# Quantized Calculus and Applications

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Our aim in this paper is to give a general introduction to noncommutative geometry and describe in some detail an example of the quantized calculus.

Many of the tools of the differential calculus acquire their full power when formulated at the level of variational calculus where the original space $X$ one is dealing with, is replaced by a functional space $\mathcal{F}(X)$ of functions or fields on $X$. The original space $X$ is involved only indirectly in $\mathcal{F}(X)$ to write down for instance the right hand side $F(\varphi)$ of a non linear evolution equation, $\frac{d\varphi}{dt} = F(\varphi)$, $\varphi \in \mathcal{F}(X)$. In $F(\varphi)$ the partial differentiation and the pointwise product of functions $\varphi$ on $X$ are being used.

The essence of noncommutative geometry is the existence of many examples of situations in which $\mathcal{F}(X)$ makes perfectly good sense while $X$ is no longer a usual space described in the set theoretic sense of points $p \in X$ and coordinates. The basic structure on the space $\mathcal{F}(X)$ of (real or complex valued) functions on a set $X$ is the pointwise product of functions. Given two functions $f_1, f_2$ we can form a new function $f_1 f_2$ given by:

$$(f_1 f_2)(p) = f_1(p)\ f_2(p) \qquad \forall p \in X\ . \tag{1}$$

In noncommutative geometry we still have a product on $\mathcal{F}(X)$ but we drop the commutativity property of (1):

$$f_1 f_2 = f_2 f_1 \qquad \forall f_j \in \mathcal{F}(X)\ . \tag{2}$$

It is this commutativity property which signals that $X$ is an ordinary set. When we drop it we are no longer dealing with a set $X$ but essentially with a set endowed with relations between different points. For instance if we consider a set $Y$ consisting of two points $1, 2$ and the relation which identifies 1 and 2 then $\mathcal{F}(Y, \mathrm{rel})$ is the space $M_2(\mathbf{C})$ of $2 \times 2$ complex matrices with the product

$$(f_1 f_2)(i, j) = \sum f_1(i, k)\ f_2(k, j) \tag{3}$$

which is the usual product of matrices.

In this simple example the space $\{1, 2\}$ given by the two points without any relation is described by the subalgebra of diagonal matrices and it is the "off diagonal" matrices such as $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ which describe the relation. This construction of the algebra $\mathcal{F}(X)$ extends easily to a pseudogroup of transformations of a manifold and to the holonomy pseudogroup of a foliation ([Co]). It encodes by a non commutative algebra the structure given by the space and relations, and applies in particular to the case of a smooth manifold with its full diffeomorphism group [C-M].

As another simple example we can consider the case of a single point divided by a discrete group $\Gamma$, then the corresponding algebra $\mathcal{F}$ is the group ring, whose elements $f$ are functions (with finite support) on $\Gamma$,

$$g \to f_g \in \mathbf{C} \tag{4}$$

with the product given by linearization of the group law $g_1, g_2 \to g_1 g_2$ in $\Gamma$:

$$(f_1 f_2)_g = \sum_{g_1 g_2 = g} f_{1,g_1}\ f_{2,g_2}\ . \tag{5}$$

When we described the functional space $\mathcal{F}(X)$ associated to an ordinary space $X$ we have been careless in fixing the degree of regularity of the functions $f \in \mathcal{F}(X)$ as functions of $p \in X$. To various degrees of regularity correspond various branches of the general theory of noncommutative associative algebras. The latter are assumed to be algebras over $\mathbf{C}$ and to be involutive, i.e. endowed with an antilinear involution

$$f \to f^* \quad , \quad (f_1 f_2)^* = f_2^*\ f_1^*\ . \tag{6}$$

The two degrees of regularity for which the corresponding algebraic theory is satisfactory are:

*Measurability.* It corresponds to the theory of von Neumann algebras.

*Continuity.* It corresponds to the theory of $C^*$ algebras.

In both theories of von Neumann and $C^*$ algebras, the Hilbert space plays a key role. Both types of algebras are faithfully represented as algebras of operators in Hilbert space with suitable closure hypothesis (cf. [Co]). One can trace this role of Hilbert space to the simple fact that *positive* complex numbers are those of the form

$$\lambda = z^* z\ . \tag{7}$$

In any of the above algebras functional analysis provides the existence (by Hahn Banach arguments) of sufficiently many linear functionals $L$ which are positive

$$L(f^* f) \geq 0 \tag{8}$$

and a Hilbert space is easily constructed from such an $L$, together with a representation, by left multiplication, of the original algebra.

Next, many of the tools of *differential topology* such as the de Rham theory of differential forms and currents, the Chern character etc..., are well captured by cyclic cohomology ([Co]) applied to *pre $C^*$ algebras,* i.e. to dense subalgebras of $C^*$ algebras which are stable under the holomorphic functional calculus:

$$f \to h(f) = \frac{1}{2i\pi} \int \frac{h(z)}{f - z}\, dz \tag{9}$$

where $h$ is holomorphic in a neighbourhood of Spec$(f)$. The prototype of such an algebra is the algebra $C^\infty(X)$ of smooth functions on a manifold $X$. The cyclic cohomology construction then gives back the usual differential forms, the de Rham complex of currents etc.... But it also applies to the highly non commutative examples of group rings, in which case the group cocycles give rise to cyclic cocycles, with direct application to the Novikov conjecture for the homotopy invariance of the higher signatures of non simply connected manifolds with given fundamental group (cf. [Co]).

If one wants to go beyond differential topology and reach the geometric structure itself, including the metric and the real analytic aspects, it turns out that the most fruitful point of view is that of *spectral geometry.*

What we mean by this is that while our measure theoretic understanding of the space $X$ was encoded by a (von Neumann) algebra of operators $\mathcal{A}$ acting in the Hilbert space $\mathcal{H}$ the *geometric* understanding of the space $X$ will be encoded, not by a suitable subalgebra of $\mathcal{A}$, but by an operator in Hilbert space:

$$D = D^* \quad \text{selfadjoint unbounded operator in } \mathcal{H}\ . \tag{10}$$

In the compact case (i.e. $X$ compact) the operator $D$ will have discrete spectrum, with (real) eigenvalues $\lambda_n$, $|\lambda_n| \to \infty$, when $n \to \infty$.

The first example of such a triple is provided by the Dirac operator on a compact Riemannian (Spin) manifold. In that case $\mathcal{H}$ is the Hilbert space of $L^2$ spinors on the manifold $M$, $\mathcal{A}$ is the algebra of (smooth) functions acting in $\mathcal{H}$ by multiplication operators and $D$ is the (selfadjoint) Dirac operator.

One can easily check in this case that no information has been lost in trading the geometric space $M$ for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ indeed (cf. [Co]) one recovers:

1) The space $M$ as the spectrum Spec$(\overline{\mathcal{A}})$, of the norm closure of the algebra $\mathcal{A}$ of operators in $\mathcal{H}$.

2) The geodesic distance $d$ on $M$ from the formula:

$$d(p,q) = \mathrm{Sup}\,\{|f(p) - f(q)|\ ;\ \|[D,f]\| \le 1\} \quad , \quad \forall p,q \in M\ .$$

The right hand side of 2) continues to make sense in general and the simplest non Riemannian example where it applies is the 0-dimensional situation in which the geometric space is finite. In that case both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ are finite dimensional, so that $D$ is a selfadjoint matrix. For instance for a two point space one lets $\mathcal{A} = \mathbf{C} \oplus \mathbf{C}$ acting in the 2-dimensional Hilbert space $\mathcal{H}$ by

$$f \in \mathcal{A} \to \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}$$

while $D = \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}$. Formula 2 gives $d(a,b) = 1/\mu$.

As a slightly more involved 0-dimensional example, one considers the algebraic structure provided by the elementary Fermions, i.e. the three families of quarks (and leptons). Thus one lets $\mathcal{H}$ be the finite dimensional Hilbert space with orthonormal basis labeled by the left handed and right handed elementary quarks such as $u_L^r$, $u_R^b \ldots$. The algebra $\mathcal{A}$ is $\mathbf{C} \oplus \mathbf{H}$ where the complex number $\lambda$ in $(\lambda, q) \in \mathcal{A}$, acts on the right handed part by $\lambda$ on up particles and $\overline{\lambda}$ on down particles. The isodoublet structure of the left handed (up, down) pairs allows the quaternion $q$ to act on them by the matrix

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \qquad q = \alpha + \beta j \quad ; \quad \alpha, \beta \in \mathbf{C} \; .$$

Then the Yukawa coupling matrix of the standard model provides the selfadjoint matrix $D$.

In [C-L] we went along and developed the theory of matter fields in the above generality, with the finite dimension hypothesis that the characteristic values of $D^{-1}$ are $O(n^{-1/d})$ for some finite $d$.

This allows to define the action functional of Quantum Electrodynamics in the above generality (cf. [Co]). The striking fact there is that if one replaces the usual picture of space time by its product by the above 0-dimensional example, the QED action functional gives the Glashow Weinberg Salam standard model Lagrangian with its Higgs fields and symmetry breaking mechanism.

In developing this theory we made use of the tools of the quantized calculus, in particular of the Dixmier trace which replaces the Lebesgue integral in this context.

