \in \cap \operatorname{Dom} \delta^n. \tag{2}$$

Except for the nuance between scalar and nonscalar pseudodifferential operators, i.e. allowing coefficients like $[D, a]$, $a \in \mathcal{A}$, all the computations of Theorem 2 are done within the following algebra $\mathcal{C}$ with derivation δ

$$\mathcal{C} = \mathrm{OP}^0 \cap \Psi^*(\mathcal{A}), \qquad \delta(\cdot) = [|D|, \cdot]. \tag{3}$$

Moreover, with $|D|^{-1} \in \mathcal{L}^{(p,\infty)}$ for some finite p, the functionals τ_k (Proposition 1) all vanish on the two-sided ideal $\mathcal{C}_\alpha = \mathrm{OP}^{-\alpha} \cap \Psi^*(\mathcal{A})$ of $\mathcal{C}$ for $\alpha > p$. This ideal is invariant under δ and, thus, the relevant algebra for our computations is the quotient of $\mathcal{C}$ by $\mathcal{C}_\alpha$. The derivation δ continues to make sense on this quotient as well as the functionals τ_k.

Any element of OP^β, $\beta < 0$, in $\mathcal{C}/\mathcal{C}_\alpha$ is now nilpotent. To capture the 'semisimple' part of this algebra we just pass to the associated C^*-algebra:

DEFINITION 2. We let $S^*\mathcal{A}$ be the C^*-algebra $S^*\mathcal{A} = \bar{\mathcal{C}}/\mathcal{K}$.

Here we let $\bar{\mathcal{C}}$ be the norm closure of $\mathcal{C}$ acting in the Hilbert space $\mathcal{H}$ and we divide it by the compact operators, i.e. we take its image by the quotient map

$$\mathcal{L}(\mathcal{H}) \xrightarrow{\pi} \mathcal{L}(\mathcal{H})/\mathcal{K} \tag{4}$$

of $\mathcal{L}(\mathcal{H})$ to the Calkin algebra.

We endow $S^*\mathcal{A}$ with the one-parameter group of automorphisms given by

$$\alpha_t(T) = e^{it|D|}Te^{-it|D|}. \tag{5}$$

One has to be careful about domain problems with the unbounded derivation δ which is the infinitesimal generator of the group α_t and, in the nonanalytic case, one may have to saturate $S^*\mathcal{A}$ by (5).

In any case, we thus obtain a C^* dynamical system

$$S^*\mathcal{A}, \ (\alpha_t)_{t\in\mathbb{R}}. \tag{6}$$

It is easy to check in the Riemannian case with the Dirac spectral triple, that

PROPOSITION 2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be the Dirac spectral triple associated to a compact Riemannian manifold M. Then the C^*-algebra $S^*\mathcal{A}$ is canonically isomorphic to the algebra $C(S^*M)$ of continuous functions on the unit cosphere bundle of M, $S^*M = \{(x, \xi) \in T^*M; ||\xi|| = 1\}$. Moreover, the one-parameter group α_t is given by the geodesic flow.*

Proof. The proof follows from Egorov's theorem on pseudodifferential operators of order 0. First, our algebra $\Psi^*(\mathcal{A})$ is contained in the algebra of usual pseudodifferential operators. The principal symbol map thus yields a homomorphism

$$C \xrightarrow{\sigma} C(S^*M) \tag{7}$$

which extends to the norm closure C and vanishes on compact operators. Thus, one obtains a homomorphism

$$S^*\mathcal{A} \xrightarrow{\sigma} C(S^*M). \tag{8}$$

One uses Egorov's theorem to show that this homomorphism σ is equivariant with respect to the actions of $\mathbb{R}$ given by α_t (5) on $S^*\mathcal{A}$ and by the usual geodesic flow on S^*M. The latter flow appears as the restriction of a Hamiltonian flow to the space of functions on T^*M/M, (the complement of the 0-section in the cotangent bundle of M), which are homogeneous of degree 0:

$$f(\lambda\xi) = f(\xi), \quad \forall\lambda \in \mathbb{R}_+^*. \tag{9}$$

The Hamiltonian flow is generated by the function

$$H_0(\xi) = ||\xi||, \quad \xi \in T^*M/M \tag{10}$$

which comes from the symbol of $|D|$. Thus, for any function f on T^*M/M homogeneous of degree 0, (9), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}G_t(f) = \{H_0, f\}, \tag{11}$$

where G_t is the geodesic flow.

