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Almost Periodic States and Factors of Type III_1

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We construct a factor of type III_1 which has no almost-periodic state (or weight). We exhibit a factor N of type II_∞ and two automorphisms θ_1, θ_2 of N which are not in the same conjugacy class in Out N = Aut N/Int N though $\tau\theta_1 = \lambda\tau$, $\tau\theta_2 = \lambda\tau$ with $\lambda \in]0, 1[$, τ = Trace on N. We introduce and study two invariants Sd and τ for factors of type III_1. We relate the closedness of Int M in Aut M to the absence of central sequences in the von Neumann algebra M.

INTRODUCTION

In [4] we proved that an arbitrary factor of type $\neq III_1$ is the crossed product of a semifinite von Neumann algebra by the group $\mathbb{Z}$ of integers. In [13] Takesaki showed that any factor of type III_1 is the cross product of a semifinite von Neumann algebra by $\mathbb{R}$, the additive group of real numbers. Due to the obvious greater technical simplicity of discrete cross products it was natural to ask whether a decomposition as cross product of a semifinite von Neumann algebra by a discrete abelian group was always possible for factors of type III_1. We shall show (Corollary 5.5) that such a decomposition may fail to exist, even for factors acting in a separable Hilbert space, proving at the same time that factors of type III_1 may fail to have any almost-periodic state [4, Problem 4].

To study factors of type III_1 we define two invariants Sd and τ. The point modular spectrum $Sd(M)$ is the intersection of the point spectra of all almost-periodic weights (if any) on M. It is always a denumerable subgroup of $\mathbb{R}_+^*$, when it is not $\mathbb{R}_+^*$ and we shall see (Corollary 4.4) that it can be any denumerable subgroup of $\mathbb{R}_+^*$. There is a large class of factors for which it is easy to compute and is reasonably significant. In fact for any full factor (see definition below) the following hold, with φ an almost-periodic weight on M.

(1) $Sd(M) = \bigcap$ point spectrum of Δ_{φ_e} with e projection, $e \in M_\varphi$, $e \neq 0$.

(2) There exists an almost-periodic weight ψ, $\psi(1) = +\infty$ such that $Sd(M) =$ point spectrum Δ_ψ.

(3) The ψ of (2) is unique up to inner automorphisms and multiplication by a scalar.

(4) $\overline{Sd(M)} = S(M)$

Property 1 does not hold in general (for nonfull factors), which then makes the computability problem hard.

The class of full factors appears when looking for a topological structure on the group Out $M =$ Aut M/Int M. When M_* is separable, the group Aut M gifted with the topology of pointwise norm convergence in M_* (topology studied in [1] and [8]) becomes a polish space as well as a topological group, which shows the significance of this u-topology. Of course the topological group Out M is hausdorff iff Int M is closed in Aut M. By definition, a von Neumann algebra M is full when Int M is closed in Aut M.

Obviously all factors of type I are full, having no outer automorphism. A factor of type II_1 is full iff it does not have property Γ of von Neumann. For instance the hyperfinite factor of type II_1: R_1 is not full, in fact Aut $R_1 = \overline{\text{Int } R_1}$, while the factor coming from the left regular representation of the free group of two generators is full.

An arbitrary factor M is full iff all sequences $(x_n)_{n\in\mathbb{N}}$, $\| x_n \|$ bounded, $x_n \in M$ such that $\|[x_n, \varphi]\| \to_{n\to\infty} 0$, $\forall\varphi \in M_*$ are trivial.

Due to their description [4, Section V], factors of type III_0 are never full, in fact they always have property L of Pukanszky and for each $t \in \mathbb{R}$, the modular automorphism σ_t^φ belongs to $\overline{\text{Int } M}$, $\forall\varphi$. For $\lambda \in]0, 1[$, the Pukanszky's factor P_λ is full. It then follows that there exists a full factor N_0 (resp. N_1) of type II_1 (resp. II_∞) with $\lambda \in$ fundamental group $G(N_0)$ (resp. $G(N_1)$). Whence $G(N) \neq \{1\}$ does not imply $N \otimes R_1$ isomorphic to N.

It also follows that there exists a factor N_1 of type II_∞ and two automorphisms θ_a, θ_b of N_1 which both satisfy $\tau\theta_a = \lambda\tau$, $\tau\theta_b = \lambda\tau$, but are not in the same conjugacy class in Out N_1. In particular $M_a =$ cross product of N_1 by θ_a, and M_b are nonisomorphic factors of type III_λ with $M_a\tilde{} = M_b\tilde{}$ in the notations of [4, Section IV]. The existence of full factors M of type III_1 having almost periodic states gives a negative answer to a conjecture in [13]: the range of the modular homomorphism δ_M can be different from center of out M.

Finally for full factors of type III_1 we show that the topology $\tau(M)$ on $\mathbb{R}$, coming from the modular homomorphism δ of $\mathbb{R}$ in the topological group Out M, can be any topology associated with a unitary representation of $\mathbb{R}$. Let us first recall that an almost periodic weight φ on a von Neumann algebra M is a faithful semifinite normal weight φ whose modular operator Δ_φ is diagonal: $\Delta_\varphi = \sum_{\lambda>0} \lambda E_\lambda$.

PROPOSITION 1.1. *Let Λ be a subgroup of $\mathbb{R}_+^*$, β the canonical injection of Λ in $\mathbb{R}_+^*$, G the dual of Λ when Λ is gifted with its discrete topology, and $\tilde{\beta}$ the transpose of β. Let also M be a von Neumann algebra, ψ a faithful semifinite normal weight on M. The following conditions are then equivalent:*

(a) *ψ is almost periodic and (point spectrum Δ_ψ) $\subset \Lambda$*

(b) *There exists a (necessarily unique, because $\tilde{\beta}(\mathbb{R})$ is dense in G) representation $\sigma^{\psi,\Lambda}$ of G in M such that $\sigma^{\psi,\Lambda}_{\tilde{\beta}(t)} = \sigma_t^\psi$, $\forall t \in \mathbb{R}$;*

(c) *ψ is strictly semifinite and there is a generating subset $\mathscr{S} \subset M$ such that: $\forall x \in \mathscr{S}$ the function $t \to \sigma_t^\psi(x)$ extends to a $*$ strongly continuous map from G to M.*

Proof.

(a) $\Rightarrow$ (b) See [4] Lemma 2.7.3.

(b) $\Rightarrow$ (c) is straightforward, using [2].

(c) $\Rightarrow$ (b) By [2] the family $(\sigma_t^\psi)_{t\in\mathbb{R}}$ of maps from the unit ball of with $*$ strong topology, to itself, is equicontinuous.

Hence for each $s \in G$ the $*$ subalgebra of M: $\mathfrak{A}_s = \{x \in M, \sigma_t^\psi(x)$ converges $*$ strongly when $\tilde{\beta}(t) \to s\}$ is strongly closed. By hypothesis each $\mathfrak{A}_s$ contains $\mathscr{S}$ hence $\mathfrak{A}_s = M$, for any $s \in G$. It is then easy to conclude, using the density of $\tilde{\beta}(\mathbb{R})$ in G, that (b) holds.

(b) $\Rightarrow$ (a) By [4] Lemma 2.1.6 the set of $x \in M$ which for some $\lambda \in \Lambda$ satisfy $\sigma_t^\psi(x) = \lambda^{it}x \ \forall t \in \mathbb{R}$ is total in M. This yields the desired diagonalisation of Δ_ψ. We note moreover that

(1) Point spectrum $\Delta_\psi = \mathrm{Sp}\, \sigma^{\psi,\Lambda}$

A Λ-almost periodic weight ψ on a von Neumann algebra is by definition a faithful semifinite normal weight satisfying the equivalent conditions in Proposition 1.1.