Writing the exact conditions that one requires for the triples $(\mathcal{A}, \mathcal{H}, D)$ is a bit like writing the axioms of noncommutative geometry. If we let $F$ and $|D|$ be the elements of the polar decomposition of $D$,

$$D = F|D| \quad , \quad |D|^2 = D^2 \quad , \quad F = \text{Sign } D \tag{11}$$

these operators $F$ and $|D|$ will play the role of the measurements of angles and of length in Hilbert's axioms of geometry.

In particular the operator $F = \text{Sign } D$ captures the conformal aspect while $D$ describes the full geometric situation.

Considering $F$ alone we developed (cf. [Co]) the quantized calculus which replaces the usual differential and integral calculus.

This new calculus can be succinctly described by the following dictionary. We fix a pair $(\mathcal{H}, F)$ where $\mathcal{H}$ is an infinite dimensional separable Hilbert space and $F$ is a selfadjoint operator of square 1 in $\mathcal{H}$. Giving $F$ is the same as giving the decomposition of $\mathcal{H}$ as the direct sum of the two orthogonal closed subspaces:

$$\{\xi \in \mathcal{H} \; ; \; F\xi = \pm\xi\}.$$

Assuming, as we shall, that both subspaces are infinite dimensional, we see that all such pairs $(\mathcal{H}, F)$ are unitarily equivalent. The dictionary is then the following:

| CLASSICAL | QUANTUM |
|---|---|
| Complex variable | Operator in $\mathcal{H}$ |
| Real variable | Selfadjoint operator in $\mathcal{H}$ |
| Infinitesimal | Compact operator in $\mathcal{H}$ |
| Infinitesimal of order $\alpha$ | Compact operator in $\mathcal{H}$ whose characteristic values $\mu_n$ satisfy $\mu_n = O(n^{-\alpha}) \quad n \to \infty$ |
| Differential of real or complex variable | $df = [F, f] = Ff - fF$ |
| Integral of infinitesimal of order 1 | Dixmier trace $\mathrm{Tr}_\omega(T)$. |

Let us comment in some detail each entry of the dictionary.

The range of a complex variable corresponds to the *spectrum* $\mathrm{Sp}(T)$ of an operator. The holomorphic functional calculus for operators in Hilbert space gives meaning to $f(T)$ for any holomorphic function $f$ defined on $\mathrm{Sp}(T)$ and only holomorphic functions act in that generality. This reflects the need for holomorphy in the theory of complex variables. For real variables the situation is quite different. Indeed when the operator $T$ is selfadjoint, $f(T)$ now makes sense for any borel function $f$ on the line.

The role of infinitesimal variables is played by the compact operators $T$ in $\mathcal{H}$. First $\mathcal{K} = \{T \in \mathcal{L}(\mathcal{H})\ ;\ T \text{ compact}\}$ is a two sided ideal in the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators in $\mathcal{H}$, and it is the largest non trivial ideal. An operator $T$ in $\mathcal{H}$ is compact iff for any $\varepsilon > 0$ the size of $T$ is smaller than $\varepsilon$ except for a finite dimensional subspace of $\mathcal{H}$. More precisely one lets for $n \in \mathbf{N}$:

$$\mu_n(T) = \mathrm{Inf}\{\|T - R\|\ ;\ R \text{ operator of rank } \leq n\} \tag{12}$$

where the rank of an operator is the dimension of its range. Then $T$ compact $\Leftrightarrow \mu_n(T) \to 0$ when $n \to \infty$. Moreover the $\mu_n(T)$ are the eigenvalues, ordered in decreasing size, of the absolute value $|T| = (T^*T)^{1/2}$ of $T$. The rate of decay of the $\mu_n(T)$ as $n \to \infty$ is a precise measure of the *size* of the infinitesimal $T$.

In particular for each positive real $\alpha$ the condition:

$$\mu_n(T) = O(n^{-\alpha}) \quad n \to \infty \tag{13}$$

(i.e. there exists $C < \infty$ such that $\mu_n(T) \leq C\, n^{-\alpha} \quad \forall n \geq 1$) defines the infinitesimals of order $\alpha$. They form a two sided ideal as is easily checked using the formula (12) for $\mu_n(T)$. Moreover if $T_1$ is of order $\alpha_1$, $T_2$ of order $\alpha_2$ then $T_1T_2$ is of order $\alpha_1 + \alpha_2$.

It could seem at this point that since the size of an infinitesimal is governed by a sequence $\mu_n \to 0$ we could dispense with operators and take the algebra $\ell^\infty(\mathbf{N})$ of bounded sequences as the notion of real or complex variables together with its ideal $C_0(\mathbf{N})$ of sequences $\mu_n$, $\mu_n \to 0$ when $n \to \infty$ as infinitesimal variables. The first possibility that we would loose in doing so would be to have variables with a continuous range (and even with a Lebesgue spectrum). Indeed there are bounded operators in $\mathcal{L}(\mathcal{H})$ with arbitrary spectral measure while the elements of $\ell^\infty(\mathbf{N})$ all have pure point spectrum.

The second very important point that would be lost is the use of commutators, crucial in the following notion of differential.

The differential $df$ of a real or complex variable, usually given by the differential geometric expression:

$$df = \Sigma \, \frac{\partial f}{\partial x^i} \, dx^i \tag{14}$$

is replaced in the new calculus by the commutator:

$$df = [F, f]. \tag{15}$$

The passage from the classical formula to the above operator theoretic one is analogous to the quantization of the Poisson brackets $\{f, g\}$ of classical mechanics as commutators: $[f, g]$. This is at the origin of the name "quantized calculus". The Leibnitz rule $d(fg) = (df)g + f\, dg$ still holds.

The equality $F^2 = 1$ is used to show that the differentials $df$ have vanishing anticommutator with $F$.

The next key ingredient of our calculus is the analogue of integration, it is given by the Dixmier trace. The Dixmier trace is a general tool designed to read in a classical manner a data of quantum mechanical nature. It is given as a positive linear form $\mathrm{Tr}_\omega$ on the ideal of infinitesimals of order 1 and is a *trace*:

$$\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS) \quad \forall T \text{ of order } 1, S \text{ bounded.} \tag{16}$$

In the classical differential calculus it is a great fact that one can neglect all infinitesimals of order $> 1$. Similarly, the Dixmier trace does neglect (i.e. vanishes on) any infinitesimal of order $> 1$, i.e.

$$\mathrm{Tr}_\omega(T) = 0 \quad \text{if} \quad \mu_n(T) = o(n^{-1}) \tag{17}$$

(where the little o means, as usual, that $n\mu_n \to 0$ as $n \to \infty$).

This vanishing allows considerable simplifications to occur, similar to those of the symbolic calculus, for expressions to which the Dixmier trace is applied.

The value of $\mathrm{Tr}_\omega(T)$ is given for $T \geq 0$ by a suitable limit of the bounded sequence:

$$\frac{1}{\log N} \sum_{0}^{\mu} \mu_n(T). \tag{18}$$

It is then extended by linearity to all compact operators of order 1.

In general the above sequence does not converge so that $\mathrm{Tr}_\omega$ a priori depends on a limiting procedure $\omega$. However, in all the applications one can prove the independence of $\mathrm{Tr}_\omega(T)$ on $\omega$. Such operators $T$ will be called measurable. For instance when $T$ is a pseudodifferential operator on a manifold it is measurable and its Dixmier trace coincides with the Manin-Wodzicki-Guillemin residue computed by a local formula. In general the term residue $(T)$ for the common value of $\mathrm{Tr}_\omega(T)$, $T$ measurable, would be appropriate since for $T > 0$ it coincides with the residue at $s = 1$ of the Dirichlet series $\zeta(s) = \mathrm{Trace}(T^s)$, $s \in \mathbf{C}$, $\mathrm{Re}(s) > 1$.

We have now completed our description of the framework of the quantized calculus. To use it for a given non commutative space $X$ we need a representation of the algebra $\mathcal{A}$ of functions on $X$ in the Hilbert space $\mathcal{H}$. The compatibility of this representation with the operator $F$ is simply that all operators $f$ in $\mathcal{H}$ coming from $\mathcal{A}$ have infinitesimal differential:

$$[F, f] \in \mathcal{K} \quad \forall f \in \mathcal{A}. \tag{19}$$

Such a representation is called a Fredholm module, and these are the basic cycles for the $K$-homology of $A$ when $A$ is a $C^*$-algebra.

To see how the new calculus works and allows operations not doable in distribution theory we shall start by a simple example. There is a unique way to quantize in the above sense the calculus of functions of one real variable (i.e. for $X = \mathbf{R}$) in a translation and scale invariant manner. It is given by the representation of functions as multiplication operators in $L^2(\mathbf{R})$ while $F$ is the Hilbert transform. Similarly for $X = S^1$ one lets $L^\infty(S^1)$ act in $L^2(S^1)$ by multiplication, while $F$ is again the Hilbert transform, given by the multiplication by the sign of $n$ in the Fourier basis $(e_n)_{n\in\mathbf{Z}}$ of $L^2(S^1)$, with $e_n(\theta) = \exp(in\theta) \quad \forall \theta \in S^1$.