The final step of the proof of Proposition 2 is to show that the homomorphism σ of (8) is surjective. This follows from the Stone–Weierstrass theorem by observing that the subalgebra of $C(S^*M)$ generated by the functions

$$f(p(G_t(x,\xi))), \quad f \in C^\infty(M), t \in \mathbb{R} \tag{12}$$

(where G is the geodesic flow and $p: S^*M \to M$ is the canonical projection) does separate the points of S^*M. □

The spectral triple associated ([6]) to hypo-elliptic operators provides a more sophisticated example where the analogue of Proposition 2 holds (Example 6 of the introduction).

Another very interesting example is provided by nilpotent discrete groups (Example 5 of the introduction) where our cosphere bundle (Definition 2) is the Gromov compactification (cf. Remark 5(b) below).

Let us now describe within our general framework the action of the geodesic flow on the complement of the 0-section in T^*M.

First, the analogue of the algebra of continuous functions vanishing at ∞ on T^*M/M is the suspension of $S^*\mathcal{A}$

$$S(S^*\mathcal{A}) = S^*\mathcal{A} \otimes C_0(\mathbb{R}_+^*). \tag{13}$$

On the right-hand side, one could use any of the pairwise isomorphic C^*-algebras $C_0(I)$ where I is an open interval but the choice of $\mathbb{R}_+^*$ corresponds to the description:

$$T^*M/M = S^*M \times \mathbb{R}_+^*. \tag{14}$$

The C^*-algebra (13) admits a very natural representation by asymptotic operators in $\mathcal{H}$. An asymptotic operator is a norm continuous map $\varepsilon \to k(\varepsilon)$ from $]0,1]$ to the algebra $\mathcal{K}$ of compact operators on $\mathcal{H}$ such that:

$$\lim_{\varepsilon \to 0} \operatorname{Sup} \|k(\varepsilon)\| < \infty. \tag{15}$$

We endow the algebra of asymptotic operators with the C^* norm given by (15). (More pedantically, we are taking its quotient by those $k(\varepsilon)$ with $\|k(\varepsilon)\| \to 0$ $\varepsilon \to 0$.) The following representation of $S(S^*\mathcal{A})$ as asymptotic operators in $\mathcal{H}$ is canonically associated by [1] to the exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to \Psi^0(\mathcal{A}) \to S^*(\mathcal{A}) \to 0, \tag{16}$$

where $\Psi^0(\mathcal{A})$ is $\pi^{-1}(S^*(\mathcal{A}))$ using (4).

PROPOSITION 3. *There exists a unique representation ρ of $S(S^*\mathcal{A})$ as asymptotic operators in $\mathcal{H}$ such that*

$$(\rho(P\otimes f))_\varepsilon = Pf(\varepsilon|D|), \quad \forall P\in \mathcal{C},\ f\in C_0(\mathbb{R}_+^*).$$

Proof. This follows from [1], since the homeomorphism

$$u\colon \mathbb{R}_+^* \to]0,1[, \quad u(x) = \frac{1}{1+x}, \tag{17}$$

transforms $\varepsilon|D|$ into a quasicentral approximate unit $u_\varepsilon = u(\varepsilon|D|)$ for $\Psi^0(\mathcal{A})$. □

In particular, the class of the asymptotic operator $Pf(\varepsilon|D|)$ only depends upon the class of P modulo compact operators. Thus, Proposition 3 gives us a good representation of the analogue of the algebra of functions on T^*M/M. It remains to identify the action of the geodesic flow on this algebra. The usual geodesic flow on T^*M/M is not defined by the Hamiltonian H_0, whose only merit was to preserve homogeneity, but by the Hamiltonian H,

$$H(x,\xi) = \tfrac{1}{2}\|\xi\|^2, \quad \forall (x,\xi)\in T^*M. \tag{18}$$

The effect of this normalization is that on the invariant submanifold $S^*M\times\{\lambda\} = \{(x,\xi)\in T^*M;\ \|\xi\| = \lambda\}$, one replaces G_t by $G_{\lambda t}$.