DEFINITION 1.2. Let M be a factor, then the point modular spectrum of M is the subset of $\mathbb{R}_+^*$ defined by

$$Sd(M) = \bigcap_{\psi \text{ almost periodic weight on } M} \text{point spectrum } \Delta_\varphi$$

THEOREM 1.3. *Let M be a factor then*:

(a) $Sd(M) = \bigcap \Gamma(\sigma^{\varphi,\mathbb{R}_+^*})$ *when* φ *runs through all almost-periodic weights*. (*See* [4], *Section* 2).

(b) $Sd(M)$ *is a subgroup of* $\mathbb{R}_+^*$.

Proof. Clearly (a) $\Rightarrow$ (b) using [4] Theorem 2.2.4. So we need only to prove (a): Let G be the dual of $\mathbb{R}_+^*$ when $\mathbb{R}_+^*$ has its discrete topology and let $\tilde{\beta}$ be the transpose of β: $\beta(\lambda) = \lambda$, $\forall \lambda \in \mathbb{R}_+^*$. Let U be a representation of G on M, with $U \sim \sigma^{\varphi,\mathbb{R}_+^*}$, in the sense of [4] Def. 2.3.3, for some almost-periodic φ. Then ([4] Lemma 3.4.3) $U \circ \tilde{\beta} \sim \sigma^{\varphi}$, hence ([4] Theorem 1.2.4) there exists a semifinite faithful normal weight ψ on M such that $\sigma^{\psi} = U \circ \tilde{\beta}$. But (Proposition 1.1) ψ is then $\mathbb{R}_+^*$-almost periodic and (1), Sp U = Sp $\sigma^{\psi,\mathbb{R}_+^*}$ = point spectrum of Δ_ψ. From [4] Proposition 2.3.17 it follows that:

$$\Gamma(\sigma^{\varphi,\mathbb{R}_+^*}) \supset \bigcap_{\psi \text{ almost periodic}} \text{point spectrum } \Delta_\psi$$

As point spectrum $\Delta_\varphi \subseteq \Gamma(\sigma^{\varphi,\mathbb{R}_+^*})$ the equality (a) follows.

Remark 1.4. If M_* is separable and if $Sd(M) \neq \mathbb{R}_+^*$ then $Sd(M)$ is countable.

Proof. The point spectrum Λ of an almost-periodic weight φ on M is necessarily countable for $\Delta_\varphi = \sum \lambda E_\lambda$ where the E_λ are pairwise orthogonal projections in the separable Hilbert space $\mathscr{H}_\varphi$.

THEOREM 1.5. *Let M be a countably decomposable factor of type* III_0, *and Γ be a dense subgroup of* $\mathbb{R}_+^*$. *Then the set of Γ-almost-periodic states on M is norm dense in the set of normal states on M.*

Proof. Let (See [4] Corollary 5.3.6) N be a type II_∞ von Neumann subalgebra of M satisfying the following conditions

(a) $N' \cap M$ = Center of N.

(b) N is the range of a normal conditional expectation E.

(c) There exists an homomorphism $\epsilon \rightarrow u_\epsilon$ of $(\mathbb{Z}/2)^{(\mathbb{N})}$ onto a subgroup $\mathscr{G}$ of the unitary group $\mathscr{N}(E)$, and a decreasing sequence of projections (e_k) $k = 1, 2,\ldots$, $e_k \in C$ such that N and $\mathscr{G}$ generate M and that $e_1 = 1$,

$$\sum_{\epsilon=0,1} \text{Ad}\, u(\overbrace{0,\ldots,0}^{k}, \epsilon, 0,\ldots)\, e_{k+1} = e_k \qquad \forall k = 1, 2,\ldots$$

Our first aim is, given a system $(N, E, u, (e_k))$ to build a faithful normal trace τ' on N such that the weight $\tau' \circ E$ is Γ-almost periodic.

We let $\mathbb{R}$ be identified by the map β of Proposition 1.1 to a dense subgroup of the dual group G of Γ. Also for each k we put

$$\underline{k} = (\overbrace{0, 0, \dots, 1}^{k}, 0, \dots) \in (\mathbb{Z}/2)^{(\mathbb{N})}$$

LEMMA 1.6. *Let $\underline{N}$ be a von Neumann algebra of type II_∞, $C =$ Center of $\underline{N}$. $\theta \in \operatorname{Aut} \underline{N}$ with $\theta^2 = 1$, and $e \in C$ be a projection with $e + \theta(e) = 1$, also τ a faithful semifinite normal trace on $\underline{N}$ and $\epsilon > 0$. Then there exists a $k \in C$, $e^{-\epsilon} \leqslant k \leqslant e^{\epsilon}$ such that, with $\tau' = \tau(k\cdot)$ the function $t \to (D\tau' \circ \theta, D\tau')_t$ extends to a $*$ strongly continuous mapping from G to the unitary group of C.*

Proof. We have $\tau = \tau''(h\cdot)$ where τ'' is θ-invariant and h is affiliated to C. Let (f_λ), $\lambda \in \Gamma$ be a family of projections in C with $\sum f_\lambda = 1$ and $e^{-\epsilon} \leqslant (\sum \lambda f_\lambda)\, h^{-1} \leqslant e^{\epsilon}$. Put $k = (\sum \lambda f_\lambda)\, h^{-1}$ then $\tau' = \tau(k\cdot)$ is deduced from the θ-invariant trace τ'' by the density $\sum \lambda f_\lambda$ hence the lemma follows.

Now let τ be a semifinite faithful normal trace on N, and for $k \in \mathbb{N}$, θ_k be the restriction of Ad $u_{\underline{k}}$ to N. Applying Lemma 1.6 to the restriction of θ_k to N_{e_k} proves the existence of a sequence $(\rho_n)_{n \in \mathbb{N}}$ of elements of C with

(1) $\operatorname{Ad} u_{(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, 0 \dots)} \rho_n = \rho_n$, $\epsilon_j = 0, 1, j = 1, 2, \dots, n$

(2) $e^{-2^{-n}} \leqslant \rho_n \leqslant e^{2^{-n}}$

(3) For each n the restriction τ_n' to N_{e_n} of $\tau_n = \tau(\prod_1^n \rho_j)$ is such that $(D\tau_n' \circ \theta_n; D\tau_n')_t$ extends to G as in Lemma 1.6. Let $\rho = \prod_1^\infty \rho_j$. Condition (1) shows that $\prod_{n+1}^\infty \rho_j$ is θ_n invariant for each n, hence that, with $\tau' = \tau(\rho\cdot)$ one has:

$$(D\tau' \circ \theta_n : D\tau') = (D\tau_n \circ \theta_n : D\tau_n)$$

Moreover (3) shows that $(D\tau_n \circ \theta_n; D\tau_n)\, e_n$ extends to G. An induction hypothesis then yields for each $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, 0, \dots)$ that $(D\tau' \circ \theta_\epsilon \circ \theta_n; D\tau' \circ \theta_\epsilon)\, e_n$ extends to G, with $\theta_\epsilon = \prod \theta_j^{\epsilon_j}$ — $(D\tau' \circ \theta_n : D\tau')\, \theta_\epsilon(e_n)$ hence extends to G.

As $(D\tau' \circ \theta_1; D\tau') = (D\tau_1' \circ \theta_1; D\tau_1')$ extends to G, and as

$$\sum \theta_{(\epsilon_1,\ldots,\epsilon_{n-1},0)}(e_n) = 1,$$

we see that $(D\tau' \circ \theta_n; D\tau')$ extends to G for all n.

It then follows from condition (c) on the u_ϵ and [4] Lemma 1.4.5(a) that condition 1.1(c) is fulfilled by the weight $\varphi' = \tau' \circ E$ hence that φ' is Γ-almost-periodic.

