NONCOMMUTATIVE DIFFERENTIAL GEOMETRY AND THE STRUCTURE OF SPACE TIME

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1 GENERALITIES

The basic data of Riemannian geometry consists of a manifold M whose points are locally labeled by a finite number of real coordinates $\{x^\mu\}$ and a *metric*, which is given by the infinitesimal line element:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\,. \tag{1.1}$$

The distance between two points $x, y \in M$ is given by

$$d(x,y) = \mathrm{Inf}\{\text{Length } \gamma \,|\gamma \text{ is a path between } x \text{ and } y\} \tag{1.2}$$

where

$$\text{Length } \gamma = \int_\gamma ds\,. \tag{1.3}$$

Riemannian geometry is flexible enough to give a good description of space–time in general relativity (up to a sign change). The essential point here is that the differential and integral calculus allows to go from the local to the global, while simple notions of Euclidean geometry continue to make sense. For instance the idea of a straight line gives rise to the notion of geodesic. The geodesic equation

$$\frac{d^2\, x^\mu}{dt^2} = -\Gamma^\mu_{\nu\rho}\, \frac{dx^\nu}{dt}\, \frac{dx^\rho}{dt} \tag{1.4}$$

where $\Gamma^\mu_{\nu\rho} = \frac{1}{2}\, g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha})$, gives the Newton equation of the motion of a particle in the Newtonian potential V provided one uses the metric $dx^2+dy^2+dz^2-(1+2V(x,y,z))dt^2$ (cf. [W] for the more precise formulation). Recent experimental data

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on binary pulsars confirms through general relativity [D-T] that Riemannian geometry works well as a model for space–time on a sufficiently large scale. However, it is not clear ([R]) whether this geometry is adequate for the description of the small scale structure of space–time. The Planck length:

$$\ell_p = (G\hbar/c^3)^{1/2} \sim 10^{-33}\text{cm} \tag{1.5}$$

is considered as a natural lower limit for the precision at which coordinates of an event in the space–time make sense. (See for example [F] or [D-F-R] for a physical argument establishing this limit using quantum mechanics).

In these lectures we shall present a new notion of a geometric space where *points* do not play the central role, thus giving much more freedom for describing the small scale texture of space–time. The proposed framework is sufficiently general to treat discrete spaces, Riemannian manifolds, configuration spaces of quantum field theory, and the duals of discrete groups which are not necessarily commutative. The main problem is to show that the usual geometrical ideas and the tools of the infinitesimal calculus do adapt to this more general framework. It turns out that the operator formalism of quantum mechanics together with the analysis of logarithmic divergences of traces of operators give the generalization of the differential and integral calculus that we are looking for. Several direct applications of this approach are described in theorems 1, 2, and 4.

We consider a geometric space to be a *spectral triple:*

$$(\mathcal{A}, \mathcal{H}, D) \tag{1.6}$$

where $\mathcal{A}$ is an involutive algebra of operators in a Hilbert space $\mathcal{H}$ and D is a selfadjoint operator on $\mathcal{H}$. The involutive algebra $\mathcal{A}$ corresponds to a given space M like in the classical duality "Space $\leftrightarrow$ Algebra" in algebraic geometry. The operator $D^{-1} = ds$ corresponds to the infinitesimal line element in Riemannian geometry.

One can see the difference between this *spectral geometry* and Riemannian geometry in two ways. Firstly it is very important that one does not assume that the algebra $\mathcal{A}$ is commutative anymore. Secondly the infinitesimal ds in spectral geometry becomes an operator and does not commute with elements of $\mathcal{A}$ even if the algebra $\mathcal{A}$ is commutative itself.

As we will see, simple commutation relations between ds and elements of $\mathcal{A}$, together with Poincaré duality, characterize the spectral triples (1.6) which come from Riemannian manifolds (Theorem 6). When the algebra $\mathcal{A}$ is commutative the spectrum of its norm closure $\bar{\mathcal{A}}$ in bounded operators on $\mathcal{H}$ is a compact space M. A point of M is a character of $\bar{\mathcal{A}}$, i.e. a homomorphism from $\bar{\mathcal{A}}$ to $\mathbb{C}$,

$$\chi : \bar{\mathcal{A}} \to \mathbb{C} \,,\; \chi(a+b) = \chi(a) + \chi(b) \,,\; \chi(\lambda a) = \lambda\,\chi(a) \,,\; \chi(ab) = \chi(a)\,\chi(b) \,, \tag{1.7}$$
$$\forall\, a, b \in \bar{\mathcal{A}} \,,\; \forall\, \lambda \in \mathbb{C}.$$

As an example let us take $\mathcal{A}$ to be the group algebra $\mathbb{C}\Gamma$ for a discrete group Γ acting on the Hilbert space $\mathcal{H} = \ell^2(\Gamma)$ by the regular (left) representation of Γ. When the group Γ and hence the algebra $\mathcal{A}$ are commutative then the characters of $\bar{\mathcal{A}}$ are elements of the Pontryagin dual of Γ,

$$\hat{\Gamma} = \{\chi : \Gamma \to U(1) \;;\; \chi(g_1 g_2) = \chi(g_1)\,\chi(g_2) \qquad \forall\, g_1, g_2 \in \Gamma\}\,. \tag{1.8}$$

Elementary notions of differential geometry for the space $\hat{\Gamma}$ continue to make sense in the general case when Γ is no longer commutative. The right column in the following dictionary does not use the commutativity of the algebra $\mathcal{A}$:

Space X	**Algebra** $\mathcal{A}$
Vector bundle	Finite projective module
Differential form of degree k	Hochschild cycle of dimension k
De Rham current of dimension k	Hochschild cocycle of dimension k
De Rham homology	Cyclic cohomology of $\mathcal{A}$

The power of this generalization to the noncommutative case is demonstrated for example in the proof of Novikov conjecture [N] for hyperbolic groups Γ [C-M1].

In the general case the notion of a point given by (1.7) is not of much interest; but the notion of probability measure keeps its full meaning. Such a measure φ, also called a state, is a normalized positive linear form on $\mathcal{A}$ such that $\varphi(1) = 1$,

$$\varphi : \bar{\mathcal{A}} \to \mathbb{C} \, , \ \varphi(a^*a) \geq 0 \, , \quad \forall a \in \bar{\mathcal{A}} \, , \ \varphi(1) = 1 \, . \tag{1.9}$$

Instead of measuring distances between points using the formula (1.2) we measure distances between states φ, ψ on $\bar{\mathcal{A}}$ by a dual formula. This dual formula involves *sup* instead of *inf* and does not use paths in the space

$$d(\varphi, \psi) = \text{Sup} \, \{ |\varphi(a) - \psi(a)| \, ; \ a \in \mathcal{A} \, , \ \|[D, a]\| \leq 1 \} \, . \tag{1.10}$$

Let us show that this formula indeed gives the geodesic distance in the Riemannian case. Let M be a Riemannian compact manifold with a K-orientation, i.e. a spin structure. The associated spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by the representation

$$(f\,\xi)(x) = f(x)\,\xi(x) \qquad \forall x \in M \, , \ f \in \mathcal{A} \, , \ \xi \in \mathcal{H} \tag{1.11}$$

of the algebra $\mathcal{A}$ of functions on M in the Hilbert space

$$\mathcal{H} = L^2(M, S) \tag{1.12}$$

of the square integrable sections of the spinor bundle.

The operator D is the Dirac operator (cf. [L-M]). One can check that the commutator $[D, f]$, for $f \in \mathcal{A} = C^\infty(M)$ is an operator which is the Clifford multiplication by the gradient ∇f and that its operator norm is:

$$\|[D, f]\| = \ \text{Sup}_{x \in M} \|\nabla f(x)\| = \text{Lipschitz norm } f \, . \tag{1.13}$$

Let $x, y \in M$ and φ, ψ be the corresponding characters: $\varphi(f) = f(x)$, $\psi(f) = f(y)$ for $\forall \ f \in \mathcal{A}$. Then formula (1.10) gives the same result as formula (1.2), i.e. it gives the geodesic distance between x and y.

Unlike the formula (1.2) its dual formula (1.10) makes sense in general, namely, for example for discrete spaces and even totally disconnected spaces.

The usual notion of *dimension* of a space is replaced by the *dimension spectrum* which is a subset of $\mathbb{C}$ with real part not bigger than some $\alpha > 0$ if

$$\lambda_n^{-1} = O(n^{-\alpha}) \tag{1.14}$$

where λ_n is the nth eigenvalue of $|D|$.

The relation between local and global data is given by the local index formula (Theorem 4) ([C-M2]).

The characteristic property of *differentiable manifolds* which is carried over to the noncommutative case is *Poincaré duality.* Poincaré duality in ordinary homology is not sufficient to describe homotopy type of manifolds ([Mi-S]) but D. Sullivan ([S2]) showed (in the simply connected PL case of dimension ≥ 5 ignoring 2–torsion) that it is sufficient to replace ordinary homology by KO–homology.

Moreover, K-homology (cf. for example [At]) thanks to the work of Brown, Douglas, Fillmore, Atiyah, and Kasparov admits a fairly simple definition in algebraic terms, given by

Space X	Algebra $\mathcal{A}$
$K_1(X)$	Stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$
$K_0(X)$	Stable homotopy class of $\mathbb{Z}/2$ graded spectral triple

(i.e. for K_0 we suppose that $\mathcal{H}$ is $\mathbb{Z}/2$–graded by γ, where $\gamma = \gamma^*$, $\gamma^2 = 1$ and $\gamma a = a\gamma \quad \forall\, a \in \mathcal{A}$, $\gamma D = -D\gamma$).

This description works for the complex K–homology which is 2–periodic.