Thus, in our context, the analogue of the geodesic flow on T^*M/M is given by

$$(\beta_t(b))(\lambda) = \alpha_{\lambda t}(b(\lambda)), \quad \forall b\in S(S^*\mathcal{A}) = C_0(\mathbb{R}_+^*, S^*\mathcal{A}). \tag{19}$$

The one-parameter group of automorphisms $(\beta_t)_{t\in\mathbb{R}}$ of $S(S^*\mathcal{A})$ is very simply implemented in the representation ρ,

PROPOSITION 4. *For any $b\in S(S^*\mathcal{A})$ one has*

$$\rho(\beta_t(b)) = (\mathrm{e}^{it\varepsilon\frac{D^2}{2}}\rho(b)_\varepsilon\,\mathrm{e}^{-it\varepsilon\frac{D^2}{2}})_{\varepsilon\in]0,1]}.$$

In other words, the action of the 'geodesic flow' β_t on asymptotic operators is obtained by conjugation

$$k(\varepsilon)\to \mathrm{e}^{it\varepsilon\frac{D^2}{2}}k(\varepsilon)\,\mathrm{e}^{-it\varepsilon\frac{D^2}{2}}. \tag{20}$$

The proof of Proposition 4 is done by a direct calculation using the invariance of the algebra of asymptotic operators by the flow (20).

In the classical Riemannian case, the knowledge of the geodesic flow in its infinitesimal form is that of the equation of geodesics:

$$\frac{d^2x^j}{dt^2} = \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt} \tag{21}$$

and this is equivalent to the knowledge of the Levi-Civita connection Γ^j_{kl}.

In our case, the infinitesimal generator of the geodesic flow is given by (20), i.e. by the operation

$$\nabla k(\varepsilon) = \tfrac{1}{2}\varepsilon[D^2, k(\varepsilon)]. \tag{22}$$

We shall thus end this section by pointing out the general properties of the derivation

$$\nabla(k) = \tfrac{1}{2}[D^2, k] \tag{23}$$

used repeatedly in the proof of Theorem 2, which allows us to get a relevant general analogue of the Levi-Civita connection. The latter, when acting on spinors, is characterized by the differential operators ∇_X of covariant differentiation with respect to arbitrary vector fields X. These operators ∇_X are densely defined

$$\nabla_X : L^2(M, S) \to L^2(M, S) \tag{24}$$

and depend on X in a $C^\infty(M)$ linear way

$$\nabla_{aX} = a\nabla_X, \quad \forall a \in C^\infty(M). \tag{25}$$

Within our framework, we shall define covariant differentiation operators $\nabla_\omega : \mathcal{H} \to \mathcal{H}$ for any $\omega \in \Omega^1(\mathcal{A})$ in the following way. We recall that $\Omega^*(\mathcal{A})$ denotes the universal differential algebra of $\mathcal{A}$ and $\pi : \Omega^*(\mathcal{A}) \to \mathcal{L}(\mathcal{H})$ its canonical representation as operators in $\mathcal{H}$ (cf. [1]).

$$\pi(a^0 da^1 \ldots da^n) = a^0[D, a^1]\ldots[D, a^n], \quad \forall a^j \in \mathcal{A}. \tag{26}$$

For any $\omega \in \Omega^1(\mathcal{A})$ we let

$$\nabla_\omega = \tfrac{1}{2}(D\pi(\omega) + \pi(\omega)D - \pi(d\omega)). \tag{27}$$