Our next aim is to show that any normal state φ on M is a norm limit of states φ_k on M such that $\varphi_k \circ E$ is Γ-almost-periodic. We let τ be a faithful semifinite normal trace such that $\tau \circ E$ is Γ almost-periodic and $h \in L^1(N, \tau)$ such that $\varphi = \tau(h \cdot)$. Let $\lambda > 1$, $\lambda \in \Gamma$, and for $n \in \mathbb{Z}$, let p_n be the spectral projection of h corresponding to $]\lambda^n, \lambda^{n+1}]$. We may assume φ to be faithful, hence h to be nonsingular. Then $\sum p_n = 1$, $p_n \in N$, $\sum \lambda^n p_n \leqslant h$, $h - \sum \lambda^n p_n \leqslant (\lambda - 1)h$ and with $\varphi_\lambda = \tau((\sum \lambda^n p_n) \cdot)$ we have $\| \varphi_\lambda/\varphi_\lambda(1) - \varphi \| \leqslant 2(\lambda - 1)$. Using the density of Γ in $\mathbb{R}_+{}^*$ and the fact that $\varphi_\lambda \circ E$ is Γ-almost-periodic (it is deduced from $\tau \circ E$ by the density $\sum \lambda^n p_n$ affiliated to $M_{\tau\circ E}$), we get the desired conclusion.

We shall now end the Proof of Theorem 5. Let ψ be a normal state on M, and ψ_0 be a faithful normal state on N. For each $k = 1, 2,\ldots$, let N_k be the von Neumann subalgebra of M generated by N and the $u_{(\epsilon_1,\ldots,\epsilon_k,0,0\ldots)}$, $\epsilon_j = 0, 1$. Then it is easy to check that each N_k satisfies condition (a) (b) (c) above and that UN_k is dense in M. Using the Gelfand Segal construction relative to $\varphi_0 = \psi_0 \circ E$ we see that ψ is a norm limit of states of the form $\varphi_0(x \cdot x^*)$, where x belongs to UN_k. But φ_0 commutes with E_k (because $EE_k = E$), and $E_k x = x$ for $x \in N_k$, hence any state $\varphi_0(x \cdot x^*)$, $x \in N_k$ is of the form $\varphi_1 \circ E_k$ where φ_1 is a state on N_k. It is then clear that any state $\varphi_0(x \cdot x^*)$, $x \in N_k$ is a norm limit of Γ-almost periodic states of the form $\varphi_\lambda \cdot E_k$.

COROLLARY 1.7. *Let M be a factor, then $Sd(M) \subset S(M)$.*

Proof. We can assume that M is countably decomposable. Then if M is of type I or II, it is clear that $Sd(M) = \{1\} \subset S(M)$. If M is of type III_0 then theorem 1.5 shows that $Sd(M) = \{1\}$ is included in $S(M)$. If M is of type III_λ, $\lambda \in]0, 1[$, then by [4] Theorem 3.4.1, one has $Sd(M) \subset \{\lambda^n, n \in \mathbb{Z}\}^- = S(M)$. Finally if M is of type III_1, the above inclusion is obvious, for $S(M) = [0, +\infty[$

COROLLARY 1.8. *Let M be a Krieger's factor then $Sd(M) = \{1\}$.*

Proof. Use [5]. This last corollary shows that the invariant Sd has no interest for Krieger's factors.

II. Asymptotic Centraliser of von Neumann Algebras

We generalize the construction of Mc. Duff [7] for Type III factors. Let M be a von Neumann algebra, M_* its predual. For $x \in M$, $\varphi \in M_*$ let $x\varphi \in M_*$, $\varphi x \in M_*$, $[x, \varphi] \in M_*$ be such that $(x\varphi)(y) = \varphi(yx)$, $(\varphi x)(y) = \varphi(xy)\, \forall y \in M$, $[x, \varphi] = x\varphi - \varphi x$. For $x \in M$, $\varphi \in M_*$ we let $\| x \|_\varphi = (\varphi(x^*x))^{1/2} = \| \prod_\varphi(x)\, \xi_\varphi \|$ (On the Gelfand Segal construction of φ) and $\| x \|_\varphi^\# = \varphi(x^*x + xx^*)^{1/2}$.

Lemma 2.1. (the verification is left to the teader). *For $x, y \in M$ and $\varphi \in M_*{}^+$, $\varphi(1) = 1$ one has*:

(a) $\|[x, \varphi]\| = \|[x^*, \varphi]\|$

(b) $\| x\varphi \| \leqslant \| x \|_\varphi$

(c) $\| \varphi x \| \leqslant \| x^* \|_\varphi$

(d) $\|[xy, \varphi]\| \leqslant \| x \| \, \|[y, \varphi]\| + \| y \| \, \|[x, \varphi]\|$

(e) $\varphi(y^*x^*xy) \leqslant \| y \| \, \| x \|^2 \, \|[y, \varphi]\| + \| y \|^2 \, \| x \| \, \| x \|_\varphi$

(f) *If* $\| x \| \leqslant 1$, $\| y \| \leqslant 1$ *then* $(\| xy \|_\varphi^\#)^2 \leqslant \|[x, \varphi]\| + \|[y, \varphi]\| + \| x \|^\# + \| y \|_\varphi^\#$.

Proposition 2.2. *Let M be a von Neumann algebra, φ a faithful normal state on M, $\beta\mathbb{N}$ the Stone–Chech compactification of the integers and $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$. Then*:

(1) *The subset $A_{\varphi,\omega}$ of $l^\infty(\mathbb{N}, M)$ of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $\|[x_n, \varphi]\| \to 0$ when $n \to \omega$ is a norm closed $*$ subalgebra of $l^\infty(\mathbb{N}, M)$.*

(2) *Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ belong to $l^\infty(\mathbb{N}, M)$ and assume $x_n - y_n \to 0$ $*$ strongly when $n \to \infty$ then $(x_n)_{n\in\mathbb{N}} \in A_{\varphi,\omega} \Leftrightarrow (y_n)_{n\in\mathbb{N}} \in A_{\varphi,\omega}$.*

(3) *The functional φ_ω, $\varphi_\omega((x_n)_{n\in\mathbb{N}}) = \lim_\omega \varphi(x_n)$ is a trace on $A_{\varphi,\omega}$*

(4) *$\varphi_\omega[(x_n)_{n\in\mathbb{N}}^*(x_n)_{n\in\mathbb{N}}] = 0 \Leftrightarrow x_n \to 0$ strongly when $n \to \omega$*

(5) *The quotient of the C^*-algebra $A_{\varphi,\omega}$ by the two-sided ideal $\mathscr{I}_\omega \cap A_{\varphi,\omega}$, $\mathscr{I}_\omega = \{(x_n)_{n\in\mathbb{N}}, x_n \to 0 * \text{ strongly when } n \to \omega\}$, is a finite von Neumann algebra noted $M_{\varphi,\omega}$.*

Proof. (1) By construction $A_{\varphi,\omega}$ is a linear subspace of $l^\infty(\mathbb{N}, M)$ and using (2.1a) and (2.1d) it is a $*$ subalgebra of $l^\infty(\mathbb{N}, M)$. It is easy to check that if $(x_n)_{n\in\mathbb{N}} \in \bar{A}_{\varphi,\omega}$ (norm closure) then $\lim_{n\to\omega} \|[x_n, \varphi]\| < \epsilon$, $\forall \epsilon > 0$.

(2) One has $\| x_n - y_n \|_\varphi^\# \to 0$ when $n \to \omega$ hence $\|[x_n - y_n, \varphi]\| \to 0$ when $n \to \omega$, using (2.1b) and (2.1c).

(3) Let $X = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$ be elements of $A_{\varphi,\omega}$ then $\varphi_\omega(XY) = \lim_\omega \varphi(x_n y_n)$, $\varphi_\omega(YX) = \lim_\omega \varphi(y_n x_n)$ so the equality follows from $\| \varphi y_n - y_n \varphi \| \to 0$ when $n \to \omega$, using the uniform boundedness of the sequence $(x_n)_{n\in\mathbb{N}}$.