The *fundamental class* of a noncommutative space is a class μ in the KR–homology of the algebra $\mathcal{A} \otimes \mathcal{A}^0$ equipped with the involution

$$\tau(x \otimes y^0) = y^* \otimes (x^*)^0 \qquad \forall\, x, y \in \mathcal{A} \tag{1.15}$$

where $\mathcal{A}^0$ denotes the algebra opposite to $\mathcal{A}$. The Kasparov intersection product [K] allows one to formulate Poincaré duality in terms of the invertibility of μ. In KR-homology besides the above data one has to give an anti-linear isometry J on $\mathcal{H}$ which implements the involution τ in as much as

$$JwJ^{-1} = \tau(w) \qquad \forall\, w \in \mathcal{A} \otimes \mathcal{A}^0, \tag{1.16}$$

KR-homology is periodic with period 8 and the dimension modulo 8 is specified by the following commutation rules. One has $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\gamma = \varepsilon'' \gamma J$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ and with n the dimension modulo 8,

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The anti-linear isometry J is given in Riemannian geometry by the charge conjugation operator and in the noncommutative case by the Tomita operator [Ta]. When an operator algebra $\mathcal{A}$ admits a cyclic vector which is cyclic for the commutant $\mathcal{A}'$, the Tomita operator establishes an anti-isomorphism

$$\mathcal{A}'' \to \mathcal{A}' : a \mapsto Ja^*J^{-1} . \tag{1.17}$$

The class μ specifies only the stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ equipped with the isometry J (and $\mathbb{Z}/2$–grading γ if n is even). The non-triviality of this homotopy class shows up in the intersection form

$$K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z} \tag{1.18}$$

given by the Fredholm index of D with coefficients in $K(\mathcal{A} \otimes \mathcal{A}^0)$.

In order to compare different spectral triples in the same homotopy class defined by μ we shall use the following spectral functional

$$\mathrm{Trace}\,(\varpi(D)) , \tag{1.19}$$

where $\varpi : \mathbb{R} \to \mathbb{R}_+$ is a suitable positive function.

Once the algebra $\mathcal{A}$ fixed, a spectral geometry is determined by the unitary equivalence class of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ together with the isometry J. Denote the representation of $\mathcal{A}$ in $\mathcal{H}$ by π. $(\pi_1, \mathcal{H}_1, D_1, J_1)$ is unitarily equivalent to $(\pi_2, \mathcal{H}_2, D_2, J_2)$ if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\pi_1(a)U^* = \pi_2(a) \quad \forall\, a \in \mathcal{A} \ , \ UD_1U^* = D_2 \ , \ UJ_1U^* = J_2 \tag{1.20}$$

(and $U\gamma_1U^* = \gamma_2$ for even n).

The automorphism group $\mathrm{Aut}(\mathcal{A})$ of the involutive algebra $\mathcal{A}$ acts on the variety of spectral geometries by the following composition:

$$\pi'(a) = \pi(\alpha^{-1}(a)) \qquad \forall\, a \in \mathcal{A} \ , \ \alpha \in \mathrm{Aut}(\mathcal{A}) . \tag{1.21}$$

Let $\mathrm{Aut}^+(\mathcal{A})$ be the subgroup of automorphisms which preserve the class μ. $\mathrm{Aut}^+(\mathcal{A})$ acts on the stable homotopy class determined by μ and preserves the action functional (1.19). In general this subgroup is not compact. For Riemannian manifolds it coincides with the group of diffeomorphisms preserving the K–orientation, $\mathrm{Diff}^+(M)$, i.e. the subgroup preserving the spin structure on M. On the other hand , the isotropy group of a given geometry is automatically *compact* (for a unital $\mathcal{A}$). This implies that the action functional (1.19) automatically produces the phenomenon of spontaneous symmetry breaking.

We will show that for the right choice of the algebra $\mathcal{A}$ the functional of action (1.19) added to $\langle \xi, D\xi \rangle$, $\xi \in \mathcal{H}$ gives the standard Glashow-Weinberg-Salam model coupled with gravity. The algebra $\mathcal{A}$ is the tensor product of the algebra of functions on a Riemannian space M and a finite dimensional noncommutative algebra whose spectral geometry is specified by the phenomenological data from particle physics.

Here we shall give a precise meaning to the notion of infinitesimal variable using the operator formalism of quantum mechanics.

The notion of infinitesimal is supposed to have an obvious intuitive meaning. Let us take a particular example ([B-W]). Let us consider darts thrown arbitrarily into a target Ω. Let $dp(x)$ be the probability for the dart to hit the point $x \in \Omega$. It is clear that $dp(x) < \varepsilon$ for any $\varepsilon > 0$ but nevertheless saying that $dp(x) = 0$ is unsatisfactory. The usual formalisms of measure theory or of the theory of differential forms avoid the problem by giving sense to the following expression

$$\int f(x)\, dp(x) \qquad f : \Omega \to \mathbb{C} \tag{2.1}$$

but it is insufficient to give any meaning to expressions like $e^{-\frac{1}{dp(x)}}$. The answer given by nonstandard analysis, a so called nonstandard real, is equally deceiving. From every nonstandard real number one can construct canonically a subset of the interval $[0,1]$, which is not Lebesgue measurable. No such set can be exhibited ([Ste]). This implies that not a single nonstandard real number can actually be exhibited. The formalism which we propose below will give a substantial and computable answer to the above problem.

Our framework consists of a separable Hilbert space decomposed into two orthogonal infinite dimensional subspaces. This decomposition can be described by the linear operator F on $\mathcal{H}$ which is the identity, $F\xi = \xi$, on one subspace and minus identity, $F\xi = -\xi$, on the other; so one has

$$F = F^* \ , \ F^2 = 1 \, . \tag{2.2}$$

This framework is determined uniquely up to isomorphism. The following is the beginning of a long dictionary which translates classical notions into operator language:

Classical	**Quantum**
Complex variable	Operator in $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order α	Compact operator with eigenvalues μ_n satisfying $\mu_n = O(n^{-\alpha})$, $n \to \infty$
Differential of a real real or complex variable	$đf = [F,f] = Ff - fF$
Integral of an infinitesimal of order 1	$⨍ T =$ Coefficient of logarithmic divergence in the trace of T.

The first two lines of the dictionary are familiar from quantum mechanics. The range of a complex variable corresponds to the *spectrum* of an operator. The holomorphic functional calculus gives a meaning to $f(T)$ for all holomorphic functions f on the spectrum of T. It is only holomorphic functions which operate in this generality which reflects the difference between complex and real analysis. When $T = T^*$ is selfadjoint then $f(T)$ has a meaning for all Borel functions f. Notice that every usual random variable X on the probability space (Ω, P) can be trivially considered as a selfadjoint operator. One takes $\mathcal{H} = L^2(\Omega, P)$ and T to be the operator of multiplication by X:

$$(T\xi)(p) = X(p)\,\xi(p) \qquad \forall p \in \Omega\ ,\ \xi \in \mathcal{H}. \tag{2.3}$$

Let us consider the third line of the dictionary. We look for a definition of "infinitesimal variables", i.e. operators T in $\mathcal{H}$ such that

$$\|T\| < \varepsilon \qquad \forall \varepsilon > 0\,, \tag{2.4}$$

where $\|T\| = \operatorname{Sup}\{\|T\xi\|\ ;\ \|\xi\| = 1\}$ is the operator norm. Clearly, if one takes (2.4) literally then one gets $\|T\| = 0$ and $T = 0$ as the unique solution, but one can weaken (2.4) in the following way:

$$\forall \varepsilon > 0\ ,\ \exists \text{ finite dimensional space } E \subset \mathcal{H} \text{ such that } \|T|_{E^\perp}\| < \varepsilon\ . \tag{2.5}$$

Here $E^\perp$ denotes the orthogonal complement to E in the space $\mathcal{H}$:

$$E^\perp = \{\xi \in \mathcal{H}\ ;\ \langle \xi, \eta\rangle = 0 \quad \forall \eta \in E\} \tag{2.6}$$

which is a subspace of finite codimension in $\mathcal{H}$. $T|_{E^\perp}$ denotes the restriction of T to this subspace,

$$T|_{E^\perp} : E^\perp \to \mathcal{H}\,. \tag{2.7}$$

Operators which satisfy the condition (2.5) are the *compact operators*, i.e. those for which the image of the unit ball of $\mathcal{H}$ has compact closure. The operator T is compact if and only if $|T| = \sqrt{T^*T}$ is compact, and this means that the spectrum of $|T|$ is a sequence of eigenvalues $\mu_0 \geq \mu_1 \geq \mu_2 \dots\ ,\ \mu_n \downarrow 0$.

These eigenvalues are the characteristic values of T and one has

$$\mu_n(T) = \operatorname{Inf}\{\|T - R\|\ ;\ R \text{ is an operator of rank } \leq n\} \tag{2.8}$$

$$\mu_n(T) = \operatorname{Inf}\{\|T/E^\perp\|\ ;\ \dim E \leq n\}\,. \tag{2.9}$$

Compact operators form a two-sided ideal $\mathcal{K}$ in the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators in $\mathcal{H}$.

Consider now the fourth entry in our dictionary. The size of the infinitesimal $T \in \mathcal{K}$ is governed by the order of decay of the sequence $\mu_n = \mu_n(T)$ as $n \to \infty$. In particular, for all real positive α the following condition defines infinitesimals of order α:

$$\mu_n(T) = O(n^{-\alpha}) \qquad \text{when } n \to \infty \tag{2.10}$$

(i.e. there exists $C > 0$ such that $\mu_n(T) \leq Cn^{-\alpha} \quad \forall n \geq 1$). Infinitesimals of order α also form a two-sided ideal as it can be checked using (2.8), (cf. [Co]) and moreover,

$$T_j \text{ of order } \alpha_j \Rightarrow T_1T_2 \text{ of order } \alpha_1 + \alpha_2\,. \tag{2.11}$$

(For $\alpha < 1$ this ideal is a normed ideal given by real interpolation between the ideal $\mathcal{L}^1$ of trace class operators and the ideal $\mathcal{K}$ ([Co]).)

Hence, apart from commutativity, intuitive properties of the infinitesimal calculus are fulfilled.

Since the size of the infinitesimal is measured by the sequence $\mu_n \to 0$ it might seem that one does not need the operator formalism at all, that it would be enough to replace the ideal $\mathcal{K}$ in $\mathcal{L}(\mathcal{H})$ by the ideal $c_0(\mathbb{N})$ of sequences converging to zero in the algebra $\ell^\infty(\mathbb{N})$ of bounded sequences. A variable would be a bounded sequence, and an infinitesimal a sequence μ_n, $\mu_n \downarrow 0$. However, this commutative version does not allow for the existence of variables with range a continuum since all elements of $\ell^\infty(\mathbb{N})$ have a point spectrum and a discrete spectral measure. Only *noncommutativity* of $\mathcal{L}(\mathcal{H})$ makes it possible to include in one framework variables with Lebesgue spectrum together with infinitesimal variables.

Noncommutativity of $\mathcal{L}(\mathcal{H})$ is also crucial for the next line of the dictionary. The differential df of a real or complex variable

$$df = \Sigma \frac{\partial f}{\partial x^\mu} dx^\mu \tag{2.12}$$

is replaced by the commutator

$$đ f = [F, f]. \tag{2.13}$$

Going from (2.12) to (2.13) is similar to taking the commutator $[f,g] = fg - gf$ of quantum observables instead of the Poisson bracket $\{f,g\}$ of two functions f,g in classical mechanics.