The first virtue of the new calculus is that $df$ continues to make sense, as an operator in $L^2(S^1)$ for an arbitrary measurable $f \in L^\infty(S^1)$. This of course would also hold if we define $df$ using distribution theory but the essential difference is the following. A distribution is defined as an element of the topological dual of the locally convex vector space of smooth functions, here $C^\infty(S^1)$. Thus only the latter linear structure on $C^\infty(S^1)$ is used, not the *algebra* structure of $C^\infty(S^1)$. It is consequently not surprising that distributions are incompatible with pointwise product or absolute value. Thus more precisely while, with $f$ non differentiable, $df$ makes sense as a distribution, we cannot make any sense of $|df|$ or powers $|df|^p$ as distributions on $S^1$. Let us give a concrete example where one would like to use such an expression for non differentiable $f$. Let $c$ be a complex number and let $J$ be the Julia set given by the complex dynamical system $z \to z^2 + c = \varphi(z)$. More specifically $J$ is here the boundary of the set $B = \{z \in \mathbf{C};\ \sup_{n\in\mathbf{N}} |\varphi^n(z)| < \infty\}$. For small values of $c$, the Julia set $J$ is a Jordan curve and $B$ is the bounded component of its complement. Now the Riemann mapping theorem provides us with a conformal equivalence $Z$ of the unit disk, $D = \{z \in \mathbf{C}, |z| < 1\}$ with the inside of $B$, and by a result of Caratheodory, the conformal mapping $Z$ extends continuously on the boundary $S^1$ of $D$ to a homeomorphism, which we still denote by $Z$, from $S^1$ to $J$. By a known result of D. Sullivan, the Hausdorff dimension $p$ of the Julia set is strictly bigger than 1, $1 < p < 2$. This shows that the function $Z$ is nowhere of bounded variation on $S^1$ and forbids a distribution interpretation of the naive expression:

$$\int f(Z)|dZ|^p \quad \forall f \in C(J) \tag{20}$$

that would be the natural candidate for the Hausdorff measure on $J$.

We shall show that the above expression makes sense in the quantized calculus and that it does give the Hausdorff measure on the Julia set $J$. The first essential fact is that as $dZ = [F, Z]$ is now an operator in Hilbert space one can, irrespective of the regularity of $Z$, talk about $|dZ|$, it is the absolute value $|T| = (T^*T)^{1/2}$ of the operator $T = [F, Z]$. This gives meaning to any function $h(|dZ|)$ where $h$ is a bounded measurable function on $\mathbf{R}_+$ and in particular to $|dZ|^p$. The next essential step is to give meaning to the integral of $f(Z)|dZ|^p$. The latter expression is an operator in $L^2(S^1)$ and we use a result of hard analysis due to V.V. Peller, together with the homogeneity properties of the Julia set to show that the operator $f(Z)|dZ|^p$ belongs to the domain of definition of the Dixmier trace $\mathrm{Tr}_\omega$, i.e. is an infinitesimal of order 1. Moreover, if one works modulo infinitesimals of order $> 1$ the rules of the usual differential calculus such as:

$$|d\varphi(Z)|^p = |\varphi'(Z)|^p \; |dZ|^p \tag{21}$$

are valid and show that the measure:

$$f \to \mathrm{Tr}_\omega(f(Z)|dZ|^p) \quad \forall f \in C(J) \tag{22}$$

has the right conformal weight and is a non zero multiple of the Hausdorff measure. The corresponding constant governs the asymptotic expansion for the distance, in the sup norm on $S^1$, between the function $Z$ and restrictions to $S^1$ of rational functions with at most $n$ poles outside the unit disk.

Let us now pass to the higher dimensional case.

# 1 Quantized calculus and conformal manifolds

Let $V$ be an even dimensional oriented compact manifold endowed with a conformal structure. We shall now show how to quantize in a canonical manner the calculus on $V$ by constructing a natural even Fredholm module $(\mathcal{H}, F, \gamma)$ over the algebra $C^\infty(V)$ of smooth functions on $V$.

For the construction we shall just need the $\mathbf{Z}/2$ grading of the vector bundle $\wedge^n T^*$, $n = \frac{1}{2}\dim V$, given by the $*$ operation. Recall that given a Euclidean oriented vector space $E$ of dimension $m$, the $*$ operation, $* : \wedge^* E \to \wedge^* E$ is given by the equality:

$$*(e_1 \wedge \ldots \wedge e_k) = e_{k+1} \wedge \ldots \wedge e_m \tag{1.1}$$

for any orthonormal basis $e_1, \ldots, e_m$ of $E$ compatible with the orientation. When $m$ is even, $m = 2n$, the restriction of $*$ to $\wedge^n E$ is unaffected if one replaces the

Euclidean metric of $E$ by its scalar multiples. Moreover one gets a $\mathbf{Z}/2$ grading on $\wedge^n_{\mathbf{C}} E$ given by the operator $\gamma = (-1)^{\frac{n(n-1)}{2}} i^n *$ which is of square 1.

Let $\mathcal{H}_0$ be the Hilbert space $L^2(V, \wedge^n_{\mathbf{C}} T^*)$ of square integrable sections of the complex vector bundle $\wedge^n_{\mathbf{C}} T^*$, with the inner product given by the complexification of the real inner product

$$\langle \omega_1, \omega_2 \rangle = \int_V \omega_1 \wedge * \omega_2 \qquad \forall \omega_1, \omega_2 \in L^2(V, \wedge^n T^*). \tag{1.2}$$

By construction $\mathcal{H}_0$ is a module over $C^\infty(V)$ (and also $L^\infty(V)$) with:

$$(f\omega)(p) = f(p)\, \omega(p) \qquad \forall f \in C^\infty(V) \ , \ \omega \in \mathcal{H}_0 \ , \ p \in V. \tag{1.3}$$

It is $\mathbf{Z}/2$ graded by the above operator $\gamma$ of square 1,

$$(\gamma\omega)(p) = \gamma(\omega(p)) \qquad \forall p \in V \ , \ \omega \in \mathcal{H}_0. \tag{1.4}$$

To construct the operator $F$ we need the following:

**Lemma 1.** *Let $B \subset \mathcal{H}_0$ be the closure of the image of $d : C^\infty(V, \wedge^{n-1}_{\mathbf{C}} T^*) \to C^\infty(V, \wedge^n_{\mathbf{C}} T^*)$. Then $B$ is the graph of a partial isometry $S : \mathcal{H}_0^- \to \mathcal{H}_0^+$ (resp. $S^* : \mathcal{H}_0^+ \to \mathcal{H}_0^-$) and $1 - (SS^* + S^*S)$ is the orthogonal projection on the finite dimensional space of harmonic forms.*

**Proof.** This follows in a straightforward manner from the Hodge decomposition ([Gi]) of $\mathcal{H}_0$ as the direct sum of the kernel of $d + d^*$, i.e. harmonic forms, and the image of $d + d^*$. Thus any $\omega \in \mathcal{H}_0^+$, orthogonal to harmonic forms, can be written as $\omega = \frac{1+\gamma}{2}\, d\alpha$, $\alpha \in L^2(V, \wedge^{n-1}_{\mathbf{C}} T^*)$, using the formula $d^* = - * d *$ for the adjoint of $d$. Moreover the equality $\omega = \frac{1+\gamma}{2}\, d\alpha$ determines $d\alpha$ uniquely since

$$\left\| \tfrac{1+\gamma}{2}\, d\alpha \right\|_2 = \left\| \tfrac{1-\gamma}{2}\, d\alpha \right\|_2 \qquad \forall \alpha \in L^2(V, \wedge^{n-1}_{\mathbf{C}}\, T^*). \tag{1.5}$$

We can now define the Fredholm module $(\mathcal{H}, F, \gamma)$ over $C^\infty(V)$. We let $\mathcal{H} = \mathcal{H}_0 \oplus H^n(V, \mathbf{C})$ be the direct sum of $\mathcal{H}_0$ with the finite dimensional Hilbert space of harmonic $n$-forms on $V$, which we identify with the $n$ dimensional cohomology group $H^n(V, \mathbf{C})$. We endow $H^n$ with the opposite $\mathbf{Z}/2$ grading $-\gamma$ and with the 0-module structure over $C^\infty(V)$. The direct sum $\mathcal{H} = \mathcal{H}_0 \oplus H^n$ is thus a $\mathbf{Z}/2$ graded $C^\infty(V)$ module. We let then $F$ be the operator in $\mathcal{H}$, direct sum of $\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}$ acting in $\mathcal{H}_0 \ominus H^n$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acting in $H^n \oplus H^n$. Note that $\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}$ acting in $\mathcal{H}_0 \ominus H^n$ is equal to $2P - 1$ where $P$ is the orthogonal projection on $B \subset \mathcal{H}_0 \ominus H^n$.

**Theorem 2.** a) *The triple $(\mathcal{H}, F, \gamma)$ is a Fredholm module over $C^\infty(V)$, canonically associated to the oriented conformal structure of $V$. It is $p$ summable for any $p > 2n$.*

b) *The character $ch_*(\mathcal{H}, F, \gamma)$ is $2^n$ times the Atiyah-Hirzebruch $\mathcal{L}$ genus multiplied by the fundamental class $[V]$.*

c) *The Fredholm module $(\mathcal{H}, F, \gamma)$ uniquely determines the oriented conformal structure of $V$.*

We refer to [Co] for the proof.