(4) The $*$ strong topology on bounded subsets of M is the same as the topology defined by $\| \ \|_\varphi^\#$, which gives the conclusion using (3).

(5) One has for $X \in A_{\varphi,\omega}$, the equivalence $X \in \mathcal{J}_\omega \Leftrightarrow \varphi_\omega(X^*X) = 0$ so that $\mathcal{J}_\omega \cap A_{\varphi,\omega}$ is a two-sided ideal in $A_{\varphi,\omega}$ and is norm closed. Let $M_{\varphi,\omega} = A_{\varphi,\omega}/\mathcal{J}_\omega \cap A_{\varphi,\omega}$ and $\rho_{\omega,\varphi}$ (noted ρ_ω if no confusion can arise) the canonical quotient map.

We just have to prove (using [11]) that the unit ball of the C^*-algebra $M_{\varphi,\omega}$ is complete for the norm $\| x \|_2 = \varphi_\omega(X^*X)^{1/2}$ where $\rho_\omega(X) = x$; as the functional $\tau = \varphi_\omega \circ \rho_\omega^{-1}$ is a faithful trace on $M_{\varphi,\omega}$. For convenience, given $x \in M_{\varphi,\omega}$ we call a sequence $(x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}, M)$ a representing sequence of x when $\rho_\omega((x_n)_{n\in\mathbb{N}}) = x$. Let $x^{(p)}$ be a sequence of elements of $M_{\varphi,\omega}$ such that:

$$\| x^{(p)} \| < 1, \qquad \| x^{(p+1)} - x^{(p)} \|_2 < 2^{-(p+1)}$$

Let $(x_n^{(1)})_{n\in\mathbb{N}}$ be a representing sequence for $x^{(1)}$ such that $\| x_n^{(1)} \| < 1$ for any n. Let $(x_n^{(2)})_{n\in\mathbb{N}}$ be a representing sequence for $x^{(2)}$ such that $\| x_n^{(2)} \| \leqslant 1 \ \forall n$, and $\| x_n^{(2)} - x_n^{(1)} \|^\# < 2^{-1}$ for all n. Inductively choose a representing sequence $(x_n^{(j)})_{n\in\mathbb{N}}$ of $x^{(j)}$ with:

$$\| x_n^{(j)} \| < 1 \quad \forall j, n, \qquad \| x_n^{(j+1)} - x_n^{(j)} \|_\varphi^\# < 2^{-j} \quad \forall j, n$$

Put $x_n = *$ strong limit of $x_n^{(j)}$ when $j \to \infty$. Then for any j, n $\| x_n - x_n^{(j)} \|_\varphi^\# \leqslant 2 \cdot 2^{-j}$ so that $\lim \| [x_n, \varphi] \| \leqslant 2^2 \cdot 2^{-j}$ and $(x_n)_{n\in\mathbb{N}} \in A_{\varphi,\omega}$. As $\| \rho_\omega((x_n)_{n\in\mathbb{N}} - x^{(j)} \|_2 \leqslant 2^{1-j}$ we see that $x = \rho_\omega((x_n)_{n\in\mathbb{N}})$ is a limit for the Cauchy sequence $x^{(j)}$, and finally that the unit ball of $M_{\varphi,\omega}$ is complete.

PROPOSITION 2.3. *Let M be a von Neumann algebra, φ a faithful normal state on M, and I a directed ordered set.*

(1) *Let $(x_j)_{j\in I}$ be a bounded family of elements of M such that $\| [x_j, \varphi] \| \to 0$, $j \to \infty$ then $\| \sigma_t^\varphi(x_j) - x_j \|_\varphi^\# \to 0$ uniformly on bounded subsets of $\mathbb{R}$.*

(2) *If 1 is an isolated point in $Sp\, \Delta_\varphi$, and if E_φ is the conditional expectation from M to M_φ then for any bounded sequence $(x_j)_{j\in I}$ of elements of M such that $\| [x_j, \varphi] \|_{j\to\infty} \to 0$ one has $\| x_j - E_\varphi(x_j) \|^\# \to 0$.*

(1) We shall provide several estimates which can be useful on other occasions:

LEMMA 2.4. *Let $t \in \mathbb{R}$ then there is an absolute constant C_t such that for any von Neumann algebra P, any couple φ, ψ of faithful normal states on P one has*

$$|1 - \varphi((D\psi : D\varphi)_t)| \leqslant C_t \| \psi - \varphi \|$$

Proof. Assume on the opposite that for each n there exists a von Neumann algebra P_n, and a couple φ_n, ψ_n with

$$\| \varphi_n - \psi_n \| \leqslant 2^{-n} | 1 - \varphi_n(D\psi_n : D\varphi_n)_t|$$

Using repetitions if necessary we can then assume that

$$\sum \| \varphi_n - \psi_n \| < \infty \qquad \text{while} \qquad \sum | 1 - \varphi_n((D\psi_n : D\varphi_n)_t)| = \infty$$

Then consider the Gelfand Segal construction $\mathscr{H}_n$, ξ_n relative to φ_n on P_n and let $\eta_n \in \mathscr{H}_n$, $\langle \eta_n, \xi_n \rangle \geqslant 0$, $\| \eta_n - \xi_n \|^2 \leqslant \| \varphi_n - \psi_n \|$, $\omega_{\eta_n} = \psi_n$. Put $P = \otimes_1^\infty (P_n, \varphi_n)$, acting in $\mathscr{H} = \otimes_1^\infty (\mathscr{H}_n, \xi_n)$. Let $\Phi_k = \psi_1 \otimes \cdots \otimes \psi_k \otimes \varphi_{k+1} \otimes \cdots$. Then when $k \to \infty$, Φ_k is a norm convergent sequence in P_*, because $(\eta_1 \otimes \cdots \otimes \eta_k \otimes \xi_{k+1})_{k=1,2..}$ is a norm convergent sequence in $\mathscr{H}$.

So using [1] or [3] we see that $(D\Phi_k; D\varphi)_t$ is a strongly convergent sequence in P, so that:

$$(D\psi_1 : D\varphi_1)_t \otimes \cdots \otimes (D\psi_n : D\varphi_n)_t \otimes 1 \otimes \cdots$$

has to be a strongly convergent sequence in P. But this contradicts the divergence of the serie $\sum | 1 - \langle (D\psi_n : D\varphi_n)_j \xi_n, \xi_n \rangle |$.

LEMMA 2.5. *Let $t \in \mathbb{R}$, M and φ as in Proposition 2.3, C_t as in Lemma 2.4, then for any unitary $v \in M$ one has*

$$(\| \sigma_t^\varphi(v) - v \|_\varphi^\#)^2 \leqslant 4C_t \| [v, \varphi] \|$$

Proof. Apply Lemma 2.4 to $\varphi_v = v^* \varphi v$ and φ on M, using the equality $(D\varphi_v : D\varphi)_t = v^* \sigma_t^\varphi(v)$. It yields $\|(v - \sigma_t^\varphi(v)) \xi_\varphi \|^2 \leqslant 2C_t \| \varphi_v - \varphi \|$.