For a given algebra $\mathcal{A}$ of operators in $\mathcal{H}$ the *dimension* of the corresponding space (in the sense of the dictionary on page 47) is governed by the size of the differentials $đ f$, for $f \in \mathcal{A}$. In dimension p one has

$$đ f \quad \text{of order } \frac{1}{p}, \text{ for any } f \in \mathcal{A}. \tag{2.14}$$

We shall see below (Theorem 1) concrete examples where p is the Hausdorff dimension of Julia sets. Simple algebraic manipulations on the functional

$$\tau(f^0, \ldots, f^n) = \text{Trace}\,(f^0\, đ f^1 \ldots đ f^n) \qquad n \text{ odd},\ n > p \tag{2.15}$$

show that τ is a cyclic cocycle and moreover that the cocycle τ is *integral*, i.e. that

$$\langle \tau, K_1(\mathcal{A}) \rangle \subset \mathbb{Z}. \tag{2.16}$$

This integrality result is a very powerful tool, see [Co].

If the dictionary would have ended here the calculus would lack a vital tool, the *locality*, namely the possibility of neglecting infinitesimals of order > 1. These are contained in the following two–sided ideal

$$\left\{ T \in \mathcal{K}\ ;\ \mu_n(T) = o\left(\frac{1}{n}\right) \right\}. \tag{2.17}$$

where little o has the usual meaning, namely, that $n\mu_n(T) \to 0$ when $n \to \infty$.

Hence if we use the trace like in (2.15) for integration we would encounter two problems:

1. infinitesimals of order 1 are not in the domain of the trace.

2. the trace of infinitesimals of order > 1 is not zero.

The natural domain of the trace is the two-sided ideal $\mathcal{L}^1(\mathcal{H})$ of trace-class operators

$$\mathcal{L}^1 = \left\{ T \in \mathcal{K} \; ; \; \sum_o^\infty \mu_n(T) < \infty \right\} . \tag{2.18}$$

The trace of an operator $T \in \mathcal{L}^1(\mathcal{H})$ is given by the sum

$$\text{Trace}\,(T) = \sum \langle T\xi_i, \xi_i \rangle \tag{2.19}$$

independent of the choice of the orthonormal basis (ξ_i) of $\mathcal{H}$. One has

$$\text{Trace}\,(T) = \sum_o^\infty \mu_n(T) \qquad \text{for } \forall T \geq 0 . \tag{2.20}$$

Let $T \geq 0$ be an infinitesimal of order 1, the only condition on $\mu_n(T)$ is

$$\mu_n(T) = O\left(\frac{1}{n}\right) \tag{2.21}$$

which is not sufficient to insure the finiteness of the sum in (2.20). This shows the nature of both of the problems since the trace is not zero even on the smallest ideal in $\mathcal{L}(\mathcal{H})$, namely, the ideal of finite rank operators, $\mathcal{R}$.
The solution is obtained by employing the Dixmier trace [Dx] , i.e. by the following analysis of the logarithmic divergence of the partial traces

$$\text{Trace}_N(T) = \sum_o^{N-1} \mu_n(T) \; , \; T \geq 0 . \tag{2.22}$$

In fact, it is useful to define $\text{Trace}_\Lambda(T)$ for any positive real $\Lambda > 0$ by the interpolation formula

$$\text{Trace}_\Lambda(T) = \text{Inf}\,\{\|A\|_1 + \Lambda\|B\| \; ; \; A + B = T\} \tag{2.23}$$

where $\|A\|_1$ is the $\mathcal{L}^1$ norm of A, $\|A\|_1 = \text{Trace}\,|A|$, and $\|B\|$ is the operator norm of B. This formula coincides with (2.22) for integer Λ and gives its piecewise affine interpolation for noninteger Λ. Then also ([Co])

$$\text{Trace}_\Lambda(T_1 + T_2) \leq \text{Trace}_\Lambda(T_1) + \text{Trace}_\Lambda(T_2) \qquad \forall \Lambda > 0 \tag{2.24}$$

$$\text{Trace}_{\Lambda_1+\Lambda_2}(T_1 + T_2) \geq \text{Trace}_{\Lambda_1}(T_1) + \text{Trace}_{\Lambda_2}(T_2) \qquad \forall \Lambda_1, \Lambda_2 > 0 \tag{2.25}$$

where T_1, T_2 are positive in (2.25).
Define for all order 1 operators $T \geq 0$

$$\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_e^\Lambda \frac{\text{Trace}_\mu(T)}{\log \mu} \, \frac{d\mu}{\mu} \tag{2.26}$$

which is the Cesaro mean of the function $\frac{\text{Trace}_\mu(T)}{\log \mu}$ over the scaling group $\mathbb{R}_+^*$.
For $T \geq 0$, an infinitesimal of order 1, one has

$$\text{Trace}_\Lambda(T) \leq C \, \log \Lambda \tag{2.27}$$

so that $\tau_\Lambda(T)$ is bounded. The essential property is the following *asymptotic additivity* of the coefficient $\tau_\Lambda(T)$ of the logarithmic divergence (2.27):

$$|\tau_\Lambda(T_1+T_2)-\tau_\Lambda(T_1)-\tau_\Lambda(T_2)|\leq 3C\,\frac{\log(\log\Lambda)}{\log\Lambda} \tag{2.28}$$

for $T_j\geq 0$.

An easy consequence of (2.28) is that any limit point τ of the nonlinear functionals τ_Λ for $\Lambda\to\infty$ defines a positive and linear trace on the two–sided ideal of infinitesimals of order 1,

$$\begin{aligned}\tau(\lambda_1 T_1+\lambda_2 T_2)=\lambda_1\ \tau(T_1)+\lambda_2\ \tau(T_2)\qquad &\forall\,\lambda_j\in\mathbb{C}\\ \tau(T)\geq 0\qquad &\forall\, T\geq 0\\ \tau(ST)=\tau(TS)\qquad &\text{for any bounded } S\\ \tau(T)=0 \text{ if } \mu_n(T)=o\left(\frac{1}{n}\right).\end{aligned} \tag{2.29}$$

In practice the choice of the limit point τ is irrelevant because in all important examples (in particular as a corollary of the axioms in the general framework cf. Section 4) T is a *measurable* operator, i.e.:

$$\tau_\Lambda(T) \text{ converges when } \Lambda\to\infty\,. \tag{2.30}$$

Thus the value $\tau(T)$ is independent of the choice of the limit point τ and is denoted

$$⨍\, T\,. \tag{2.31}$$

The first interesting example is provided by pseudodifferential operators T on a differentiable manifold M. When T is of order 1 (in the sense of (2.21)) it is measurable and $⨍ T$ is the non-commutative residue of T ([Wo], [Ka]). It has a local expression in terms of the distribution kernel $k(x,y)$, $x,y\in M$. For T of order 1 the kernel $k(x,y)$ diverges logarithmically near the diagonal,

$$k(x,y)=-a(x)\log|x-y|+0(1)\ (\text{for}\ \ y\to x) \tag{2.32}$$

where $a(x)$ is a 1–density independent of the choice of Riemannian distance $|x-y|$. Then one has (up to normalization),

$$⨍\, T=\int_M a(x). \tag{2.33}$$

The right hand side of this formula makes sense for all pseudodifferential operators (cf. [Wo]) since one can see that the kernel of such an operator is asymptotically of the form

$$k(x,y)=\sum a_k(x,x-y)+a(x)\log|x-y|+0(1) \tag{2.34}$$

where $a_k(x,\xi)$ is homogeneous of degree $-k$ in ξ, and the 1–density $a(x)$ is defined intrinsically.

The same principle of extension of $⨍$ to infinitesimals of order <1 works for hypoelliptic operators and more generally (cf. Theorem 4) for spectral triples whose dimension spectrum is simple.

After this description of the general framework let us discuss examples. The infinitesimal variable $dp(x)$, which is the probability in the darts game, is given by the operator

$$dp = \Delta^{-1} \tag{2.35}$$

where Δ is the Dirichlet Laplacian for the domain Ω, acting on the Hilbert space $L^2(\Omega)$. Also the algebra of functions $f(x_1, x_2)$, $f : \Omega \to \mathbb{C}$ acts by multiplication operators on $L^2(\Omega)$(cf. (2.3)). From the H. Weyl theorem on the asymptotic behavior of eigenvalues of Δ it follows that dp is of order 1, $f\,dp$ is measurable, and that

$$⨍ f\, dp = \int_\Omega f(x_1, x_2)\, dx_1 \wedge dx_2 \tag{2.36}$$

gives the usual expression for the probability. Note that $e^{-\frac{1}{dp(x)}}$ has now full meaning as the heat kernel.

Now let us show how this infinitesimal calculus can be used to make sense of some expressions like the area of for 4–dimensional manifolds, which do not exist in the usual calculus.

There is a canonical quantization of the infinitesimal calculus over $\mathbb{R}$ which is invariant under translations and dilations. It is given by the representation of the algebra of functions f on $\mathbb{R}$ as multiplication operators in $L^2(\mathbb{R})$ (cf. (2.3)), while the operator F in $\mathcal{H} = L^2(\mathbb{R})$ is the Hilbert transform ([St])

$$(f\xi)(s) = f(s)\xi(s) \qquad \forall\, s \in \mathbb{R}\,,\ \xi \in L^2(\mathbb{R})\,,\ (F\xi)(t) = \frac{1}{\pi i}\int \frac{\xi(s)}{s-t}\, ds\,. \tag{2.37}$$

One has a unitary equivalent description for $S^1 = P_1(\mathbb{R})$ with $\mathcal{H} = L^2(S^1)$ and

$$F\, e_n = \mathrm{Sign}\,(n)\, e_n\ ,\ e_n(\theta) = \exp\,(in\theta) \qquad \forall\, \theta \in S^1\ ,\ (\mathrm{Sign}\,(0) = 1)\,. \tag{2.38}$$

The operator $đ f = [F, f]$, for $f \in L^\infty(\mathbb{R})$, is represented by the kernel $\frac{1}{\pi i}\, k(s,t)$, with

$$k(s,t) = \frac{f(s) - f(t)}{s-t}\,. \tag{2.39}$$

Since f and F are bounded operators, $đ f = [F, f]$ is also bounded for all bounded measurable f on S^1, and it makes sense to talk about $|đ f|^p$ for all $p > 0$. Let us give an example where one has to use such an expression for a non differentiable f. Let J be the Julia set associated to the iterations of the map ($c \in \mathbb{C}$)

$$\varphi(z) = z^2 + c\ ,\ J = \partial B\ ,\ B = \{z \in \mathbb{C}\ ;\ \sup_{n\in\mathrm{N}} |\varphi^n(z)| < \infty\}\,. \tag{2.40}$$

For small c, J is a Jordan curve and B is the bounded component of its complement. The Riemann mapping theorem provides us with a conformal equivalence $D \sim B$ of the unit disc $D = \{z \in \mathbb{C},\ |z| < 1\}$ with B. By a theorem of Caratheodory it extends continuously to a homeomorphism $Z : S^1 \to J$ where $S^1 = \partial D$. Since (by a result of D. Sullivan) the Hausdorff dimension p of J is strictly bigger than 1 (for $c \neq 0$) the function Z is nowhere of bounded variation on S^1 and $|Z'|$, the absolute value of the derivative of Z, does not make sense as a distribution. However $|đ Z|$ is well defined and one has:

Theorem 1 .