Let us work out in more detail the simplest example of the construction of the Fredholm module $(\mathcal{H}, F, \gamma)$ associated by theorem 2 to an oriented conformal manifold. Thus let $V = P_1(\mathbf{C})$ be the Riemann sphere. The Hilbert space $\mathcal{H}$ is the space of square integrable 1-forms, i.e. the direct sum $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ of the spaces of square integrable forms of type $(1,0)$ and $(0,1)$. Using the complex coordinate $z$ in $P_1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ we can write any element $\xi \in \mathcal{H}^\pm$ as $\xi(z)dz$ (resp. $\xi(z)d\bar{z}$) where $\xi$ is a square integrable function on $\mathbf{C}$. With these notations the unitary operator $S$, $\mathcal{H}^- \to \mathcal{H}^+$ such that $F = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}$, is the complex Hilbert transform, given by:

$$(S\xi)(z') = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\xi(z)}{(z-z')^2} \, dz \, d\bar{z} \tag{1.6}$$

where the integral is defined as a Cauchy principal value, i.e. as the limit for $\varepsilon \to 0$ of the integral over $|z - z'| \geq \varepsilon$.

The operator $S$ is canonically associated to the conformal structure of $P_1(\mathbf{C})$. Thus the differential form $(z - z')^{-2} \, dz \, dz'$ on $P_1(\mathbf{C}) \times P_1(\mathbf{C})$ is $SL(2, \mathbf{C})$ invariant.

**Remark**

(1) The construction of theorem 2 applies to arbitrary quasi-conformal topological manifolds ([Co-S-T]) and yields local formulae for rational Pontrjagin classes.

(2) This construction extends to an arbitrary local field such as the $p$-adic number field $\mathbf{Q}_p$. It is interesting in that respect to relate the Polyakov action discussed below with the $p$-adic string action discussed in []. The latter is obtained using the usual trace instead of the Dixmier trace, due to the 0-dimensionality of the $p$-adic situation.

## 1.1 Perturbation of Fredholm modules by the commutant von Neumann algebra

Let $M$ be a von Neumann algebra and $M_2(M) = M_2(\mathbf{C}) \otimes M$. Let

$$G = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in M_2(M); \begin{bmatrix} a^* & -b^* \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a^* & -b^* \\ -b^* & a^* \end{bmatrix} = 1 \right\}. \tag{1.7}$$

In other words $a, b$ are elements of $M$ which fulfill the conditions:

$$a^*a - b^*b = 1 \ , \ a^*b = b^*a \ , \ aa^* - bb^* = 1 \ , \ ba^* = ab^*. \tag{1.8}$$

**Proposition 4.** a) *$G$ is a subgroup of $GL_2(M)$, and is isomorphic to $GL_1(M)$*
b) *Let $\mu = \mu^* \in M$, $\|\mu\| < 1$. Then $g(\mu) \in G$ where*

$$g(\mu) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \ , \ a = (1-\mu^2)^{-1/2} \ , \ b = \mu(1-\mu^2)^{-1/2}.$$

c) *Let $\mathcal{U} = \left\{ \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \ ; \ u \in M \ , \ u^*u = uu^* = 1 \right\}$ be the unitary group of $M$ viewed as a subgroup of $G$. Then every element $g \in G$ is uniquely decomposed as $g = u\, g(\mu)$ for some $u \in \mathcal{U}$, $\mu \in M$, $\mu = \mu^*$, $\|\mu\| < 1$.*

**Proof.** a) Let $g_1, g_2 \in G$, with $g_j = \begin{bmatrix} a_j & b_j \\ b_j & a_j \end{bmatrix}$. Then one has

$$g_1 g_2 = \begin{bmatrix} a_1a_2 + b_1b_2 & a_1b_2 + b_1a_2 \\ a_1b_2 + b_1a_2 & a_1a_2 + b_1b_2 \end{bmatrix}$$

$$g_2^{-1} g_1^{-1} = \begin{bmatrix} a_2^* & -b_2^* \\ -b_2^* & a_2^* \end{bmatrix} \begin{bmatrix} a_1^* & -b_1^* \\ -b_1^* & a_1^* \end{bmatrix} = \begin{bmatrix} a_2^*a_1^* + b_2^*b_1^* & -b_2^*a_1^* - a_2^*b_1^* \\ -b_2^*a_1^* - a_2^*b_1^* & a_2^*a_1^* + b_2^*b_1^* \end{bmatrix}.$$

These equalities show that $g_1g_2 \in G$. They also show that the map $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a + b$ is an isomorphism $G \simeq GL_1(M)$.

b) Since $\|\mu\| < 1$, $(1-\mu^2)^{-1/2}$ makes sense. By construction $a = a^*$, $b = b^*$ all commute with each other and $a^2 - b^2 = 1$. Thus $g(\mu) \in G$.

c) Let $g = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in G$. One has $a^*a = 1 + b^*b \geq 1$, $aa^* = 1 + bb^* \geq 1$. Thus $a$ is invertible and we let $u$ be the unitary of its polar decomposition: $a = u(a^*a)^{1/2}$. Replacing $g$ by $\begin{bmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{bmatrix} g$, one can assume that $a$ is positive. It follows then, using the equalities $b^*b = a^*a - 1 = aa^* - 1 = bb^*$, that $b$ is normal, and $|b| = (a^2-1)^{1/2}$. Let $b = v|b|$ be the polar decomposition of $b$. Then $v$ commutes with $|b|$, so that $b$ commutes with $a$. The equality $ba = ab^*$ then shows that $b = b^*$, and it follows that $g = g(\mu)$ where $\mu = ba^{-1}$. One has $\|\mu\| < 1$ since $|b| = (a^2-1)^{1/2}$ and $a$ is bounded.

**Definition 5.** *Let $M$ be a von Neumann algebra. We let $\mu(M)$ be the above subgroup of $GL_2(M)$. If $M$ is $\mathbf{Z}/2$ graded we let $\mu_{\mathrm{ev}}(M)$ be the subgroup of $\mu(M)$ determined by the conditions:*

$$g = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in \mu_{\mathrm{ev}} \ \ iff \ \ a \text{ is even and } b \text{ is odd.}$$

Let now $(\mathcal{H}, F)$ be a Fredholm module over a $C^*$-algebra $A$, and let $M$ be the commutant of $A$ in $\mathcal{H}$. By construction $M$ is a von Neumann algebra, and it is $\mathbf{Z}/2$ graded when the Fredholm module is even. We shall now describe a natural action of the group $\mu(M)$ (resp. $\mu_{\mathrm{ev}}(M)$ in the even case), on the space of $F$'s yielding a Fredholm module over $A$.

**Proposition 6.** *Let* $g = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in \mu(M)$ *(resp.* $\mu_{\mathrm{ev}}(M)$ *in the even case). Then with* $F' = g(F) = (aF+b)(bF+a)^{-1}$*, the pair* $(\mathcal{H}, F')$ *is a Fredholm module over* $A$*. It is even if* $g \in \mu_{\mathrm{ev}}(M)$*. Moreover for any* $x \in A$*, the commutator* $[F', x]$ *belongs to the two sided ideal generated by* $[F, x]$*.*

**Proof.** The equality $g(F) = (aF+b)F(aF+b)^{-1}$ shows that $g(F)^2 = 1$. To show that $g(F)^* = g(F)$ one has to check that

$$(aF+b)^*(bF+a) = (bF+a)^*(aF+b).$$

But this equality follows from the relations $a^*b = b^*a$, $a^*a - b^*b = 1$. To conclude, we just need to compute $[F', x]$ in terms of $[F, x]$. One has $[(aF+b)(bF+a)^{-1}, x] = a[F,x](bF+a)^{-1} - (aF+b)(bF+a)^{-1}b[F,x](bF+a)^{-1} = (a - F'b)[F,x](bF+a)^{-1} = (bF+a)^{*-1}[F,x](bF+a)^{-1}$. We have used the equality $(a - F'b)^{-1} = (bF+a)^*$.

**Example 7.** Let $(\mathcal{H}, F, \gamma)$ be the even Fredholm module on the $C^*$-algebra $C(P_1(\mathbf{C}))$ associated by theorem 2 to the Riemann sphere, $V = P_1(\mathbf{C})$. The commutant $M = A'$ of $A = C(P_1(\mathbf{C}))$ in $\mathcal{H}$ is the von Neumann algebra of $2 \times 2$ matrices:

$$a = \begin{bmatrix} f & u \\ v & g \end{bmatrix}$$

where $f$ and $g$ are measurable bounded functions on $V = P_1(\mathbf{C})$ and $u, v$ are measurable bounded Beltrami differentials: $u(z,\bar{z})dz/d\bar{z}$, $v(z,\bar{z})d\bar{z}/dz$ [Ber$_1$]. In particular the *odd* elements $\mu \in M$, $\mu = \mu^*$ with $\|\mu\| < 1$ correspond exactly to a single Beltrami differential $v(z,\bar{z})d\bar{z}/dz$ with $\|v\|_\infty < 1$, and $v$ measurable, by the equality:

$$\mu = \begin{bmatrix} 0 & v^* \\ v & 0 \end{bmatrix}.$$

Now by proposition 4 c) all the relevant perturbations of a Fredholm module by the action of $\mu_{\mathrm{ev}}(M)$ are obtained using the elements $g(\mu)$, $\mu$ odd, of proposition 4 c). (The action of $\mathcal{U}$ just conjugates the Fredholm module to an equivalent one.) One checks by a direct calculation that for any $g(\mu) \in \mu_{\mathrm{ev}}(M)$ the perturbed Fredholm module $(\mathcal{H}, g(\mu)(F))$ over $A = C(P_1(\mathbf{C}))$ is canonically isomorphic to the Fredholm module over $A$ associated to the perturbed conformal structure on $P_1(\mathbf{C})$ associated to the measurable Beltrami differential $v(z,\bar{z})d\bar{z}/dz$.