LEMMA 2.6. *Let $t \in \mathbb{R}$, φ be a faithful normal state on a von Neumann algebra M, C_t as in Lemma 2.4.*

(a) $\forall x \in M$, $0 \leqslant x \leqslant 1/2$ one has $|||[1 - x^2)^{1/2}, \varphi]||| \leqslant 2/3|||[x, \varphi]|||$

(b) $\forall x \in M$, $\| x \| \leqslant 1$ *one has* $\| \sigma_t^\varphi(x) - x \|^\# \leqslant 16C_t^{1/2}|||[x, \varphi]|||^{1/2}$

Proof. (a) For each n one has $|||[x^n, \varphi]||| \leqslant n \| x \|^{n-1}|||[x, \varphi]|||$ (2.1d)) hence $|||[x^n, \varphi]||| \leqslant n \cdot 2^{-n+1} |||[x, \varphi]|||$. Then

$$|||[(1 - x^2)^{1/2}, \varphi]||| \leqslant \sum_0^\infty \left| \frac{1/2(1/2 - 1) \cdots (1/2 - n)}{(n + 1)!} \right| |||[x^{2n+2}, \varphi]|||$$

$$\leqslant \sum_{n=0}^\infty 2^{-(2n+1)} |||[x, \varphi]||| = 2/3 |||[x, \varphi]|||$$

(b) Put $|||[x, \varphi]||| = \epsilon$. Then put $a = (x + x^*)/2$, $b = (x - x^*)/2i$ One has $|||[a, \varphi]||| \leqslant \epsilon$, $|||[b, \varphi]||| \leqslant \epsilon$, $0 \leqslant (1 + a)/4 \leqslant 1/2$, $0 \leqslant (1 + b)/4 \leqslant 1/2$. And with $u_1 = (1 + a)/4 + i(1 - ((1 + a)/4)^2)^{1/2}$, $u_2 = u_1{}^*$ it follows from (a) that $|||[u_j , \varphi]||| \leqslant 2 |||[(1 + a)/4, \varphi]||| \leqslant \epsilon/2$ for $j = 1, 2$. Hence (2.5) we get:

$$\| \sigma_t^\varphi(u_j) - u_j \|_\varphi^\# \leqslant 2^{1/2}C_t^{1/2}\epsilon^{1/2},$$

$$\| \sigma_t^\varphi(a) - a \|_\varphi^\# = 2 \left\| \sigma_t^\varphi \left(\frac{1 + a}{2}\right) - \left(\frac{1 + a}{2}\right)\right\|_\varphi^\#$$

$$= 2 \| \sigma_t^\varphi(u_1 + u_2) - (u_1 + u_2)\|_\varphi^\# \leqslant 8C_t^{1/2}\epsilon^{1/2}.$$

Also $\| \sigma_t^\varphi(b) - b \|_\varphi^\# \leqslant 8C^{1/2}\epsilon_t^{1/2}$ and using $x = a + ib$ we get (2.6b).

LEMMA 2.7. *There exists a constant $C_0 < \infty$ such that for any von Neumann algebra M, and any faithful normal state φ on M one has*:

$$\| \sigma_t^\varphi(x) - x \|_\varphi^\# \leqslant C_0(1 + | t |) |||[x, \varphi]|||^{1/2}$$

Proof. Put $K(t) = \operatorname{Inf} \lambda$, $\| \sigma_t^\varphi(x) - x \|_\varphi^\# \leqslant \lambda |||[x, \varphi]|||^{1/2}$, $\forall M, \varphi, x$. Then K is lower semicontinuous, $K(-t) = K(t)$ $\forall t \in \mathbb{R}$, $K(0) = 0$, $K(t + t') \leqslant K(t) + K(t')$ so that $K(t) \leqslant C_0(1 + | t |)$, for some $C_0 > 0$. The proof of 2.3.1 is immediate using Lemma 2.7.

(2) Let $f \in L^1(\mathbb{R})$, $x \in M$, $\| x \| \leqslant 1$ then assume $\int f(t)\, dt = 1$

$$\| \sigma^\varphi(f)x - x \|_\varphi^\# = \left\| \int_{\mathbb{R}} (\sigma_t^\varphi(x) - x) f(t)\, dt \right\|_\varphi^\#$$

$$\leqslant \int_{\mathbb{R}} C_0(1 + (t)) \, | f(t)| \, dt \, |||[x, \varphi]|||^{1/2}.$$

So for $f \in L^1(\mathbb{R})$, $\int | t | \, | f(t)| \, dt < \infty$ we get an inequality

$$\| \sigma^\varphi(f)x - x \|_\varphi^\# \leqslant C_f |||[x, \varphi]|||^{1/2}, \qquad \forall x \in M, \quad \| x \| \leqslant 1.$$

Now choose f such that Support $\hat{f} \cap Sp\,\Delta_\varphi = \{1\}$. It follows that $\sigma^\varphi(f)x = E_\varphi(x)$ (Use $\sigma^\varphi(f)\,M \subset M_\varphi$) for any $x \in M$ hence (2).

PROPOSITION 2.8. *Let M be a countably decomposable von Neumann algebra, I be an ordered directed set, and $(x_j)_{j\in I}$ be a uniformly bounded family of elements of M then the following conditions are equivalent*:

(α) *There exists a faithful $\varphi \in M_*^+$ and a weakly dense subset $\mathscr{S} \subset M$ with $\|[x_j, \varphi]\| \to_{j\to\infty} 0$, $[x_j, y] \to_{j\to\infty} 0$ strongly $\forall y \in \mathscr{S}$.*

(β) *There exists a total subset $\mathscr{D} \subset M_*$ such that $\forall \psi \in \mathscr{D}$, $\|[x_j, \psi]\| \to_{j\to\infty} 0$.*

(γ) *$\|[x_j, \psi]\| \to_{j\to\infty} 0, \forall \psi \in M_*$ and $[x_j, y] \to_{j\to\infty} * 0$ strongly, $\forall y \in M$.*

Proof. (γ) $\Rightarrow$ (α) is clear. Let us prove (α) $\Rightarrow$ (β). Take $\epsilon > 0$, $x, y \in M$ such that $\|[x, y]\|_\varphi < \epsilon$ and $\|[x, \varphi]\| < \epsilon$ then for any $z \in M$ we have:

$$|\varphi(zxy - zyx)| \leqslant \|z\|\,\epsilon, \qquad |\varphi(zyx - xzy)| \leqslant \epsilon\,\|z\|\,\|y\|$$

hence $|((y\varphi)x)(z) - (x(y\varphi))(z)| \leqslant \epsilon(1 + \|y\|)\,\|z\|$. And it follows easily that for each $y \in \mathscr{S}$, and $\psi = y\varphi \in M_*$ we have $\|[\psi, x_j]\| \to 0$ when $j \to \infty$, hence (β). (β) $\Rightarrow$ (γ) It follows from the following inequality: $a, x \in M$, $\|a\| \leqslant 1$, $\|x\| \leqslant 1$

$$|\varphi([a, x]^*\,[a, x])| \leqslant 4\,\mathrm{Sup}\,\|[\varphi, x]\|, \|[a\varphi, x]\|, \|[a\varphi, x^*]\|, \qquad \forall \varphi \in M_*^+$$

We assume that, with $\epsilon > 0$, we have $\|[\varphi, x]\| < \epsilon$, $\|[a\varphi, x]\| < \epsilon$, $\|[a\varphi, x^*]\| < \epsilon$ then for any $y \in M$ the following inequalities are true:

$$|\varphi(xy - yx)| \leqslant \epsilon\,\|y\|, \qquad |\varphi(xya) - \varphi(yxa)| \leqslant \epsilon\,\|y\|,$$
$$|\varphi(x^*ya) - \varphi(yx^*a)| \leqslant \epsilon\,\|y\|$$

hence $|\varphi(a^*x^*xa) - \varphi(xa^*x^*a)| \leqslant \epsilon$, $|\varphi(xa^*x^*a) - \varphi(a^*x^*ax)| \leqslant \epsilon$ which gives:

$$|\varphi(a^*x^*[x, a])| = |\varphi(a^*x^*(xa - ax))| = |\varphi(a^*x^*xa) - \varphi(a^*x^*ax)| \leqslant 2\epsilon.$$

Moreover

$$|\varphi(x^*a^*ax) - \varphi(xx^*a^*a)| \leqslant \epsilon \quad \text{and} \quad |\varphi(x^*a^*xa) - \varphi(xx^*a^*a)| \leqslant \epsilon$$

so that $|\varphi(x^*a^*[a, x])| \leqslant 2\epsilon$.