1. *$|đ Z|$ is an infinitesimal of order $\frac{1}{p}$.*

2. *For every continuous function h on J, the operator $h(Z)\,|đ Z|^p$ is measurable.*

3. *$\exists\,\lambda > 0$,*
$$\int\!\!\!\!\!- h(Z)\,|đ Z|^p = \lambda \int_J h\, d\Lambda_p \qquad \forall\, h \in C(J)$$
where $d\Lambda_p$ denotes the Hausdorff measure on J.

The first statement of the theorem uses a result of V.V. Peller which characterizes functions f for which $\mathrm{Trace}\,(|đ f|^\alpha) < \infty$. The constant λ is determined by the asymptotic expansion in $n \in \mathbb{N}$ for the distance in $L^\infty(S^1)$ between Z and restrictions to S^1 of rational fractions with at most n poles outside the unit disc. This constant is of order $\sqrt{p-1}$ and so it is zero for $p = 1$. This is related to a specific feature of dimension 1 manifolds, namely, the differential $đ f$ of a function $f \in C^\infty(S^1)$ is not just of order $(\dim S^1)^{-1} = 1$ but is even trace-class, with

$$\mathrm{Trace}\,(f^0\, đ f^1) = \frac{1}{\pi i} \int_{S^1} f^0\, df^1 \qquad \forall\, f^0, f^1 \in C^\infty(S^1)\,. \tag{2.41}$$

In fact, by a classical result of Kronecker $đ f$ is of finite rank if and only if f is a rational fraction (cf. [P]).

The quantized calculus can be used in the same manner to describe the projective space $P_1(K)$ over any local field K (i.e. non discrete locally compact). This calculus is invariant under projective transformations. The special cases of $K = \mathbb{C}$ and $K = \mathbb{H}$ (the field of quaternions) are examples of the calculus on oriented conformal even–dimensional compact manifolds, $M = M^{2n}$. The calculus is defined as follows:

$$\mathcal{H} = L^2(M, \Lambda^n T^*)\ ,\ (f\xi)(p) = f(p)\,\xi(p) \qquad \forall\, f \in L^\infty(M)\ ,\ F = 2P - 1. \tag{2.42}$$

where the scalar product on the Hilbert space of differential forms of degree $n = \frac{1}{2}\dim M$ is given by $\langle \omega_1, \omega_2 \rangle = \int \omega_1 \wedge * \omega_2$ which only depends on the conformal structure on M. The operator P is the orthogonal projection on the subspace of exact forms.

Consider $n = 1$, i.e. M being a Riemann surface. An easy calculation shows that

$$\int\!\!\!\!\!- đ f\, đ g = -\frac{1}{\pi} \int df \wedge * dg \qquad \forall\, f, g \in C^\infty(M)\,. \tag{2.43}$$

Let X be a smooth map from M to the space $\mathbb{R}^N$ equipped with a Riemannian metric $g_{\mu\nu}\, dx^\mu\, dx^\nu$. The components X^μ of the map X are functions on M. One has

$$\int\!\!\!\!\!- g_{\mu\nu}\, đ X^\mu\, đ X^\nu = -\frac{1}{\pi} \int_M g_{\mu\nu}\, dX^\mu \wedge * dX^\nu \tag{2.44}$$

where the right hand side is the Polyakov action in string theory. However, the equality (2.44) does not hold for $n = 4$: the right hand side is not very interesting because it is not conformally invariant but the left hand side is still conformally invariant, because $đ X^\mu = [F, X^\mu]$ and F are conformally invariant. It defines the natural conformal analog of the Polyakov action in the 4-dimensional case. A calculation yields:

Theorem 2 . *Let X be a smooth map from M^4 to $(\mathbb{R}^N, g_{\mu\nu}\, dx^\mu\, dx^\nu)$,*

$$\int\!\!\!\!\!- g_{\mu\nu}(X)\, đX^\mu\, đX^\nu = (16\pi^2)^{-1} \int_M g_{\mu\nu}(X)$$
$$\left\{\tfrac{1}{3} r\langle dX^\mu, dX^\nu\rangle - \Delta\langle dX^\mu, dX^\nu\rangle + \langle\nabla dX^\mu, \nabla dX^\nu\rangle - \tfrac{1}{2}(\Delta X^\mu)(\Delta X^\nu)\right\} dv$$

where for the right hand side one uses a Riemannian structure η on M, compatible with the given conformal structure. The scalar curvature r, the Laplacian Δ, the Levi–Civita connection ∇, and the measure dv are defined by η but the result is independent of its choice.

Theorem 2 is related with the following formula expressing the Hilbert–Einstein action as the area of the four dimensional manifold (cf. [Kas] [K-W])

$$\int\!\!\!\!\!- ds^2 = \frac{-1}{96\pi^2} \int_{M_4} r\sqrt{g}\; d^4x \tag{2.45}$$

($dv = \sqrt{g}\; d^4x$ is the volume form and $ds = D^{-1}$ the length element, i.e. the inverse of the Dirac operator).

When the metric $g_{\mu\nu}\, dx^\mu\, dx^\nu$ on $\mathbb{R}^N$ is invariant under translations the action functional of Theorem 2 is given by the Paneitz operator on M. It is a fourth order operator which plays the role of the Laplacian in conformal geometry ([B-O]). Its conformal anomaly was computed by T. Branson [B].

Let us go back to the case $n = 2$ and modify the conformal structure on M by a Beltrami differential $\mu(z,\bar z)\, d\bar z/dz$, $|\mu(z,\bar z)| < 1$. Thus if z is a conformal local coordinate, we now measure angles at $z \in M$ by the identification

$$T_z(M) \to \mathbb{C}:\;\; X \mapsto \langle X, dz + \mu(z,\bar z)\, d\bar z\rangle \tag{2.46}$$

instead of $X \mapsto \langle X, dz\rangle$. The quantized calculus on M associated to the new conformal structure has the same $\mathcal{H}$, $\mathcal{A}$, and representation of $\mathcal{A}$ in $\mathcal{H}$ unchanged but the operator F is replaced by F' with

$$F' = (\alpha F + \beta)(\beta F + \alpha)^{-1}\;,\;\; \alpha = (1-m^2)^{-1/2}\;,\;\; \beta = m(1-m^2)^{-1/2}. \tag{2.47}$$

Here m is the operator in $\mathcal{H} = L^2(M, \Lambda^1 T^*)$ given by the endomorphism of the vector bundle $\Lambda^1 T^* = \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$ with matrix:

$$m(z,\bar z) = \begin{bmatrix} 0 & \bar\mu(z,\bar z)\, d\bar z/dz \\ \mu(z,\bar z)\, dz/d\bar z & 0 \end{bmatrix}. \tag{2.48}$$

The crucial properties of the operator $m \in \mathcal{L}(\mathcal{H})$ are as follows:

$$\|m\| < 1\;,\;\; m = m^*\;,\;\; mf = fm \qquad \forall f \in \mathcal{A} = C^\infty(M) \tag{2.49}$$

and the deformation (2.47) is a particular case of the

Proposition 3 . *Let $\mathcal{A}$ be an involutive algebra of operators in $\mathcal{H}$ and*

$$N = \mathcal{A}' = \{T \in \mathcal{L}(\mathcal{H})\,;\, Ta = aT \quad \forall a \in \mathcal{A}\}$$

the von Neumann algebra commutant of $\mathcal{A}$.

1. *The following formula defines an action of the group* $G = GL_1(N)$ *of invertible elements of* N *on the operators* F, *that satisfy* $F = F^*$, $F^2 = 1$

$$g(F) = (\alpha F + \beta)(\beta F + \alpha)^{-1} \qquad \forall g \in G$$

where $\alpha = \frac{1}{2}(g - (g^{-1})^*)$, $\beta = \frac{1}{2}(g + (g^{-1})^*)$.

2. *One has*

$$[g(F), a] = Y[F, a]Y^* \quad \forall a \in \mathcal{A}, \quad \textit{where} \qquad Y = (\beta F + \alpha)^{*-1}.$$

The last equality shows that the transformation $F \to g(F)$ preserves the condition

$$[F, a] \in J \tag{2.50}$$

for all two–sided ideals $J \subset \mathcal{L}(\mathcal{H})$. Only *measurability* of the Beltrami differential μ is needed for m to satisfy (2.50), and similarly one only needs that the conformal structure on M is measurable in order to define the associated quantized calculus. More to that, the second equality in proposition 3 shows that the regularity condition on $a \in L^\infty(M)$ imposed by (2.50) only depends on the quasi-conformal structure on the manifold M ([C-S-T]). A local homeomorphism φ of $\mathbb{R}^n$ is *quasi-conformal* if and only if there exists $K < \infty$ such that

$$H_\varphi(x) = \quad \text{Lim sup}_{r \to 0} \frac{\max\{|\varphi(x) - \varphi(y)|\,;\, |x - y| = r\}}{\min\{|\varphi(x) - \varphi(y)|\,;\, |x - y| = r\}} \leq K \ , \ \forall x \in \text{Domain}\, \varphi\,. \tag{2.51}$$

A quasi–conformal structure on a topological manifold M^n is given by a quasi–conformal atlas. The discussion above applies to the general case (n even) ([C-S-T]) and shows that the quantized calculus is well defined on all quasi–conformal manifolds. The result of D. Sullivan [S1] based on [Ki] shows that all topological manifolds M^n, $n \neq 4$ admit a quasi–conformal structure. Using the quantized calculus and cyclic cohomology instead of the differential calculus and Chern–Weil theory one gets ([C-S-T]) a local formula for the topological Pontryagin classes of M^n.