The same interpretation of the construction of proposition 6 holds for arbitrary Riemann surfaces. But the above case of $P_1(\mathbf{C})$ is particularly significant since the measurable Riemann mapping theorem ([Ber]) is equivalent in that case to the *stability* of $(\mathcal{H}, F)$ under perturbations, i.e. the existence for any $g \in \mu_{\mathrm{ev}}(M)$ of a unitary operator $U(g)$ in $\mathcal{H}$ such that:

$\alpha$) $U(g)\ A\ U(g)^* = A$

$\beta$) $U(g)\ F\ U(g)^* = g(F)$.

(Such a unitary is uniquely determined modulo the automorphism group $U(1) \times PSL(2, \mathbf{C})$ of the module $(\mathcal{H}, F)$.)

We refer to [Ber$_1$] for a proof of the measurable Riemann mapping theorem based on the 2-dimensional Hilbert transform, i.e. on the above operator $F$.

## 2 The 4-dimensional analogue of the Polyakov action

We shall use the quantized calculus to find the analogue in dimension 4 of the 2-dimensional Polyakov action, namely:

$$I = \frac{1}{2\pi} \int_\Sigma \eta_{ij}\ dX^i \wedge *\ dX^j \tag{2.1}$$

for a Riemann surface $\Sigma$ and a map $X$ from $\Sigma$ to a $d$-dimensional space $M$.

Our first task will be to write the Polyakov action (1) as the Dixmier trace of the operator:

$$\Sigma\ \eta_{ij}\ dX^i\ dX^j \tag{2.2}$$

where now $dX = [F, X]$ is the quantum differential of $X$ taken using the canonical Fredholm module $(\mathcal{H}, F)$ of the Riemann surface $\Sigma$.

The same expression will then continue to make sense in dimension 4, i.e. with $\Sigma$ replaced by a 4-dimensional conformal manifold. The action we shall get will be conformally invariant by construction and intimately related to the Einstein action of gravity.

In general, given an even dimensional conformal manifold $\Sigma$, $\dim \Sigma = n = 2m$, we let $\mathcal{H} = L^2\left(\Sigma, \wedge^m_{\mathbf{C}}\ T^*\right)$ be the Hilbert space of square integrable forms of middle dimension, in which functions on $\Sigma$ act as multiplication operators.

We let $F = 2P - 1$ be the operator in $\mathcal{H}$ obtained from the orthogonal projection $P$ on the image of $d$. It is clear that both $\mathcal{H}$ and $F$ only depend upon the conformal structure of $\Sigma$, which we assume to be *compact.*

In terms of an arbitrary Riemannian metric compatible with the conformal structure of $\Sigma$ one has the formula:

$$F = (dd^* - d^*d)(dd^* + d^*d)^{-1} \quad \text{on} \quad L^2(\Sigma, \wedge^m\ T^*) \tag{2.3}$$

which ignores the finite dimensional subspace of harmonic forms, irrelevant in our later computations.

By construction $F$ is a pseudodifferential operator of order 0, whose principal symbol is given by:

**Lemma 1.** *The principal symbol $\sigma_0(F)$ is given by:*

$$\sigma_0(x, \xi) = (e_\xi\ i_\xi - i_\xi\ e_\xi)\ \|\xi\|^{-2}\ ,\ \forall (x, \xi) \in T^*(\Sigma).$$

We have denoted by $e_\xi$ (resp. $i_\xi$) the exterior multiplication (resp. interior) by $\xi$.

When $n = \dim \Sigma = 2$, one has $\wedge^m_{\mathbf{C}} T^* = T^*_{\mathbf{C}}$ and $\sigma_0$ associates to any $\xi \neq 0$, $\xi \in T^*_x(\Sigma)$, the symmetry with axis $\xi$. For any function $f \in C^\infty(\Sigma)$, the operator $[F, f]$ is pseudodifferential of order $-1$. Its principal symbol is the Poisson bracket $\{\sigma_0, f\}$,

$$\{\sigma_0, f\}(x,\xi) = 2\left(e_{df}\ i_\xi + e_\xi\ i_{df} - 2e_\xi\ i_\xi\ \langle \xi, df\rangle\ \|\xi\|^{-2}\right)\ \|\xi\|^{-2}. \tag{2.4}$$

For $\|\xi\| = 1$, decompose $df$ as $\langle df, \xi\rangle\xi + \eta$ where $\eta \perp \xi$. Then $\{\sigma_0, f\}(x,\xi) = 2(e_\eta\ i_\xi + e_\xi\ i_\eta)$, and its Hilbert Schmidt norm, for $n = 2$, is given by:

$$\operatorname{trace}(\{\sigma_0, f\}(x,\xi)^*\ \{\sigma_0, f\}(x,\xi)) = 8\|\eta\|^2\ ,\ \eta = df - \langle df, \xi\rangle\xi.$$

The Dixmier trace $\operatorname{Tr}_\omega(f_0[F, f_1]^*\ [F, f_2])$ is thus easy to compute for $n = 2$, as the integral on the unit sphere $S^*\Sigma$ of the cotangent bundle of $\Sigma$, of the function:

$$\operatorname{trace}(f_0\ \{\sigma_0, f_1\}^*\ \{\sigma_0, f_2\}) = 8\ f_0(x)\ \langle df_1^\perp, df_2^\perp\rangle$$

where $df^\perp = df - \langle df, \xi\rangle\xi$ by convention. One thus gets:

**Proposition 2.** *Let $\Sigma$ be a compact Riemann surface ($n = 2$), then for any smooth map $X = (X^i)$ from $\Sigma$ to $\mathbf{R}^d$ and metric $\eta_{ij}(x)$ on $\mathbf{R}^d$ one has*

$$\frac{1}{2\pi}\int_\Sigma \eta_{ij}\ dX^i \wedge *\ dX^j = -1/2\ \operatorname{Tr}_\omega\left(\eta_{ij}\ [F, X^i][F, X^j]\right).$$

Both sides of the equality have obvious meaning when the $\eta_{ij}$ are constants. In general one just views them as functions on $\Sigma$ namely $\eta_{ij} \circ X$.

Let us now pass to the more involved 4-dimensional case. We want to compute the following action defined on smooth maps $X : \Sigma \to \mathbf{R}^d$ of a 4-dimensional compact conformal manifold $\Sigma$ to $\mathbf{R}^d$, endowed with the metric $\eta_{ij}\ dx^i\ dx^j$.

$$I = \operatorname{Tr}_\omega\left(\eta_{ij}\ [F, X^i][F, X^j]\right). \tag{2.5}$$

Here we are beyond the natural domain of the Dixmier trace $\operatorname{Tr}_\omega$ but we can use the remarkable fact, due to Wodzicki, that it extends uniquely as a trace on the algebra of pseudodifferential operators (cf. [Wo]). For practical purposes the local formula for this extension, which we still denote by $\operatorname{Tr}_\omega$, is given as follows (up to normalization):

$$\operatorname{Tr}_\omega(P_\sigma) = \int_{S^*\Sigma} \sigma_{-4}(x,\xi)\ d^4x\ d^3\xi \tag{2.6}$$

where $P_\sigma$ is a pseudodifferential operator whose total symbol

$$\sigma(x,\xi) = \sigma_0(x,\xi) + \sigma_{-1}(x,\xi) + \sigma_{-2}(x,\xi) + \cdots \tag{2.7}$$

has $\sigma_{-4}(x,\xi)$, as the component of order $-4$.

This formula makes sense for scalar pseudodifferential operators, defined in local coordinates $x^j$ by the usual formula:

$$(P_\sigma)(x,y) = \int e^{i\langle x-y,\xi\rangle}\, \sigma(x,\xi)\, d^4\xi \tag{2.8}$$

but, by [Wo2] it is independent of the choice of local coordinates and defines a trace, $\mathrm{Tr}_\omega$, on the algebra of scalar pseudodifferential operators.