THEOREM 2.9. *Let M be a countably decomposable von Neumann algebra and $\omega \in \beta\mathbb{N}/\mathbb{N}$.*

(1) *The quotient of the C^*-algebra $A_\omega \subset l^\infty(\mathbb{N}, M)$ of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $\|[x_n, \varphi]\| \to 0$, $n \to \omega$, $\forall \varphi \in M_*$ by the two-sided ideal of sequences converging $*$ strongly to 0 when $n \to \omega$, is a finite von Neumann algebra M_ω (called asymptotic centraliser of M at ω).*

(2) *For any automorphism $\theta \in \operatorname{Aut} M$, the automorphism $(x_n) \to (\theta(x_n))$ of A_ω defines an automorphism θ_ω of M_ω, and θ_ω depends only on the canonical image $\epsilon(\theta)$ of θ in* Out M.

(3) *For any $t \in \mathbb{R}$ one has $(\delta(t))_\omega = 1$ where δ is the modular homomorphism.*

Proof. (1) By construction $A_\omega = \bigcap A_{\varphi,\omega}$, φ faithful normal state on M. Let φ be a given faithful normal state on M and $\mathcal{D}$ the set of faithful normal states on M with $\alpha\varphi \leqslant \psi \leqslant \alpha^{-1}\varphi$ for some $\alpha > 0$. Then $A_\omega = \bigcap_{\psi\in\mathcal{D}} A_{\psi,\omega}$ (Use 2.8). Moreover considering M_ω as a subset of $M_{\varphi,\omega}$ we get, using (2.2.2):

$$M_\omega = \bigcap_{\psi\in\mathcal{D}} \rho_{\omega,\varphi}(A_{\varphi,\omega} \cap A_{\psi,\omega})$$

As on $\rho_{\omega,\varphi}(A_{\varphi,\omega} \cap A_{\psi,\omega})$ the norms corresponding to $\lim_\omega \varphi(x_n{}^* x_n)^{1/2}$ and $\lim_\omega \psi(x_n^* x_n)$ are equivalent it is easy to conclude that M_ω is a weakly closed $*$ subalgebra of M_ω hence a von Neumann algebra.

(2) We just have to show that for any unitary $u \in M$ and any sequence $(x_n)_{n\in\mathbb{N}} \in A_\omega$ one has $ux_nu^* - x_n \to_{n\to\omega} 0$ $*$ strongly, which follows from Proposition 2.8.

(3) Follows from Proposition (2.3.1).

As an application we shall prove:

THEOREM 2.10. (a) *Let $\lambda \in]0, 1[$ then there exists a factor of Type* II_1 N_0 *acting in a separable Hilbert space, having λ in its fundamental group but $1 \notin r_\infty(N_0)$* (i.e., $N_0 \otimes R_1$ *not isomorphic to* N_0).

(b) *Let $\lambda \in]0, 1[$ then there exists a factor of type* II_∞ *such that the set C_λ of conjugacy classes in* Out N *of elements j such that $\gamma(f) = \lambda$ contains at least two elements.*

Proof. (a) Let P_λ be the Pukanszky's factor of type III_λ. By construction there exists a finite measure space Ω, μ and an ergodic group $\mathcal{G}$ of non singular transformations of Ω, μ such that $P_\lambda = W^*(\mathcal{G}, \Omega)$. Now let $I(L^\infty(\Omega, \mu))$ be the canonical abelian maximal subalgebra of P_λ, E the corresponding conditional expectation from P_λ, $\varphi = \mu \circ I^{-1} \circ E$ the faithful normal state on P_λ corresponding to μ, and for $s \in \mathscr{G}$, U_s the corresponding unitary in P_λ. From [4] page 207 and [12] page 193 it follows that if g_1, g_2 are the two generators of the free group G_2 and $s_1 = \Phi_{g_1}$, $s_2 = \Phi_{g_2}$ ([4] page 207) are the corresponding elements of $\mathscr{G}$ one has:

(1) $\operatorname{Sp} \Delta_\varphi = \{\lambda^n, n \in \mathbb{Z}\}^-$

(2) $\forall x \in P_\lambda$, $| \varphi(x)|^2 \geqslant \| x \|_\varphi^2 - 5.14^2 \sup_{j=1,2} \|[x, U_{s_j}]\|^2$

(3) $U_{s_j} \in (P_\lambda)_\varphi \quad j = 1, 2$

Let $N_0 = (P_\lambda)_\varphi$ then in $P_\lambda \otimes \mathscr{L}(\mathscr{H})$, $N_0 \otimes \mathscr{L}(\mathscr{H})$ is the centraliser of the weight $\varphi \otimes$ Trace which is generalized trace on $P_\lambda \otimes \mathscr{L}(\mathscr{H})$ (see [4]), hence by [4] 4.4.5, we have $\lambda \in$ Fundamental group of N_0. Let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}} \in A_\omega(N_0)$. Then by Proposition 2.8 one has $[x_n, y] \to_{n\to\omega} 0$ $*$ strongly, $\forall y \in N_0$.

But as the U_{s_j}, $j = 1, 2$ belong to N_0 it is easy to conclude from (2) that there exists a sequence $\lambda_n \in \mathbb{C}$ such that $x_n - \lambda_n \to 0$ $*$ strongly hence that $N_{0,\omega} = \mathbb{C}$. Assertion (a) follows easily [7].

LEMMA 2.11. *Let Q_1 be a factor, $\varphi_0, \ldots, \varphi_p$ be faithful normal states on Q_1, $b_1, \ldots, b_p$ be elements of Q_1 such that for some $K > 0$, and any $\epsilon > 0$, any $x \in Q_1$: $\|[x, b_j]\|_{\varphi_j} < \epsilon, \forall j = 1, \ldots, p \Rightarrow \| x - \varphi_0(x)\|_{\varphi_0} \leqslant K\epsilon$ then*

(a) *For any von Neumann algebra with separable predual Q_2 and any faithful normal state φ on Q_2, any $X \in Q_2 \otimes Q_1$ one has*

$$\| X - (1 \otimes \varphi_0)(X)\|_{\varphi\otimes\varphi_0}^2 \leqslant K^2 \sum_{k=1}^{p} \|[X, 1 \otimes b_k]\|_{\varphi\otimes\varphi_k}^2$$

(b) *For any Q_2 like in* (a) *and any $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ the canonical homomorphism π_ω corresponding to π: $x \in Q_2 \to 1_{Q_1} \otimes x$ is an isomorphism of $Q_{2,\omega}$ onto $(Q_1 \otimes Q_2)_\omega$.*

Proof. (a) Let $a_1, a_2, \ldots, a_n, \ldots$ be an orthonormal basis of the pre-Hilbert space Q_2 with scalar product $(x, y) \to \varphi(y^*x)$, such that the linear span of the a_j is a $*$ subalgebra of Q_2 (Use the Schmidt orthogonalisation process).