3 LOCAL INDEX FORMULA AND THE TRANSVERSE FUNDAMENTAL CLASS.

In this section we show how the infinitesimal calculus allows to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$. We will apply the general result to the cross product of a manifold by a group of diffeomorphisms.

Let us make the following regularity hypothesis on $(\mathcal{A}, \mathcal{H}, D)$

$$a \text{ and } \ [D, a] \ \in \ \cap\, \text{Dom}\, \delta^k, \ \forall a \in \mathcal{A} \tag{3.1}$$

where δ is the derivation $\delta(T) = [|D|, T]$ for any operator T.

Let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$, $\delta^k([D, a])$. The *dimension* of a spectral triple is bounded above by $p > 0$ if and only if $a(D + i)^{-1}$ is an infinitesimal of order $\frac{1}{p}$ for any $a \in \mathcal{A}$. When $\mathcal{A}$ is unital it depends only on the spectrum of D.

The precise notion of dimension is given by the subset $\Sigma \subset \mathbb{C}$ of singularities of the analytic functions

$$\zeta_b(z) = \text{Trace}\,(b|D|^{-z}) \qquad \text{Re}\, z > p \;,\; b \in \mathcal{B}\,. \tag{3.2}$$

We assume that Σ is discrete and simple, i.e. that ζ_b can be extended to $\mathbb{C}/\Sigma$ with simple poles in Σ.

We refer to [C-M2] for the case of a spectrum with multiplicities.

The Fredholm index of the operator D determines an additive map $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$ given by the equality

$$\varphi([u]) = \text{Index}\,(PuP) \;,\; u \in GL_1(\mathcal{A}) \tag{3.3}$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}\,(D)$.

This map is calculated by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, \ldots, a^n) = \text{Trace}\,(a^0[F, a^1] \ldots [F, a^n]) \qquad \forall\, a^j \in \mathcal{A} \tag{3.4}$$

where $F = \text{Sign}\; D$ and $n \geq p$ is an odd integer.

It is difficult to compute τ in general because the formula (3.4) employs the ordinary trace instead of the local trace $⨍$.

This problem is solved by the following general formula:

Theorem 4 . *([C-M2]).*

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple satisfying the hypothesis (3.1) and (3.2). Then

1. *The equality*

$$⨍ P = \text{Res}_{z=0}\, \text{Trace}\,(P|D|^{-z})$$

defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.

2. *There is only a finite number of non-zero terms in the following formula. It defines the odd components $(\varphi_n)_{n=1,3,\ldots}$ of a cocycle in the bicomplex (b, B) of $\mathcal{A}$,*

$$\varphi_n(a^0, \ldots, a^n) = \sum_k c_{n,k} ⨍ a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)}\, |D|^{-n-2|k|} \qquad \forall\, a^j \in \mathcal{A}$$

where the following notations are used: $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2T - TD^2$, k is a multi-index, $|k| = k_1 + \ldots + k_n$,

$$c_{n,k} = (-1)^{|k|}\sqrt{2i}(k_1! \ldots k_n!)^{-1}\,((k_1+1)\ldots(k_1+k_2+\ldots+k_n+n))^{-1}\,\Gamma\left(|k| + \frac{n}{2}\right).$$

3. *The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.*

Let us remind that the bicomplex (b, B) is given by the following operators acting on multi-linear forms on $\mathcal{A}$,

$$(b\varphi)(a^0, \ldots, a^{n+1}) =$$
$$\textstyle\sum_0^n (-1)^j\, \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1}\, \varphi(a^{n+1}a^0, a^1, \ldots, a^n) \tag{3.5}$$

$$B = AB_0 \;,\; B_0\,\varphi(a^0, \ldots, a^{n-1}) = \varphi(1, a^0, \ldots, a^{n-1}) - (-1)^n\, \varphi(a^0, \ldots, a^{n-1}, 1)$$
$$(A\psi)(a^0, \ldots, a^{n-1}) = \textstyle\sum_0^{n-1}(-1)^{(n-1)j}\, \psi(a^j, a^{j+1}, \ldots, a^{j-1})\,. \tag{3.6}$$

For the normalization of the pairing between HC^* and $K(\mathcal{A})$ see [Co].

Remarks.

(a) The statement of Theorem 4 remains valid when D is replaced by $D|D|^\alpha$, $\alpha \geq 0$.

(b) In the even case, i.e. when $\mathcal{H}$ is $\mathbb{Z}/2$ graded by γ,

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma a = a\gamma \quad \forall\, a \in \mathcal{A}, \; \gamma D = -D\gamma,$$

there is an analogous formula for a cocycle (φ_n), n even, which gives the Fredholm index of D with coefficients in K_0. However, φ_0 is not expressed in terms of the residue $\int\!\!\!\!\!\!-$ because it is not local for a finite dimensional $\mathcal{H}$ (cf. [C-M2]).

(c) There exists an analogous formula for the case when the dimension spectrum Σ has multiplicities. There are some correction terms, their number is finite and bounded independently of the multiplicity (cf. [C-M2]).

The dimension spectrum of a manifold M is the set $\{0, 1, \ldots, n\}$, $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z \notin \mathbb{R}$ in the dimension spectrum.

Starting from a manifold M we are going to perform a general geometrical construction which will yield a spectral triple satisfying the above hypothesis (3.1) and (3.2) and give the fundamental class in K–homology of a K–oriented manifold M without breaking the symmetry of $\mathrm{Diff}^+(M)$, the group of diffeomorphisms of M which preserve K–orientation. More precisely, we are looking for a spectral triple, $(C^\infty(M), \mathcal{H}, D)$ in the same K–homology class as the Dirac operator associated to a Riemannian metric (cf. (1.12) and (1.13)) but which is equivariant under the action of the group $\mathrm{Diff}^+(M)$ in the sense of [K]. This means that one has a unitary representation $\varphi \to U(\varphi)$ of $\mathrm{Diff}^+(M)$ in $\mathcal{H}$ such that

$$U(\varphi)\, f\, U(\varphi)^{-1} = f \circ \varphi^{-1} \qquad \forall\, f \in C^\infty(M)\ ,\ \varphi \in \mathrm{Diff}^+(M) \tag{3.7}$$

and

$$U(\varphi)\, D\, U(\varphi)^{-1} - D \text{ is bounded for all } \varphi \in \mathrm{Diff}^+(M)\,. \tag{3.8}$$

When D is the Dirac operator associated to a Riemannian structure, the principal symbol of D in turn determines the Riemannian metric and hence diffeomorphisms which satisfy (3.8) are isometries.

A solution to this problem is essential in order to define the transverse geometry of foliations. It is obtained in two steps. The first step consists of using the negative curvature metric on $GL(n)/O(n)$ and the "dual Dirac" operator of Miscenko and Kasparov ([Col]) to reduce the problem to the action of $\mathrm{Diff}^+(M)$ on the total space P of the bundle of metrics over M. The second step, following the idea of Hilsum and Skandalis ([H-S]), is to use hypoelliptic operators to construct D on P.

Although the equivariant geometry obtained on P is finite dimensional and satisfies hypothesis (3.1) and (3.2), the geometry obtained for M by using the intersection product with the "dual Dirac" is infinite dimensional and θ-summable,

$$\mathrm{Trace}\,(e^{-\beta D^2}) < \infty \qquad \forall\, \beta > 0\,. \tag{3.9}$$

By construction the fiber of $P \xrightarrow{\pi} M$ is the quotient $F/O(n)$ of the $GL(n)$-principal bundle F of frames on M by the action of the orthogonal group $O(n) \subset GL(n)$. The space P admits a canonical foliation: the vertical foliation $V \subset TP$, $V = \operatorname{Ker} \pi_*$ and on the fibers V and on $N = (TP)/V$ the following Euclidean structures. A choice of $GL(n)$-invariant Riemannian metric on $GL(n)/O(n)$ determines a metric on V. The metric on N is defined tautologically: for every $p \in P$ one has a metric on $T_{\pi(p)}(M)$ which is isomorphic to N_p by π_*.

This construction is functorial for diffeomorphisms on M.

The hypoelliptic calculus adapted to this structure is a particular case of the pseudodifferential calculus on Heisenberg manifolds ([B-G]). One simply modifies the notion of homogeneity of symbols $\sigma(p,\xi)$ by using the following homotheties:

$$\lambda \cdot \xi = (\lambda \xi_v, \lambda^2 \xi_n) \ , \ \forall \lambda \in \mathbb{R}_+^* \tag{3.10}$$

where ξ_v, ξ_n are vertical and perpendicular to vertical components of the covector ξ. The formula (3.10) depends on local coordinates (x_v, x_n) adapted to the vertical foliation, but the corresponding pseudodifferential calculus does not depend upon this choice. The principal symbol of a hypoelliptic operator of order k is a function, homogeneous of degree k in the sense of (3.10) on the fiber $V^* \oplus N^*$. The distribution kernel $k(x,y)$ of a pseudodifferential operator T in this hypoelliptic calculus has the following behavior near the diagonal

$$k(x,y) \sim \sum a_j(x, x-y) - a(x) \log |x-y|' + 0(1) \tag{3.11}$$

where a_j is homogeneous of degree $(-j)$ in $(x-y)$ in the sense of (3.10) and where the metric $|x-y|'$ is locally of the form

$$|x-y|' = ((x_v - y_v)^4 + (x_n - y_n)^2)^{1/4} . \tag{3.12}$$

Like in the ordinary pseudodifferential calculus the residue is extended to operators of all degrees and is given by the equality

$$⨍ T = \frac{1}{v+2m} \int a(x) \tag{3.13}$$

where the 1–density $a(x)$ does not depend on the choice of metric $|\ |'$ and where $v = \dim V$, $m = \dim N$ and $v + 2m$ is the Hausdorff dimension of the metric space $(P, |\ |')$.

The operator D is defined by the equation $D|D| = Q$ where Q is a differential hypoelliptic operator of degree 2. For v even, Q is obtained by combining the signature operator $d_V d_V^* - d_V^* d_V$ with the transverse Dirac operator, where d_V is the vertical differential. (We use the metaplectic cover $M\ell(n)$ of $GL(n)$ to define the spin structure on M.) The explicit formula for Q uses an affine connection on M. The choice of this connection does not affect the *principal hypoelliptic symbol* of Q and therefore of D which ensures that D is invariant as in (3.8) under diffeomorphisms of M.