When we consider a vector bundle $E$ over a manifold $\Sigma$, and a pseudodifferential operator $P$ acting on sections of $E$, we compute $\mathrm{Tr}_\omega(P)$ as follows. Choose local coordinates $x^j$ and local basis of sections $\alpha_k$ for the bundle $E$. Then $P$ appears as a matrix $P_k^\ell$ of scalar pseudodifferential operators:

$$P(f^k\, \alpha_k) = (P_k^\ell\, f^k)\, \alpha_\ell.$$

The expression $\mathrm{Tr}_\omega(P) = \mathrm{Tr}_\omega(P_k^k)$ is then independent of the choice of the local basis $(\alpha_k)$ of $E$ and defines a trace.

It is clear that to compute the action $I$ we just need to compute the following trilinear form $\tau$ on $C^\infty(\Sigma)$.

$$\tau(f_0,f_1,f_2) = \mathrm{Tr}_\omega\left(f_0\, [F,f_1][F,f_2]\right) \qquad \forall f_j \in C^\infty(\Sigma). \tag{2.9}$$

By construction $\tau$ is a Hochschild 2-cocycle on $C^\infty(\Sigma)$. We let $\Omega(f_1,f_2)$ be the 4-dimensional differential form on $\Sigma$ uniquely determined by the equation:

$$\tau(f_0,f_1,f_2) = \int_\Sigma f_0\, \Omega(f_1,f_2) \qquad \forall f_0 \in C^\infty(\Sigma). \tag{2.10}$$

The existence of $\vee$ follows from the general formula for the total symbol of the product of two pseudodifferential operators $P_{\sigma_1}$, $P_{\sigma_2}$, in terms of $\sigma_1$ and $\sigma_2$:

$$\sigma(x,\xi) = \sum \frac{1}{\alpha!}\, \partial_\xi^\alpha\, \sigma_1(x,\xi)\, D_x^\alpha\, \sigma_2(x,\xi) \tag{2.11}$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a multiindex, $\alpha! = \alpha_1!\ \alpha_2!\ \alpha_3!\ \alpha_4!$ and $D_x^\alpha = (-i)^{|\alpha|}\, \partial_x^\alpha$.

This formula, applied with $P_{\sigma_1} = f_0$, $P_{\sigma_2} = [F,f_1][F,f_2]$ shows the existence of $\vee$.

Our task is to compute, given $x \in \Sigma$, the value of the differential form $\Omega(f_1,f_2)$ at $x$, in terms of $f_1, f_2$ and the conformal structure of $\Sigma$.

We shall take local coordinates $x^j$ around $x$ and let $\omega^\alpha = dx^i \wedge dx^j$ be the corresponding basis for our vector bundle $E = \wedge^2_{\mathbf{C}}\, T^*$ over $\Sigma$.

Let $P = [F,f_1][F,f_2]$. It is a pseudodifferential operator of order $-2$ and in terms of its symbol up to order $-4$:

$$\sigma = \sigma_{-2} + \sigma_{-3} + \sigma_{-4} \tag{2.12}$$

where we have omitted the $\alpha,\beta$ matrix indices, we get the following formula for $\Omega(f_1,f_2)$ at $x$:

$$\Omega(f_1,f_2) = \left(\int_{S^3} \mathrm{trace}\left(\sigma_{-4}(x,\xi)\right) d^3\xi\right)\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{2.13}$$

where $S^3$ is the unit sphere in the $\xi$ variable and $d^3\xi$ the normalized volume on $S^3$.

Next the total symbol $\sigma$, up to order $-4$ included, is obtained by formula (2.11) (and with more matrix indices) from the total symbols $\sigma([F, f_1])$, $\sigma([F, f_2])$ which we only need to know up to order $-3$ included. To compute them we again use formula (2.11) for $Ff - fF$ and we thus only need to know the total symbol of $F$ up to order $-2$.

The computation is done using formula (2.3) and (2.11). What matters is the way the variables $g_{ij}$ enter the formula:

**Lemma 3.** *The total symbol $\sigma^F$ of $F$, up to order $-2$ included, is a $6 \times 6$ matrix of the form:*

$$\sigma^F = \sigma_0^F + \sigma_{-1}^F + \sigma_{-2}^F$$

*where $\sigma_0^F$ only invokes $g_{ij}(x)$, $\sigma_{-1}^F$ is linear in the 1-jet of the metric (at $x$) with coefficients depending smoothly on $g_{ij}(x)$, $\sigma_{-2}^F$ is linear in the 2-jet of the metric + quadratic in the 1-jet of the metric, with coefficients depending smoothly on $g_{ij}(x)$.*

**Proof.** Both operators $dd^* - d^*d$ and $\Delta = dd^* + d^*d$ acting on $\wedge_{\mathbf{C}}^2 T^* = E$ can be expanded in our local basis in the form:

$$\sigma(dd^* - d^*d) = q_2 + q_1 + q_0$$

$$\sigma(\Delta) = p_2 + p_1 + p_0 \tag{2.14}$$

where $p_2, q_2$ only invoke the $g_{ij}(x)$, and $p_1, q_1, p_0, q_0$ have the properties indicated in the lemma for $\sigma_{-1}^F$, $\sigma_{-2}^F$ (cf. for instance [Gi] Lemma 2.4.2 p.118).

Now to compute the total symbol $\sigma(\Delta^{-1})$ up to order $-2$ let us denote by $\circ$ the product of symbols as defined by formula (2.11). One has:

$$\sigma(\Delta^{-1}) = p \circ (1 - \varepsilon_{-1} - \varepsilon_{-2} + \varepsilon_{-1}^2) \tag{2.15}$$

where, with $p(x, \xi) = (p_2(x, \xi))^{-1}$ one lets

$$\Delta \circ p = 1 + \varepsilon_{-1} + \varepsilon_{-2} \tag{2.16}$$

be the total symbol of $\Delta \circ p$ up to order $-2$ included. By construction $p$ only depends upon the $g_{ij}(x)$, so that by the formula (2.11) the symbols $\varepsilon_{-1}$, $\varepsilon_{-2}$ satisfy the conditions of lemma 3 (with $\varepsilon_{-k}$ linear in the $k$-jet of the metric + square of 1-jet for $k = 2$).

It thus follows from the formula (2.15) that $\sigma(\Delta^{-1})$ has a similar expansion:

$$\sigma(\Delta^{-1}) = \sigma_{-2}(\Delta^{-1}) + \sigma_{-3}(\Delta^{-1}) + \sigma_{-4}(\Delta^{-1}) \tag{2.17}$$

with $\sigma_{-2-k}(\Delta^{-1})$ linear in the $k$-jet of the metric + eventual quadratic terms for $k = 2$.

Finally when we compute the composition

$$\sigma(dd^* - d^*d) \circ \sigma(\Delta^{-1}) = \sigma_0^F + \sigma_{-1}^F + \sigma_{-2}^F$$

we get, using formulas (2.14) and (2.11) the required property.

Now the total symbol of $[F, f]$, up to order $-3$ included, is of the form:

$$\sum_{|\alpha|=1} \partial_\xi^\alpha \, \sigma_0^F \, D^\alpha f \; + \sum_{|\alpha|=2} \tfrac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma_0^F \, D^\alpha f \; + \sum_{|\alpha|=3} \tfrac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma_0^F \, D^\alpha f$$
$$+ \sum_{|\alpha|=1} \partial_\xi^\alpha \, \sigma_{-1}^F \, D^\alpha f \; + \sum_{|\alpha|=2} \tfrac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma_{-1}^F \, D^\alpha f$$
$$+ \sum_{|\alpha|=1} \partial_\xi^\alpha \, \sigma_{-2}^F \, D^\alpha f. \tag{2.18}$$

Note that the differentiation $\partial_\xi^\alpha$ does not alter the properties of $\sigma_{-k}^F$ stated in the lemma, so that for instance $\partial_\xi^\alpha \, \sigma_{-1}^F$ is linear in the 1-jet of the metric.

To compute $\Omega(f_1, f_2)$ we need the component of order $-4$ of the total symbol of $[F, f_1][F, f_2]$. This component $\sigma_{-4}$ is obtained by composition (i.e. using formula (2.11)) of the expressions (2.18) applied to $f_1$ and $f_2$. We thus get:

$$\sigma_{-4} = \sum \left( \frac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma_{-k}^F \, D^\alpha \, f_1 \right) \circ \left( \frac{1}{\beta!} \, \partial_\xi^\beta \, \sigma_{-\ell}^F \, D^\beta \, f_2 \right) \tag{2.19}$$

where the sum is restricted to $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha| + k + |\beta| + \ell \leq 4$, and one takes in the composition $\circ$ of the symbols, its component of degree $-4$ only. In other words, using (2.11) and $\partial_\xi^\alpha(D^\alpha f_1) = 0$ we get:

$$\sigma_{-4} = \sum \frac{1}{\alpha!} \, \frac{1}{\beta!} \, \frac{1}{\gamma!} \, \frac{1}{\delta!} \, (D^\alpha f_1) \left( \partial_\xi^{\alpha+\gamma+\delta} \, \sigma_{-k}^F \right) \left( \partial_\xi^\beta \, D^\gamma \, \sigma_{-\ell}^F \right) \left( D^{\beta+\delta} f_2 \right) \tag{2.20}$$

where the sum is restricted to $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha|+|\beta|+|\gamma|+|\delta|+k+\ell = 4$. The inequality $k + |\gamma| + \ell \leq 2$ allows to write $\sigma_{-4}$ as a sum of 3 terms according to the value of $k+|\gamma|+\ell \in \{0,1,2\}$. The term $\sigma_{-4}^{(0)} = \sum\limits_{k=\ell=|\gamma|=0}$ depends only upon the $g_{ij}(x)$. The term $\sigma_{-4}^{(1)} = \sum\limits_{k+\ell+|\gamma|=1}$ is linear in the 1-jet of the metric with coefficients depending smoothly on the $g_{ij}$. The term $\sigma_{-4}^{(2)} = \sum\limits_{k+\ell+|\gamma|=2}$ is the sum of a linear term in the 2-jet of the metric and of a quadratic term in the 1-jet, both with coefficients depending smoothly on the $g_{ij}$. Since $|\alpha| + |\beta| + |\delta| = 2$ if $k + \ell + |\gamma| = 2$ we see that $\sigma_{-4}^{(2)}$ only involves the 1-jet of $f_1$ and $f_2$ at $x$.