The algebraic tensor product of this $*$ algebra by Q_1 is a dense subalgebra of $Q_2 \otimes Q_1$ hence we can assume that:

$$X = \sum_{j=1}^{n} a_j \otimes x_j, \qquad x_j \in Q_1, \qquad j = 1, 2, \ldots, n.$$

Then

$[X, 1 \otimes b_k] = \sum_{j=1}^{n} a_j \otimes [x_j, b_k]$ and $\|[X, 1 \otimes b_k]\|^2_{\varphi \otimes \varphi_k} = \sum_{j=1}^{n} \|[x_j, b_k]\|^2_{\varphi_k}$

As for any $x \in Q_1$ we have $\| x - \varphi_0(x)\|^2_{\varphi_0} \leqslant K \sum_1^{2p} \|[x, b_k]\|^2_{\varphi_k}$, it yields

$$\sum_j \| x_j - \varphi_0(x_j)\|^2_{\varphi_0} \leqslant K^2 \sum_k \sum_j \|[x_j, b_k]\|^2_{\varphi_k}$$

hence

$$\sum_j \| a_j \otimes (x_j - \varphi_0(x_j))\|^2_{\varphi \otimes \varphi_0} \leqslant K^2 \sum_k \|[X, 1 \otimes b_k]\|^2_{\varphi \otimes \varphi_k}$$

and hence conclusion (a).

(b) Let $(X_n)_{n \in \mathbb{N}}$ be a uniformly bounded sequence of elements of $Q_2 \otimes Q_1$ such that $\|[X_n, \psi]\| \to 0_{n \to \omega}$, $\forall \psi \in (Q_2 \otimes Q_1)_*$, then by Proposition 2.8, one has $[X_n, Y] \to 0$ * strongly for any $Y \in Q_2 \otimes Q_1$ hence by (a) $X_n - (1 \otimes \varphi_0) X_n \to_{n \to \omega} 0$ strongly. Also

$$X_n{}^* - (1 \otimes \varphi_0) X_n{}^* \xrightarrow[n \to \omega]{} 0$$

strongly so that the sequence $Y_n = (1 \otimes \varphi_0) X_n$ which belongs to $Q_2 \otimes \mathbb{C}$ yields the same elements of $(Q_2 \otimes Q_1)_\omega$ as the sequence $(X_n)_{n \in \mathbb{N}}$.

(2.10b) From (2), using the equality $\| x - \varphi(x)\|^2_\varphi = \varphi(x^*x) - |\varphi(x)|^2$ it follows that N_0 satisfies the hypothesis of Lemma 2.11. Let F_∞ be a factor of type I_∞, $(e_{ij})_{i,j \in \mathbb{Z}}$ be a system of matrix units in F_∞ and for $x \in F_\infty$, $(x_{ij})_{i,j \in \mathbb{Z}}$ be the matrix components of x. Then the following inequalities show that F_∞ satisfy the hypothesis of Lemma 2.11, with $\lambda_j = 2^{-j}$, $j \geqslant 0$, $\lambda_j = 2^{j+1/2}$, $j < 0$.

$$\sum_{i \neq j} | x_{ij} |^2 2^{-3|j|} \leqslant 5 \sum_{i,j} | x_{ij}(\lambda_i - \lambda_j)|^2 2^{-|j|}$$

$$\sum_j | x_{jj} - x_{0,0} |^2 2^{-3|j|} \leqslant \sum_{i,j} | x_{i+1,j+1} - x_{i,j} |^2 2^{-2|j|}$$

So let $N = N_0 \otimes F_\infty \otimes R_1$ where R_1 is the hyperfinite factor of type II_1, it follows from Lemma 2.11 that for each $\omega \in \beta\mathbb{N} \backslash \mathbb{N}$ the homomorphism π, $x \in R_1 \to 1_{N_0} \otimes 1_{F_\infty} \otimes x$ defines an isomorphism π_ω of $R_{1,\omega}$ onto N_ω. Let θ_1 be an automorphism of $N_0 \otimes F_\infty$ such that $\gamma(\theta_1) = \lambda$ (See a)), and $\theta = \theta_1 \otimes 1$ the corresponding automorphism of N. Clearly using π_ω one has $\theta_\omega = 1$. Let α be an automorphism of R_1 such that $\alpha_\omega \neq 1$ (For instance write $R_1 = R_{1,1} \otimes R_{1,1}$

and take $\alpha(x \otimes y) = y \otimes x)$ then put $\theta' = \theta \circ (\mathrm{Identity}_{N_0 \otimes F_\infty} \otimes \alpha)$, we get $\theta_\omega' \neq 1$ using π_ω, though $\gamma(\theta') = \lambda$.

THEOREM 2.12. *Let M be a factor of type* III_0 *then for any $\omega \in \beta\mathbb{N}/\mathbb{N}$ ond has $M_\omega \neq \mathbb{C}$ and even Center of $M_\omega \neq \mathbb{C}$.*

Proof. Identify M with $P \otimes F_\infty$ where P is a factor isomorphic to M. Let [4] Lemma 5.2.4, φ_0 be a faithful normal state on P such that 1 is isolated in Sp Δ_{φ_0}. Let $(x_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of elements of M such that $\|[x_n, \psi]\|_{n\to\omega} \to 0$, $\forall \psi \in M_*$. Then Lemma 2.11 shows that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of P such that $x_n - (y_n \otimes 1)_{n\to\omega} \to 0$ $*$ strongly. It follows that $\|[y_n, \varphi_0]\|_{n\to\omega} \to 0$ hence (Proposition 2.3) that there exists a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in P_{\varphi_0}$ such that $x_n - (z_n \otimes 1)_{n\to\omega} \to 0$ $*$ strongly. Let $\varphi_0 = \varphi_0 \otimes \mathrm{Trace}$, it is a faithful semifinite normal weight on M which satisfies the conditions of Lemma 5.3.2 of [4] on M. It hence follows from [4] p. 235–238 that the centralizer $M_\varphi = N$ of φ in M satisfies conditions (a)(b)(c) in the proof of Theorem 1.5.

To finish the proof of 2.12 we need only construct a sequence $(v_n)_{n\in\mathbb{N}}$ of elements of the center of $M_\varphi = N$, a faithful normal state ψ_0 on M such that (a) $\|[v_n, \psi_0]\|_{n\to\infty} \to 0$, (b) there exists a strongly dense subset $\mathscr{S}$ of M such that $[v_n, y]_{n\to\omega} \to 0$ strongly $\forall y \in \mathscr{S}$, (c) $\psi_0(v_n) = 0$ for all n. We use the same notations as in the proof of Theorem 1.5 and we let ψ be a faithful normal state on $N = M_\varphi$, and $\psi_0 = \psi \circ E$. For each n there exists a unitary $v_n \in C$ such that $\psi(v_n) = 0$ and $\mathrm{Ad}\, u_{(\epsilon_1,\ldots,\epsilon_n,0,\ldots,0,\ldots)} v_n = v_n$ for all $\epsilon_j = 0, 1$. Then as $v_n \in M_{\psi_0}$ (because $v_n \in N_\psi$) the sequence (v_n) satisfies requirements (a)(b)(c).

III. COMPLETENESS OF THE GROUP OF INNER AUTOMORPHISMS

THEOREM 3.1. *Let M be a von Neumann algebra with separable predual, C =* Center M, *then the following conditions are equivalent*

(a) Int M *is a closed subgroup of* Aut M *where* Aut M *has the topology of pointwise norm convergence in M_*.*

(b) *The homomorphism $u \to$* Ad u *from the unitary group $\mathscr{U}(M)$, gifted with strong topology, to* Aut M, *gifted with topology of pointwise norm convergence in M_*, is open on its range* (Int M).

(c) *For any strong neighborhood $\mathscr{V}$ of 0 in M there exists $\varphi_1, \ldots, \varphi_n \in M_*$ and $\epsilon > 0$ such that $\forall u \in \mathscr{U}(M)$, $\| u\varphi_j u^* - \varphi_j \| < \epsilon \Rightarrow u \in \mathscr{U}(C) + \mathscr{V}$.*

(d) *For any ordered directed set I and any bounded family* $(x_j)_{j\in I}$ *of elements of M such that* $\|[x_j, \varphi]\| \to 0, j \to \infty$ *there exists a bounded family* $(z_j)_{j\in I}$ *of elements of C such that* $x_j - z_j \to_{j\to\infty} 0 *$ *strongly*

(e) *Same statement as* (d) *with* $I = \mathbb{N}$, *the integers in their usual order.*

The topology of pointwise norm convergence in M_* on Aut M, coincides with the topology of uniform weak convergence in M. It has already been fully discussed in the literature [1], [8]. Following [8] we shall call it the u-topology. It is clear that gifted with the u-topology, Aut M is a topological group.