Let us give the explicit formula for Q in the case $n = 1$, i.e. for $M = S^1$. We replace P by the suspension $SP = \mathbb{R} \times P$ in order to consider the case where the vertical dimension is even. A point of $SP = \mathbb{R} \times P$ is parameterized by three coordinates $\alpha \in \mathbb{R}$ and $p = (s, \theta)$ where $\theta \in S^1$ and $s \in \mathbb{R}$ defines the metric $e^{2s}(d\theta)^2$ for $\theta \in S^1$.

We endow SP with the measure $\nu = d\alpha\, ds\, d\theta$ and represent the algebra $C_c^\infty(SP)$ by multiplication operators on $\mathcal{H} = L^2(SP,\nu) \otimes \mathbb{C}^2$. Functoriality of the construction above gives the following unitary representation of the group $\mathrm{Diff}^+(S^1)$,

$$(U(\varphi)^{-1}\xi)(\alpha,s,\theta) = \varphi'(\theta)^{1/2}\,\xi(\alpha, s-\log\varphi'(\theta), \varphi(\theta))\,. \tag{3.14}$$

The operator Q is given by the formula

$$Q = -2\partial_\alpha\partial_s\sigma_1 + \frac{1}{i}\,e^{-s}\partial_\theta\sigma_2 + \left(\partial_s^2 - \partial_\alpha^2 - \frac{1}{4}\right)\sigma_3 \tag{3.15}$$

where $\sigma_1, \sigma_2, \sigma_3 \in M_2(\mathbb{C})$ are three Pauli matrices.

In the hypoelliptic calculus the operator ∂_θ has *degree 2* which shows the hypoellipticity of Q.

Long calculation gives the following result ([C-M3]):

Theorem 5 . *Let $\mathcal{A}$ be the crossed product of $C_c^\infty(SP)$ by* $\mathrm{Diff}^+(S^1)$.

1. *The spectral triple $(\mathcal{A},\mathcal{H},D)$ (where $\mathcal{A}$ acts on $\mathcal{H}$ by (3.14) and $D|D| = Q$) satisfies the hypotheses (3.1) and (3.2); its dimension spectrum is $\Sigma = \{0,1,2,3,4\}$.*

2. *The only nonzero term of the associated cocycle (Theorem 4) is φ_3, which is cohomological to 2ψ where ψ is the 3 cyclic cocycle of the transversal fundamental class of the crossed product.*

From Theorem 4 follows the *integrality* of 2ψ, i.e. that the pairing $\langle 2\psi, K_1(\mathcal{A})\rangle$ is an integer. The 3 cocycle ψ is given by the following formula (cf. [Co])

$$\begin{aligned} \psi(f^0U(\varphi_0)\,,\ f^1U(\varphi_1)\,,\ f^2U(\varphi_2)\,,\ f^3U(\varphi_3)) = 0 \quad &\text{if} \quad \varphi_0\varphi_1\varphi_2\varphi_3 \neq 1 \\ \text{and } = \int h^0\, dh^1 \wedge dh^2 \wedge dh^3 \ &\text{if} \quad \varphi_0\varphi_1\varphi_2\varphi_3 = 1 \\ \text{where } \ h^0 = f^0\ ,\ h^1 = (f^1)^{\varphi_0}\ ,\ h^2 = (f^2)^{\varphi_0\varphi_1}\ ,\ &h^3 = (f^3)^{\varphi_0\varphi_1\varphi_2}\,. \end{aligned} \tag{3.16}$$

The homology between φ_3 and 2ψ involves the action of the Hopf algebra generated by the following linear transformations of the algebra $\mathcal{A}$ (for the relation of δ_3 to Godbillon Vey invariant see [Co])

$$\begin{aligned} \delta_1(fU(\varphi)) = (\partial_\alpha f)\,U(\varphi)\ ,\ \delta_2(fU(\varphi)) = (\partial_s f)\,U(\varphi)\ , \\ \delta_3(fU(\varphi)) = f\,e^{-s}\,\partial_\theta \log(\varphi^{-1})'\,U(\varphi)\ ,\ X(fU(\varphi)) = e^{-s}(\partial_\theta f)\,U(\varphi)\,. \end{aligned} \tag{3.17}$$

The compatibility with the multiplication in $\mathcal{A}$ is given by the coproduct rules

$$\Delta\delta_j = \delta_j \otimes 1 + 1 \otimes \delta_j \qquad j = 1,2,3 \tag{3.18}$$

(i.e. δ_j are derivations in $\mathcal{A}$)

$$\Delta X = X \otimes 1 + 1 \otimes X - \delta_3 \otimes \delta_2\,. \tag{3.19}$$

The last equation shows that X is of degree 2 not only from its degree as a pseudodifferential operator in the hypoelliptic calculus but also in an algebraic manner since ΔX involves a tensor product of two derivations.

4 THE NOTION OF MANIFOLD AND THE AXIOMS OF GEOMETRY

Let us first characterize the spectral triples corresponding to ordinary Riemannian geometry (Theorem 6 below). Let the dimension n be given and $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with $\mathbb{Z}/2$–grading γ when n is even.

The axioms for *commutative* geometry are the following:

(1) (Dimension) $ds = D^{-1}$ is an infinitesimal of order $\frac{1}{n}$.

(2) (Order one) $[[D, f], g] = 0 \quad \forall\, f, g \in \mathcal{A}$.

(3) (Smoothness) For all $f \in \mathcal{A}$, both f and $[D, f]$ belong to $\cap_k$ Domain δ^k, where δ is the derivation $\delta(T) = [|D|, T]$.

(4) (Orientability) There exists a Hochschild cocycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ such that $\pi(c) = 1$ (n odd) or $\pi(c) = \gamma$ (n even), where π: $\mathcal{A}^{\otimes(n+1)} \to \mathcal{L}(\mathcal{H})$ is the unique linear map such that

$$\pi(a^0 \otimes a^1 \otimes \cdots \otimes a^n) = a^0[D, a^1]\ldots[D, a^n] \quad \forall\, a^j \in \mathcal{A}.$$

(5) (Finiteness) The $\mathcal{A}$-module $\mathcal{E} = \cap_k$ Domain D^k is finite projective. The following identity defines a Hermitian structure on $\mathcal{E}$,

$$\langle a\xi, \eta\rangle = \int\!\!\!\!\!\!- \, a(\xi, \eta)\, ds^n \qquad \forall\, \xi, \eta \in \mathcal{E}\ ,\ a \in \mathcal{A}\,.$$

(6) (Poincaré duality) The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ given by the composition of the Fredholm index of D with the diagonal,

$$m_* : K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to K_*(\mathcal{A} \otimes \mathcal{A}) \to K_*(\mathcal{A}),$$

is *invertible.*

(7) (Reality) There exists an anti-linear isometry J on $\mathcal{H}$ such that

$$Ja^*J^{-1} = a \quad \forall a \in \mathcal{A} \ \text{ and } \ J^2 = \varepsilon, JD = \varepsilon' DJ, J\gamma = \varepsilon'' \gamma J$$

where the values of $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are given by the table (1.16) as functions of n modulo 8.

Axioms (2) and (4) describe the presentation of the abstract algebra denoted by $(\mathcal{A}, ds)$ generated by $\mathcal{A}$ and $ds = D^{-1}$.

Theorem 6 *Let $\mathcal{A} = C^\infty(M)$, where M is a compact smooth manifold.*

1. Let π be unitary representation of $(\mathcal{A}, ds)$ satisfying Axioms (1) to (7). Then there exists a unique Riemannian structure g on M such that the geodesic distance is given by

$$d(x, y) = \operatorname{Sup}\{|a(x) - a(y)|\ ;\ a \in \mathcal{A}\ ,\ \|[D, a]\| \leq 1\}\,.$$

2. *The metric $g = g(\pi)$ depends only on the unitary equivalence class of π. Fibers of the map {unitary equivalence classes} $\to g(\pi)$ are a finite union of affine spaces $\mathcal{A}_\sigma$ parameterized by the spin structures σ on M.*

3. *The functional $⨍ ds^{n-2}$ is a positive quadratic form on each $\mathcal{A}_\sigma$ with a unique minimum π_σ.*

4. *π_σ is the representation of $(\mathcal{A}, ds)$ in $L^2(M, S_\sigma)$ given by multiplication operators and the Dirac operator associated to the Levi–Civita connection of the metric g.*

5. *The value of $⨍ ds^{n-2}$ the π_σ is the Hilbert–Einstein action of the metric g,*

$$⨍ ds^{n-2} = -c_n \int r \sqrt{g}\, d^n x \ , \ c_n = \tfrac{n-2}{12} (4\pi)^{-n/2} 2^{[n/2]} \Gamma\left(\tfrac{n}{2}+1\right)^{-1} .$$

To understand the meaning of this theorem consider the most simple example, namely, the verification that the geometry of the circle S^1 of length 2π is completely specified by the presentation:

$$U^{-1}[D, U] = 1 \ , \ \text{where } UU^* = U^*U = 1 . \tag{4.1}$$

Then the algebra $\mathcal{A}$ is the algebra of smooth functions of the single element U. One has $S^1 = \text{Spectrum}\ (\mathcal{A})$ and the equality 4.1 is the simplest case of Axiom (4).

Remarks.

(a) We should not have to assume in the statement of Theorem 6 that the algebra $\mathcal{A}$ is equal to the algebra of smooth functions on a manifold. For a commutative $\mathcal{A}$ it should in fact follow from the axioms that the spectrum of $\mathcal{A}$ is a smooth manifold M and that $\mathcal{A} = C^\infty(M)$. From Axioms (3) and (5) it follows that if $\mathcal{A}''$ is the von Neumann algebra generated by $\mathcal{A}$ ($\mathcal{A}''$ is the weak closure and the bicommutant of $\mathcal{A}$ in $\mathcal{H}$) one has

$$\mathcal{A} = \left\{ T \in \mathcal{A}'' \ ; \ T \in \bigcap_{k>0} \text{Dom}\, \delta^k \right\} \tag{4.2}$$

$\mathcal{A}$ is uniquely specified inside $\mathcal{A}''$ by fixing D (i.e. the geometry $(\mathcal{A}, \mathcal{H}, D)$ is determined by $(\mathcal{A}'', \mathcal{H}, D)$). This also implies that $\mathcal{A}$ is stable under the smooth functional calculus in its norm closure $\bar{\mathcal{A}} = A$ and in particular

$$\text{Spectrum}\, \mathcal{A} = \text{Spectrum}\, A . \tag{4.3}$$

Let $X = \text{Spectrum}\, A$. It is a compact space. One should deduce from the axioms that the map from X to $\mathbb{R}^N$ given by $a_i^j \in \mathcal{A}$ (the components of the Hochschild cocycle c given by Axiom (4)) is an embedding of X as a smooth submanifold of $\mathbb{R}^N$ (cf. [Co, Proposition 15, p.312]).