These properties of $\sigma_{-4}^{(k)}$ obviously persist after integration of the $\xi$ variable on the unit sphere $S^3$ of $T_x^*(\Sigma)$. Choosing the coordinates $x^j$ to be geodesic normal coordinates at the point $x$, we can assume that $g_{ij}(x) = \delta_{ij}$, that the 1-jet of $g_{ij}$ at $x$ vanishes and that the 2-jet is expressed in terms of the curvature tensor $R_{ijk\ell}$, at $x$. We thus get:

**Lemma 4.** *There exists a universal bilinear expression* $B(\nabla^\alpha \, df_1 \, , \, \nabla^\beta \, df_2)$ *and a trilinear form* $C(R, df_1, df_2)$ *such that:*

$$\Omega(f_1, f_2) = \left( B(\nabla^\alpha \, df_1, \nabla^\beta \, df_2) + C(R, df_1, df_2) \right) \, dv$$

*where $R$ is the curvature tensor, $\nabla$ the covariant differentiation and $dv$ the volume form of a Riemannian structure compatible with the given conformal structure.*

In order to determine the bilinear expression $B$ we just need to perform the computation of $\Omega(f_1, f_2)$ in the flat case. Note that our notation $\nabla^\alpha$ is ambiguous for $|\alpha| > 1$ since the covariant derivatives do not commute, but only $|\alpha| \leq 2$ will be involved and the corresponding ambiguity is absorbed by the term $C(R, df_1, df_2)$. We shall determine $C$ using conformal invariance of $\Omega(f_1, f_2)$, but let us begin by the computation of $\vee$ in the flat case.

In the flat case we have:

$$\sigma_0^F(x,\xi) = (e_\xi \; i_\xi - i_\xi \; e_\xi) \; \|\xi\|^{-2} \; , \; \sigma_{-k}^F = 0 \qquad \forall k > 0. \tag{2.21}$$

As $\sigma_0^F$ is independent of $x$, the formula (20) simplifies to

$$\sigma_{-4}(x,\xi) = \Sigma \; \frac{1}{\alpha!} \, \frac{1}{\beta!} \, \frac{1}{\delta!} \; \left(\partial_\xi^{\alpha+\delta} \; \sigma_0^F\right) \left(\partial_\xi^\beta \; \sigma_0^F\right) (D^\alpha \; f_1)(x) \left(D^{\beta+\delta} \; f_2\right)(x) \tag{2.22}$$

where the sum is performed for multiindices $\alpha, \beta, \delta$ such that $|\alpha| + |\beta| + |\delta| = 4$, $|\alpha| \geq 1$, $|\beta| \geq 1$.

Let us consider the function of three vector variables $\xi, \mu, \nu$ given by

$$f(\xi,\mu,\nu) = \Sigma \; \frac{1}{\alpha!} \, \frac{1}{\beta!} \, \frac{1}{\delta!} \; \text{trace}\left(\left(\partial_\xi^{\alpha+\delta} \; \sigma_0^F\right) \left(\partial_\xi^\beta \; \sigma_0^F\right)\right) \; \mu^\alpha \; \nu^{\beta+\delta} \tag{2.23}$$

where the sum is performed with the same conditions as in (22). By construction we thus have in the flat case

$$\Omega(f_1, f_2) = \left(\Sigma \; A_{\alpha,\beta} \; (D^\alpha \; f_1)(x) \left(D^\beta \; f_2\right)(x)\right) \; dx^1 \wedge \ldots \wedge dx^4 \tag{2.24}$$

where $\Sigma \; A_{\alpha,\beta} \; \mu^\alpha \; \nu^\beta = \int_{S^3} \; f(\xi,\mu,\nu) \; d^3\xi$.

To determine the function $f(\xi,\mu,\nu)$ we use the equality:

$$f(\xi,\mu,\nu) = g(\xi, \mu+\nu, \nu) + \text{ terms not involving } \mu \tag{2.25}$$

where $g(\xi,\mu,\nu) = \Sigma \; \frac{1}{\alpha!} \, \frac{1}{\beta!} \; \text{trace}\left(\partial_\xi^\alpha \; \sigma_0^F \; \partial_\xi^\beta \; \sigma_0^F\right) \; \mu^\alpha \; \nu^\beta$ with the sum performed for $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha| + |\beta| = 4$. Thus $g(\xi,\mu,\nu)$ reads of from the Taylor expansion of $h(\xi+\mu, \xi+\nu)$ with:

$$h(\xi,\eta) = \text{ trace}\left(\sigma_0^F(\xi) \; \sigma_0^F(\eta)\right) = 2\langle\xi,\eta\rangle^2 \; \|\xi\|^{-2} \; \|\eta\|^{-2} + \text{ constant.} \tag{2.26}$$

A straightforward calculation of the Taylor expansion of $h$ on the diagonal gives:

$$\begin{aligned} g(\xi,\mu,\nu) = \; & 2\|\mu\|^2 \; \langle\xi,\mu\rangle \; \langle\xi,\nu\rangle - 4\langle\xi,\mu\rangle^2 \; \langle\mu,\nu\rangle - 4\langle\xi,\mu\rangle^3 \; \langle\xi,\nu\rangle \\ & + \langle\mu,\nu\rangle^2 - \|\nu\|^2 \; \langle\xi,\mu\rangle^2 - \|\mu\|^2 \; \langle\xi,\nu\rangle^2 + \|\mu\|^2 \; \|\nu\|^2 \\ & + 2\|\nu\|^2 \; \langle\xi,\nu\rangle \; \langle\xi,\mu\rangle - 4\langle\xi,\nu\rangle^2 \; \langle\nu,\mu\rangle - 4\langle\xi,\nu\rangle^3 \; \langle\xi,\mu\rangle. \end{aligned} \tag{2.27}$$

If we use on $S^3$ the normalized volume element of integral one we have:

$$\int_{S^3} \langle \xi, \mu \rangle \, \langle \xi, \nu \rangle \, d^3\xi = \frac{1}{4} \, \langle \mu, \nu \rangle \tag{2.28}$$

$$\int_{S^3} \langle \xi, \mu \rangle^3 \, \langle \xi, \nu \rangle \, d^3\xi = \frac{1}{8} \, \langle \mu, \nu \rangle \, \|\mu\|^2. \tag{2.29}$$

We thus get:

$$\int_{S^3} g(\xi, \mu, \nu) \, d^3\xi = -\|\mu\|^2 \, \langle \mu, \nu \rangle + \langle \mu, \nu \rangle^2 + \frac{1}{2} \, \|\mu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu, \nu \rangle. \tag{2.30}$$

Using equality (2.25) we just need to determine the terms involving both $\mu$ and $\nu$ in the expression:

$$-\|\mu + \nu\|^2 \, \langle \mu + \nu, \nu \rangle + \langle \mu + \nu, \nu \rangle^2 + \frac{1}{2} \, \|\mu + \nu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu + \nu, \nu \rangle$$

and we get the desired result:

$$\Sigma \, A_{\alpha,\beta} \, \mu^\alpha \, \nu^\beta = -\|\mu\|^2 \, \langle \mu, \nu \rangle - \langle \mu, \nu \rangle^2 - \frac{1}{2} \, \|\mu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu, \nu \rangle. \tag{2.31}$$

Using equality (2.24) we get the following formula for $\Omega(f_1, f_2)$ in the flat case:

$$\Omega(f_1, f_2) = \left( \Delta \left( \langle df_1, df_2 \rangle \right) + \langle \nabla df_1, \nabla df_2 \rangle - \frac{1}{2} \, \Delta f_1 \, \Delta f_2 \right) \, dx^1 \wedge \ldots \wedge dx^4 \tag{2.32}$$

where $\Delta = -\Sigma \, \partial_j^2$ is the Laplacian and $\nabla$ the covariant derivative. We can thus use lemma 4 and summarize what we have found so far in the following:

**Lemma 5.** *There exists a universal trilinear form $C(R, df_1, df_2)$ in the curvature $R$ and the covectors $df_1, df_2$ such that, in full generality, one has:*

$$\Omega(f_1, f_2) = \left( \Delta \left( \langle df_1, df_2 \rangle \right) + \langle \nabla df_1, \nabla df_2 \rangle - \frac{1}{2} (\Delta f_1)(\Delta f_2) + C(R, df_1, df_2) \right) \, dv. \tag{2.33}$$

Our next task is to use conformal invariance of $\Omega(f_1, f_2)$ to determine the term $C(R, df_1, df_2)$. Thus let us replace the metric $g_{ij}$ of $\Sigma$ by $(1 + \delta) \, g_{ij}$ where $\delta$ is a smooth function on $\Sigma$ and compute, to first order in $\delta$, the variation of the various terms of formula (2.33).