This is intimately related to (23) since, for any $a \in \mathcal{A}$, one has

$$\nabla_{da} = \tfrac{1}{2}[D^2, a]. \tag{28}$$

One easily checks the following general rule

$$\nabla_{a\omega b} = a\nabla_\omega b, \quad \forall a, b \in \mathcal{A}, \ \omega \in \Omega^1(\mathcal{A}). \tag{29}$$

In the case of the Dirac spectral triple of a Riemannian spin manifold, a simple calculation gives for any $\omega = \sum f_i \mathrm{d}g_i \in \Omega^1(\mathcal{A})$, with associated vector field $X = \sum f_i \,\mathrm{Grad}(g_i)$ and scalar field $\rho = \sum \langle \mathrm{d}f_i, \mathrm{d}g_i \rangle$, the general equality

$$\nabla_\omega = \nabla_X + \tfrac{1}{2}\mathrm{div}(X) - \rho. \tag{30}$$

In this equality, the left-hand side is given by definition (27) and depends only upon the classical form $\pi(\omega)$ associated to ω and the 'auxiliary field' ρ [1] involved in $\pi(\mathrm{d}\omega)$. The right-hand side ∇_X is the covariant differentiation of spinors with respect to the vector field X as in (24), while the two other terms are scalar multiplication operators by the functions $\mathrm{div}(X)$ and ρ on M. The following normalization allows us to define ∇_A as a function of $A = \pi(\omega)$ alone and to eliminate the term $\frac{1}{2}\mathrm{div}(X) - \rho$ in formula (30).

$$\tau_0(\alpha\nabla_A|D|^{-1}) = 0, \quad \forall \alpha = \pi(\mathrm{d}\beta), \pi(\beta) = 0.$$

Remarks 5. (a) Proposition 3 gives a general construction of the tangent groupoid of manifolds (cf. [1]) for arbitrary spectral triples.

(b) In the case of discrete groups, with $\mathcal{A} = \mathbb{C}\Gamma$ and $(\mathcal{H}, |D|)$ given by the regular representation and word length function, the fundamental scaling $D \to \varepsilon D$ used throughout this section is the same as the method of M. Gromov [8] of looking at the group from a very far distance.

(c) In a number of examples such as the Dirac spectral triple of a Riemannian manifold or the spectral triple of the standard model [1], the following additional condition (pointed out to me by H. Moscovici) is fulfilled:

$$\gamma \in \pi(\Omega^n(\mathcal{A})) \quad (\text{for some even integer } n). \tag{31}$$

(In the odd case the analogue is

$$1 \in \pi(\Omega^n(\mathcal{A})) \quad (\text{for some odd } n).) \tag{31'}$$

This condition is quite interesting because it allows us to prove that the $\pi(\Omega^*(\mathcal{A}))$ bimodule $\mathcal{D}^1$ of differential operators of order 1 (cf. Section 1), i.e. the $\pi(\Omega^*(\mathcal{A}))$ bimodule generated by the operators

$$\tfrac{1}{2}[D^2, a] = \nabla_{\mathrm{d}a}, \quad a \in \mathcal{A} \tag{32}$$

is in fact *finitely generated*. It is indeed generated by the finitely many operators

$$\tfrac{1}{2}[D^2, a_k^j], \tag{33}$$

where the a_k^j enter the formula for γ,

$$\gamma = \sum_k a_k^0[D, a_k^1] \ldots [D, a_k^n]. \tag{34}$$

The proof is straightforward using the equality

$$D = -\gamma(\nabla_\gamma + \tfrac{1}{2}\pi(\mathrm{d}\gamma)), \tag{35}$$

where the notation ∇_ω of (27) has been extended to arbitrary elements ω of $\Omega^*(\mathcal{A})$ following the rule

$$\nabla_{\omega_1\omega_2} = (\nabla_{\omega_1})\omega_2 + (-1)^{\partial\omega_1}\omega_1\nabla_{\omega_2}. \tag{36}$$

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