(b) Let us recall that a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ is an element of $\mathcal{A}^{\otimes(n+1)}$, $c = \sum a_i^0 \otimes a_i^1 \ldots \otimes a_i^n$ such that $bc = 0$, where b is the linear map $b : \mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n}$ (cf. (3.5)). The class of the Hochschild cycle c determines the *volume form.*

(c) We use the convention that the scalar curvature r is positive for the sphere S^n, in particular, the sign of the action $⨍ ds^{n-2}$ is the correct one for the Euclidean formulation of gravity. For example for $n = 4$ the Hilbert–Einstein action

$$-\frac{1}{16\pi G}\int r\sqrt{g}\, d^4x$$

coincides with the area $\frac{1}{\ell_p^2} ⨍ ds^2$ in Planck units.

(d) When M is a spin manifold the map $\pi \to g(\pi)$ from Theorem 6 is surjective and if one fixes the cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ its image is the set of metrics whose volume form (Remark **(b)**) is given by the class of c.

(e) If one omits Axiom (7) , one gets a result analogous to Theorem 6 replacing spin structures by spin^c–structures ([L-M]), but then there will be no more uniqueness in Theorem 6 *3.*. because of the choice of spin connection.

(f) It follows from Axiom (1) (see [Co, Theorem 8, p.309]) that the operators $a\, ds^n$, $a \in \mathcal{A}$ are automatically measurable so that $⨍$ is well defined in Axiom (5).

Now let us consider the general noncommutative case. Given an involutive algebra of operators $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, Tomita's theory associates to all vectors $\xi \in \mathcal{H}$, cyclic for $\mathcal{A}$ and for its commutant $\mathcal{A}'$

$$\overline{\mathcal{A}\xi} = \mathcal{H}\ ,\ \overline{\mathcal{A}'\xi} = \mathcal{H} \tag{4.4}$$

an anti-linear isometric involution $J : \mathcal{H} \to \mathcal{H}$ obtained from the polar decomposition of the operator

$$S\, a\xi = a^*\xi \qquad \forall\, a \in \mathcal{A}\,. \tag{4.5}$$

It satisfies the following commutation relation:

$$J\mathcal{A}''J^{-1} = \mathcal{A}'\,. \tag{4.6}$$

In particular $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$ where

$$b^0 = Jb^*J^{-1} \qquad \forall b \in \mathcal{A} \tag{4.7}$$

so $\mathcal{H}$ becomes an $\mathcal{A}$-bimodule using the representation of the opposite algebra $\mathcal{A}^0$ given by (4.7). There is no difference between module and bimodule structures in the commutative case because one has $a^0 = a \quad \forall\, a \in \mathcal{A}$.

Tomita's theorem is the key ingredient which guarantees the substance of the axioms in the general case. The axioms (1), (3), and (5) are left untouched, but in the axiom of reality (7) the equality $Ja^*J^{-1} = a \quad \forall\, a \in \mathcal{A}$ is replaced by

(7′) $\qquad [a, b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$ where $b^0 = Jb^*J^{-1}$

also Axiom (2) (order one) becomes

(2′) $\qquad [[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$.

(Notice that since a and b^0 commute $(2')$ is equivalent to $[[D, a^0], b] = 0 \quad \forall\, a, b \in \mathcal{A}$.)
The Hilbert space $\mathcal{H}$ becomes an $\mathcal{A}$-bimodule by Axiom $(7')$ and gives a class μ of KR^n-homology for the algebra $\mathcal{A} \otimes \mathcal{A}^0$ equipped with the anti-linear automorphism τ,

$$\tau(x \otimes y^0) = y^* \otimes x^{*0} .$$

The Kasparov intersection product [K] allows to formulate the Poincaré duality in terms of the invertibility of μ,

$(6')$ $\exists\, \beta \in KR_n(\mathcal{A}^0 \otimes \mathcal{A})$, $\beta \otimes_{\mathcal{A}} \mu = \mathrm{id}_{\mathcal{A}^0}$, $\mu \otimes_{\mathcal{A}^0} \beta = \mathrm{id}_{\mathcal{A}}$.

It implies the isomorphism $K_*(\mathcal{A}) \xrightarrow{\cap \mu} K^*(\mathcal{A})$. The intersection form

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$$

is obtained from the Fredholm index of D with coefficients in $K_*(\mathcal{A} \otimes \mathcal{A}^0)$. Note that it is defined without using the diagonal map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, which is not a homomorphism in the noncommutative case. This form is quadratic or symplectic according to the value of n modulo 8.

The Hochschild homology with coefficients in a bimodule makes perfect sense in the general case and Axiom (4) takes the following form:

$(4')$ There exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ such that $\pi(c) = 1$ (n odd), or $\pi(c) = \gamma$ (n even).

(Where $\mathcal{A} \otimes \mathcal{A}^0$ is the $\mathcal{A}$-bimodule obtained by restriction of the structure of $\mathcal{A} \otimes \mathcal{A}^0$-bimodule of $\mathcal{A} \otimes \mathcal{A}^0$ to the subalgebra $\mathcal{A} \otimes 1 \subset \mathcal{A} \otimes \mathcal{A}^0$, i.e.

$$a(b \otimes c^0)d = abd \otimes c^0 \qquad \forall\, a, b, c, d \in \mathcal{A} .)$$

Axioms (1), (3) et (5) are unchanged in the noncommutative setting. The proof showing that the operators $a(ds)^n$, $(a \in \mathcal{A})$ are measurable stays valid in general.

We adopt Axioms (1), $(2')$, (3), $(4')$, (5), $(6')$ et $(7')$ in the general case as a definition of a *spectral manifold* of dimension n. Once fixing the algebra $\mathcal{A}$ one can talk about the spectral geometry of $\mathcal{A}$ like in (1.20) and (1.21). One can show that the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$ in $\mathcal{H}$ is automatically finite and hyperfinite and there is a complete list of such algebras up to isomorphism [Co]. The algebra $\mathcal{A}$ is stable under smooth functional calculus in its norm closure $A = \bar{\mathcal{A}}$ so that $K_j(\mathcal{A}) \simeq K_j(A)$, i.e. $K_j(\mathcal{A})$ depends only on the underlying topology (defined by the C^* algebra A). The integer $\chi = \langle \mu, \beta \rangle \in \mathbb{Z}$ gives the Euler characteristic in the form

$$\chi = \mathrm{Rang}\, K_0(\mathcal{A}) - \mathrm{Rang}\, K_1(\mathcal{A})$$

and Theorem 4 gives a local formula for it.

The group of automorphisms of the involutive algebra $\mathcal{A}$, $\mathrm{Aut}(\mathcal{A})$, in general plays the role of the group of diffeomorphisms, $\mathrm{Diff}(M)$, of a manifold M. (There is a canonical isomorphism $\mathrm{Diff}(M) \xrightarrow{\alpha} \mathrm{Aut}(C^\infty(M))$ given by

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \qquad \forall\, f \in C^\infty(M) \;,\; \varphi \in \mathrm{Diff}(M) .)$$

In the general noncommutative case, parallel to the normal subgroup $\mathrm{Int}\,\mathcal{A} \subset \mathrm{Aut}\,\mathcal{A}$ of inner automorphisms of $\mathcal{A}$,

$$\alpha(f) = ufu^* \qquad \forall f \in \mathcal{A} \tag{4.8}$$

where u is a unitary element of $\mathcal{A}$ (i.e. $uu^* = u^*u = 1$), there exists a natural foliation of the space of spectral geometries on $\mathcal{A}$ by equivalence classes of *inner deformations* of a given geometry. Such a deformation is obtained by the following formula without modifying neither the representation of $\mathcal{A}$ in $\mathcal{H}$ nor the anti-linear isometry J

$$D \to D + A + JAJ^{-1} \tag{4.9}$$

where $A = A^*$ is an arbitrary selfadjoint operator of the form

$$A = \Sigma\, a_i[D, b_i]\ ,\ a_i, b_i \in \mathcal{A}\,. \tag{4.10}$$

The newly obtained spectral triple also satisfies Axioms (1) through (7′).

The action of the group $\mathrm{Int}(\mathcal{A})$ on the spectral geometries (cf. (1.21)) is simply the following gauge transformation of A

$$\gamma_u(A) = u[D, u^*] + uAu^*\,. \tag{4.11}$$

The required unitary equivalence is implemented by the following representation of the unitary group of $\mathcal{A}$ in $\mathcal{H}$,

$$u \to uJuJ^{-1} = u(u^*)^0\,. \tag{4.12}$$

The transformation (4.9) is the identity operator for the usual Riemannian case. To get a nontrivial example it suffices to consider the product of a Riemannian triple by the unique spectral geometry on the finite-dimensional algebra $\mathcal{A}_F = M_N(\mathbb{C})$ of $N \times N$ matrices on $\mathbb{C}$, $N \geq 2$. One then has $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathrm{Int}(\mathcal{A}) = C^\infty(M, PSU(N))$ and inner deformations of the geometry are parameterized by the gauge potentials for the gauge theory of the group $SU(N)$. The space of pure states of the algebra $\mathcal{A}$, $P(\mathcal{A})$, is the product $P = M \times P_{N-1}(\mathbb{C})$ and the metric on $P(\mathcal{A})$ determined by the formula (1.10) depends on the gauge potential A. It coincide with the Carnot metric [G] on P defined by the horizontal distribution given by the connection associated to A (cf. [Co2]). The group $\mathrm{Aut}(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is the following semi–direct product

$$\mathrm{Aut}(\mathcal{A}) = \mathcal{U} \rtimes \mathrm{Diff}(M) \tag{4.13}$$

of the local gauge transformation group $\mathrm{Int}(\mathcal{A})$ by the group of diffeomorphisms. In dimension $n = 4$, the Hilbert–Einstein action functional for the Riemannian metric and the Yang–Mills action for the vector potential A appear in the asymptotic expansion in $\frac{1}{\Lambda}$ of the number $N(\Lambda)$ of eigenvalues of D which are $\leq \Lambda$. One regularizes this expression replacing it by

$$\mathrm{Trace}\ \varpi\left(\frac{D}{\Lambda}\right) \tag{4.14}$$

where $\varpi \in C_c^\infty(\mathbb{R})$ is an even function which is 1 on $[-1, 1]$, (cf. [C-C]). Other nonzero terms in the asymptotic expansion are cosmological, Weyl gravity and topological terms.