The perturbation of the Levi Civita connection is given, up to order one in $\delta$, by the following bundle map $T^* \to T^* \otimes T^*$:

$$(\nabla' - \nabla)\omega = -\frac{1}{2} \, \left( \omega \otimes d\delta + d\delta \otimes \omega - \langle d\delta, \omega \rangle \, g \right) \in T^* \otimes T^* \tag{2.34}$$

where we used the symbol $g$ for the metric viewed as an element of $T^* \otimes T^*$.

We can then compute the perturbation, up to order one:

$$\langle \nabla' df_1, \nabla' df_2\rangle' - \langle \nabla df_1, \nabla df_2\rangle =$$
$$\langle (\nabla' - \nabla) df_1, \nabla df_2\rangle + \langle \nabla df_1, (\nabla' - \nabla) df_2\rangle + \langle \nabla df_1, \nabla df_2\rangle' - \langle \nabla df_1, \nabla df_2\rangle.$$

The first term gives, using (2.34) and the equality

$$\langle \nabla df, g\rangle = -\Delta f \qquad \forall f \in C^\infty(\Sigma). \tag{2.35}$$

$$-\frac{1}{2}\ \langle (df_1 \otimes d\delta + d\delta \otimes df_1 - \langle d\delta, df_1\rangle\ g)\,, \nabla df_2\rangle$$
$$= -\frac{1}{2}\ (2\langle df_1, \nabla_{d\delta}(df_2)\rangle + \langle d\delta, df_1\rangle\ \Delta f_2)\,.$$

We can thus rewrite the sum of the first two terms as

$$-\langle d\delta, d\langle df_1, df_2\rangle\rangle - \frac{1}{2}\ \langle d\delta, df_1\rangle\ \Delta f_2 - \frac{1}{2}\ \langle d\delta, df_2\rangle\ \Delta f_1. \tag{2.36}$$

The last two terms just contribute

$$-2\delta\ \langle \nabla df_1, \nabla df_2\rangle. \tag{2.37}$$

We thus have to add (2.36) and (2.37) to get the perturbation of the middle term in (2.33).

Similarly the general formula to order one in $\delta$:

$$(\Delta' - \Delta)h = \delta\Delta(h) - \langle dh, d\delta\rangle \qquad \forall h \in C^\infty(\Sigma) \tag{2.38}$$

shows that the perturbations of the first and third terms of (33) are respectively:

$$-\Delta'\ \langle df_1, df_2\rangle' + \Delta\ \langle df_1, df_2\rangle = \langle d\delta, d\ \langle df_1, df_2\rangle\rangle + \delta\Delta\ \langle df_1, df_2\rangle + \Delta\,(\delta\ \langle df_1, df_2\rangle) \tag{2.39}$$

$$-\frac{1}{2}\,(\Delta' f_1\ \Delta' f_2 - \Delta f_1\ \Delta f_2) = \delta\ \Delta f_1\ \Delta f_2 + \frac{1}{2}\ \langle df_1, d\delta\rangle\ \Delta f_2 + \frac{1}{2}\ \langle df_2, d\delta\rangle\ \Delta f_1. \tag{2.40}$$

Adding (2.36), (2.37), (2.39) and (2.40) gives the following expression for the perturbation $T' - T$ of the sum of the first three terms of (2.33)

$$T' - T = -2\delta T - 2\,\langle d\delta, d\ \langle df_1, df_2\rangle\rangle - \Delta\,(\delta\langle df_1, df_2\rangle) + \delta\ \Delta\,(\langle df_1, df_2\rangle)\,. \tag{2.41}$$

The general identity

$$\Delta(fh) - f\ \Delta h - (\Delta f)h = -2\langle df, dh\rangle \qquad \forall f, h \in C^\infty(\Sigma) \tag{2.42}$$

applied with $f = \delta$, $h = \langle df_1, df_2\rangle$ thus gives:

$$T' - T = -2\delta T - (\Delta\delta)\ \langle df_1, df_2\rangle. \tag{2.43}$$

Thus, as up to order one in $\delta$ we have $(dv)' - dv = 2\delta\ dv$, the differential form $T\ dv$ satisfies, to order one in $\delta$:

$$T'\ (dv)' - T\ dv = -\Delta\delta\ \langle df_1, df_2\rangle\ dv. \tag{2.44}$$

We hence just need to find $C(R, df_1, df_2)$ such that

$$C'\ dv' - C\ dv = \Delta\delta\ \langle df_1, df_2\rangle\ dv. \tag{2.45}$$

The perturbation of the Riemannian curvature $R$ viewed as a linear map $R : \wedge^2 T^* \to \wedge^2 T^*$ is given by:

$$R' - R = -\delta R + 1/2\ \wedge^2 (\nabla d\delta) \tag{2.47}$$

where $\wedge^2(\nabla d\delta)$ is the natural action of the second derivative $\nabla d\delta$ on $\wedge^2 T^*$, at the Lie algebra level. The curvature scalar, $r$ = trace $R$ thus satisfies:

$$r' - r = -\delta r - 3\ (\Delta\delta). \tag{2.48}$$

We hence get the following natural solution of (45):

$$C(R, df_1, df_2) = 1/3\ r\ \langle df_1, df_2\rangle. \tag{2.49}$$

What we know so far is that $1/3\ r\ \langle df_1, df_2\rangle$ is a possible solution. It is in fact the only one since the only other invariant expression $C(R, df_1, df_2)$ that could be added is a multiple of the Ricci tensor applied to $df_1 \otimes df_2$ and one checks that it fails to give, when multiplied by $dv$, a conformally invariant answer.

We can thus summarize what we found as follows:

**Theorem 6.** *Let $\Sigma$ be a 4-dimensional conformal manifold, $X : \Sigma \to \mathbf{R}^d$ a smooth map, $\eta = \eta_{\mu\nu}\ dx^\mu\ dx^\nu$ a smooth metric on $\mathbf{R}^d$. One has:*

$$\begin{aligned}\mathrm{Tr}_\omega\ (\eta_{\mu\nu}[F, X^\mu]\ [F, X^\nu]) = (16\pi^2)^{-1} \int_\Sigma \eta_{\mu\nu} & \left\{ \frac{1}{3}\ r\ \langle dX^\mu, dX^\nu\rangle \right. \\ & \left. + \Delta\ \langle dX^\mu, dX^\nu\rangle + \langle \nabla\ dX^\mu, \nabla\ dX^\nu\rangle - \frac{1}{2}\ (\Delta X^\mu)(\Delta X^\nu) \right\} dv\end{aligned}$$

*where $r$ is the curvature scalar of $\Sigma$, $dv$ its volume form, $\nabla$ its covariant derivative, $\Delta$ its Laplacian for an arbitrary Riemannian metric compatible with the given conformal structure.*

Of course as we saw in the proof the various terms of the formula such as $\frac{1}{3}\ r\ \langle dX^\mu, dX^\nu\rangle$ are not conformally invariant themselves, only their sum is. It is also important to check that the right hand side of the formula is, like obviously the left hand side, a Hochschild 2-cocycle. This allows to double check the constants in front of the various terms, except for the first one.

Theorem 6 gives a natural 4-dimensional analogue of the Polyakov action, and in particular in the special case when the $\eta_{\mu\nu}$ are constant, a natural conformally invariant action for scalar fields $X : \Sigma \to \mathbf{R}$,

$$I(X) = \mathrm{Tr}_\omega\ \left([F, X]^2\right) \tag{2.50}$$

which by theorem 6, can be expressed in local terms, and defines an elliptic differential operator $P$ of order 4 on $\Sigma$ such that:

$$I(X) = \int_{\Sigma} P(X)\, X\, dv. \tag{2.51}$$

This operator $P$ is (up to the factor $\frac{1}{2}$) equal to the Paneitz' operator $P = \Delta^2 + d^* \{2\text{Ricci} - 4/3r\} d$ already known to be the analogue of the scalar Laplacian in 4-dimensional conformal geometry.

Equation 51 uses the volume element $dv$ so that $P$ itself is not conformally invariant, its principal symbol is:

$$\sigma_4(P)\ (x, \xi) = \frac{1}{2}\ \|\xi\|^4 \tag{2.52}$$

which is *positive.*

The conformal anomaly of the functional integral

$$\int e^{-I(X)}\ \pi\ dX(x)$$

is that of $(\det P)^{-1/2}$ and can be computed (cf. [B-O]). The above discussion gives a very clear indication that the induced gravity theory from the above scalar field theory in dimension 4 should be of great interest, in analogy with the 2-dimensional case.

In this paper we only discussed the conformal aspect of noncommutative geometry. We refer to [Co] for the metric aspect and to [C-M] for the general local index formula in the above framework.

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