A more sophisticated example of a spectral manifold is provided by the noncommutative torus $\mathbb{T}^2_\theta$. The parameter $\theta \in \mathbb{R}/\mathbb{Z}$ defines the following deformation of the algebra of smooth functions on the torus $\mathbb{T}^2$, with generators U, V. The relations

$$VU = \exp 2\pi i\theta\, UV \quad \text{and} \quad UU^* = U^*U = 1\ ,\ VV^* = V^*V = 1 \tag{4.15}$$

define the presentation of the involutive algebra $\mathcal{A}_\theta = \{\Sigma\, a_{n,m} U^n V^n\ ;\ a = (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}$ where $S(\mathbb{Z}^2)$ is the Schwartz space of sequences with rapid decay. Geometries on $\mathbb{T}^2_\theta$ are parameterized by complex numbers τ with positive imaginary part like in the case of elliptic curves. Up to isometry the geometry depends only on the orbit of τ under the action of $PSL(2,\mathbb{Z})$ [Co]. However, a new phenomenon appears in the noncommutative case, namely, the *Morita equivalence* which relates the algebras $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$ if θ_1 and θ_2 are in the same orbit of the $PSL(2,\mathbb{Z})$ action on $\mathbb{R}$ [Ri1, Ri2].

Given a spectral manifold $(\mathcal{A}, \mathcal{H}, D)$ and the Morita equivalence between $\mathcal{A}$ and an algebra $\mathcal{B}$ where

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \tag{4.16}$$

where $\mathcal{E}$ is a finite, projective, Hermitian right $\mathcal{A}$–module, one gets a spectral geometry on $\mathcal{B}$ by the choice of a *Hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the rules ([Co])

$$\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da \qquad \forall \xi \in \mathcal{E}\ ,\ a \in \mathcal{A} \tag{4.17}$$

$$(\xi, \nabla\eta) - (\nabla\xi, \eta) = d(\xi, \eta) \qquad \forall \xi, \eta \in \mathcal{E} \tag{4.18}$$

where $da = [D, a]$ and where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$–bimodule generated by operators of the form 4.10.

Any algebra $\mathcal{A}$ is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the above construction one gets exactly the inner deformations of the spectral geometry.

5 THE SPECTRAL GEOMETRY OF SPACE–TIME.

The experimental and theoretical datas which one has about the structure of space–time is summarized in the following action functional:

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_G + \mathcal{L}_{G\varphi} + \mathcal{L}_\varphi + \mathcal{L}_{\varphi f} + \mathcal{L}_f \tag{5.1}$$

where

$$\mathcal{L}_E = -\frac{1}{16\pi G}\int r\,\sqrt{g}\,d^4x$$

is the Hilbert–Einstein action while the five other terms constitute the standard model of particle physics with minimal coupling to gravity. Besides the metric $g_{\mu\nu}$ this Lagrangian involves several bosonic and fermionic fields. Spin 1 bosons are the photon γ, the intermediate bosons $W^\pm$ and Z, and the eight gluons. The zero spin bosons are the Higgs fields φ which are introduced in order to provide masses to various particles through the mechanism of spontaneous symmetry breaking without contradicting the renormalizability of nonabelian gauge fields. All fermions have spin $\frac{1}{2}$ and form three families of quarks and leptons.

The fields involved in the standard model have a priori a completely different status from the one of the metric $g_{\mu\nu}$. The symmetry group of these fields, namely, the group of local gauge transformations:

$$\mathcal{U} = C^\infty(M, U(1) \times SU(2) \times SU(3)) \tag{5.2}$$

is a priori quite different from the group $\mathrm{Diff}(M)$ of symmetries of the total Lagrangian $\mathcal{L}_E$. The natural group of symmetries of $\mathcal{L}$ is the semi-direct product $\mathcal{U} \rtimes \mathrm{Diff}(M) = G$ of $\mathcal{U}$ by the natural action of $\mathrm{Diff}(M)$ by automorphisms of $\mathcal{U}$. The first requirement if one wants to obtain a pure geometrical understanding of $\mathcal{L}$ unifying gauge theory with gravity is to find a geometric space X such that $G = \mathrm{Diff}(X)$. This determines the algebra $\mathcal{A}$ *

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F \ , \ \mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{5.3}$$

where the involutive algebra $\mathcal{A}_F$ is the direct sum of the algebras $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $M_3(\mathbb{C})$ of 3×3 complex matrices.

The algebra $\mathcal{A}_F$ corresponds to a *finite* space where the standard model fermions and the Yukawa parameters (masses of fermions and mixing matrix of Kobayashi Maskawa) determine the spectral geometry in the following manner. The Hilbert space is finite-dimensional and admits the set of elementary fermions as a basis. For example for the first generation of leptons this set is

$$e_L, e_R, \nu_L, \bar{e}_L, \bar{e}_R, \bar{\nu}_L \, . \tag{5.4}$$

The algebra $\mathcal{A}_F$ admits a natural representation in $\mathcal{H}_F$ (see [Co2]). Denote by J_F the unique anti-linear involution which exchanges f and $\bar{f}$ for all vectors in the base. One checks the commutation relation

$$[a, Jb^* J^{-1}] = 0 \qquad \forall \, a, b \in \mathcal{A}_F \, , \tag{5.5}$$

which shows that Axiom (2) holds. The operator D_F is given simply by the matrix $\begin{bmatrix} Y & 0 \\ 0 & \bar{Y} \end{bmatrix}$ where Y is the Yukawa coupling matrix. The detailed structure of Y (and in particular the fact that color is not broken) allows to check the the following relation

$$[[D_F, a], b^0] = 0 \qquad \forall \, a, b \in \mathcal{A}_F \, . \tag{5.6}$$

The natural $\mathbb{Z}/2$–grading of $\mathcal{H}_F$ gives 1 for left–handed fermions $(e_L, \nu_L \ldots)$ and -1 for right–handed fermions; one has

$$\gamma_F = \varepsilon \, \varepsilon^0 \text{ where } \varepsilon = (1, -1, 1) \in \mathcal{A}_F \, . \tag{5.7}$$

We refer to [Co2] for the verification of the axioms (1) through (7′). The only drawback of this construction is that the number of families introduces a multiplicity in the intersection form, $K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$, given by an integer multiple of the matrix

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} . \tag{5.8}$$

*taking in account the lifting of diffeomorphisms to the spinors

We will dwell on the significance of the specific spectral geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ at the end of this exposition.

The next step consists of the computation of internal deformations (formula (4.9)) of the product geometry $M \times F$ where M is a 4-dimensional Riemannian spin manifold. The computation gives the standard model gauge bosons $\gamma, W^{\pm}, Z$, the eight gluons and the Higgs fields φ with accurate quantum numbers. It also shows that

$$\mathcal{L}_{\varphi f} + \mathcal{L}_f = \langle \psi, D\psi \rangle \tag{5.9}$$

where $D = D_0 + A + JAJ^{-1}$ is the inner deformation of the product geometry (given by the operator $D_0 = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$).

The product structure of $M \times F$ gives a bigrading of Ω_D^1 and a decomposition $A = A^{(1,0)} + A^{(0,1)}$ of A which corresponds to the decomposition (5.9). The term $A^{(1,0)}$ is the sum of the vector potentials of all spin 1 bosons, the term $A^{(0,1)}$ is the Higgs boson which appears from the finite space $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ as finite difference terms. This bigrading exists also on Ω_D^2, the analogue of 2–forms [Co], and decomposes the curvature $\theta = dA + A^2$ in three terms $\theta = \theta^{(2,0)} + \theta^{(1,1)} + \theta^{(0,2)}$, which are pairwise orthogonal with respect to the scalar product

$$\langle \omega_1, \omega_2 \rangle = ⨍ \omega_1 \, \omega_2^* \, ds^4 \, . \tag{5.10}$$

Hence, the Yang-Mills action, $\langle \theta, \theta \rangle = ⨍ \theta^2 \, ds^4$, also decomposes into the sum of three terms. One can show that these terms are $\mathcal{L}_G$, $\mathcal{L}_{G\varphi}$ and $\mathcal{L}_\varphi$ respectively [Co].

The Yang-Mills action $⨍ \theta^2 \, ds^4$ uses the decomposition $D = D_0 + A + JAJ^{-1}$ and hence depends on more than just the geometry fixed by D. Since we want to unify matter with gravity by an action which is purely geometric we need a better formula that only involves D alone. In the simplest case, as it was shown in formula (4.14), the sum $\mathcal{L}_E + \mathcal{L}_G$ appears directly in the asymptotic expansion of the number of eigenvalues of D which are smaller than Λ. The same principle (cf. [C-C]) applies to the standard model and it is governed by the following functional

$$\mathrm{Trace}\left(\varpi\left(\frac{D}{\Lambda}\right)\right) + \langle \psi, D\psi \rangle \tag{5.11}$$

whose asymptotic expansion ([C-C]) gives our original Lagrangian $\mathcal{L}$ (5.1) together with a Weyl gravity term and a term in $r\varphi^2$, the only term which could be added to $\mathcal{L}$ without changing the standard model. For the physical interpretation of this result see ([C-C]).

The finite geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ above was dictated by the experimental results, encapsulated in the details of the standard model. One still has to understand its conceptual significance using as a tool the analogue of Lie groups in noncommutative geometry , i.e. the quantum groups. A simple fact (cf. [M]) is that the spin cover Spin(4) of $SO(4)$ is not the maximal cover in the quantum group category. One has Spin(4) $= SU(2) \times SU(2)$ and the group $SU(2)$ admits according to Lusztig a finite cover of the form (Frobenius at ∞):

$$1 \to H \to SU(2)_q \to SU(2) \to 1 \tag{5.12}$$

where q is a root of unity $q^m = 1$, m odd. The simplest case is $m = 3$, $q = \exp\left(\frac{2\pi i}{3}\right)$. The finite quantum group H has a finite dimensional Hopf algebra very similar to $\mathcal{A}_F$, and the spinor representation of H defines a bimodule whose structure is very similar to the $\mathcal{A}_F$–bimodule $\mathcal{H}_F$. This suggests to extend spin geometry ([L-M]) to quantum covers of the spin group. This requires for the description of principal G bundles to introduce a minimum of noncommutativity (of the type $C^\infty(M) \otimes \mathcal{A}_F$) in the algebras of functions.

At last let us mention that we neglected the important difference between Riemannian and Lorentzian signatures in this exposé.

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