LEMMA 3.2. *Let M be a von Neumann algebra with separable predual, and on* Aut M *let the u-uniform structure be the sup of the right and left uniform structures of* Aut M *with u-topology. Then with u-uniform structure* Aut M *is a complete separable metric space.*

Proof. Apply the results of [1] and [8].

LEMMA 3.3. *Let M be a von Neumann algebra with separable predual, let* $\mathscr{U}(M)$ *be the topological group of unitaries of M with the strong topology, let* $u \to \underline{u}$ *be the canonical open homomorphism of* $\mathscr{U}(M)$ *onto* $\underline{\mathscr{U}}(M) = \mathscr{U}(M)/\mathscr{U}(C)$ *where* $C =$ *Center of M.*

(a) *Let* $(\mathscr{V}_n)_{n-1,2\ldots}$ *be a basis of neighborhoods of* 0 *in M for the strong topology, then* $(\mathscr{W}_n)_{n=1,2,\ldots}$, $\mathscr{W}_n = \{\underline{u}, u \in \mathscr{U}(M) \cap \mathscr{V}_n + \mathscr{U}(C)\}$ *is a basis of neighborhoods of* 1 *in* $\underline{\mathscr{U}}(M)$.

(b) *There exists on* $\underline{\mathscr{U}}(M)$ *a metric, compatible with the topology, which makes it into a complete separable space.*

Proof. (a) The typical neighborhood of 1 in $\underline{\mathscr{U}}(M)$ is $\mathscr{W}$ where $\mathscr{W} = \mathscr{U}(C) \times \mathscr{U}(M) \cap (1 + \mathscr{V})$ where $\mathscr{V}$ is a strong neighborhood of 0 in M. As one can assume that $u\mathscr{V} = \mathscr{V}$, $\forall u \in \mathscr{U}(M)$ we get $\mathscr{W} = \mathscr{U}(M) \cap (\mathscr{U}(C) + \mathscr{V})$.

(b) Let d be a metric on $\mathscr{U}(M)$ corresponding to the sup of left and right uniform structures. Then $\mathscr{U}(M)$ is a complete separable metric space. Then clearly $d(u, v) = \mathrm{Inf}_{\underline{u'}=u,\underline{v'}=v}\, d(u', v')$ is a metric on $\underline{\mathscr{U}}(M)$, yielding the quotient topology, under which $\underline{\mathscr{U}}(M)$ is complete and separable. We now state a known lemma whose proof is included for completeness.

LEMMA 3.4. *Let* G_1 *and* G_2 *be topological groups, polish as topological spaces and f be a continuous bijective homomorphism of* G_1 *onto* G_2, *then* f^{-1} *is continuous.*

Proof. For each Borel subset A of G_1, $f(A)$ is analytic as well as $f(A)^c$ hence $f(A)$ is Borel. In particular f^{-1} has the Baire's property: there exists a meager subset $\mathcal{M} \subset G_2$ such that $f^{-1}/\mathcal{M}^c$ is continuous. Take $v_n \to_{n\to\infty} v_0$, where $v_n \in G_2$, $n = 0, 1, 2...$ There exists $u \in \bigcap_{n=0,1} \mathcal{M}^c v_n^{-1}$, hence such that $uv_n \notin \mathcal{M}$, $n = 0, 1...$ Then $f^{-1}(uv_n) \to_{n\to\infty} f^{-1}(uv_0)$ hence $f^{-1}(v_n) \to_{n\to\infty} f^{-1}(v_0)$.

Proof of (a) $\Rightarrow$ (b). Assume that Int M is closed in Aut M. Then the u-topology makes it into a polish topological group and the map $u \in \underline{\mathcal{U}}(M) \to \operatorname{Ad} u \in \operatorname{Int} M$ is a bijective homomorphism of $\underline{\mathcal{U}}(M)$ onto Int M which is obviously continuous. So Lemma 3.4 shows that this mapping is open on its range hence that (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) By hypothesis when $\operatorname{Ad} u \to 1$ in Aut M, $u \to 1$ in $\underline{\mathcal{U}}(M)$. Hence (c) holds using Lemma (3.3a).

(c) $\Rightarrow$ (d) One can assume $0 \leqslant x_j \leqslant 1/2$ $\forall j \in I$, and writing $2x_j = (x_j + i\sqrt{1 - x_j^2}) + (x_j - i\sqrt{1 - x_j^2})$ one can then assume that x_j is unitary for each j (use the estimate (2.6a)). But then $\|[x_j, \varphi]\| = \| \varphi \circ \operatorname{Ad} x_j - \varphi \|$, $\forall \varphi \in M_*$, hence $\operatorname{Ad} x_j \to 1$ in Aut M so that, using (c), there exists a sequence (z_j), $z_j \in C$ such that $x_j - z_j \to 0$ strongly when $n \to \infty$.

(e) $\Rightarrow$ (a) Let M act in the separable Hilbert space $\mathcal{H}$ with $(\xi_j)_{j=1,2}$ a dense in $\mathcal{H}$. Let $\mathcal{V}_n = \{x \in M, \| x\xi_j - \xi_j \| \leqslant 2^{-n}, j < n\}$ and for each n let $\mathcal{W}_n$ be a neighbourhood of 1 in Aut M, such that (using e)

$$u \in \mathcal{U}(M), \quad \operatorname{Ad} u \in \mathcal{W}_n \Rightarrow u \in \mathcal{U}(C) + \mathcal{V}_n$$

Let $\theta \in \overline{\operatorname{Int} M}$, and $v_n \in \mathcal{U}(M)$ be such that $\operatorname{Ad} v_{n+1}^{-1} v_n \in \mathcal{W}_n$, $\forall n = 1, 2,...$ and $\operatorname{Ad} v_n \to \theta$ when $n \to \infty$. Choose for each n, a $u_n \in \mathcal{U}(M)$ such that $\underline{u}_n = v_n$ and $u_{n+1} - u_n \in \mathcal{V}_n$. Then u_n converges strongly to some $u \in M$, and u is an isometry such that $ux = \theta(x)u$. $\forall x \in M$ so that θ is inner ([4] 1.3).

DEFINITION 3.5. A von Neumann algebra satisfying equivalent conditions in 3.1 will be called a full von Neumann algebra.

This name refers to the completeness of the group of inner automorphisms.

COROLLARY 3.6. *Let M be a factor with separable predual then: M is full $\Leftrightarrow M_\omega = \mathbb{C}$ for some $\omega \in \beta\mathbb{N}\backslash\mathbb{N} \Leftrightarrow M_\omega = \mathbb{C}\ \forall \omega \in \beta\mathbb{N}\backslash\mathbb{N}$.*

Proof. If M is full then condition 3.1(d) immediately implies

$M_\omega = \mathbb{C}\ \forall \omega \in \beta\mathbb{N}\backslash\mathbb{N}$. If M is not full, let φ be a faithful normal state on M, $\epsilon > 0$, $(x_n)_{n=1,2}$ be a bounded sequence of elements of M such that $\|[x_n, \psi]\| \to_{n\to\infty} 0$ for all $\psi \in M_*$ but $\| x_n - \mathbb{C} \|_\varphi^\# \geq \epsilon$ for all $n \in \mathbb{N}$. Then $\forall \omega \in \beta\mathbb{N}\backslash\mathbb{N}$, $\rho_\omega((x_n))$ is a non scalar element of M_ω.

COROLLARY 3.7. *Let M be a factor with separable predual then* (*all central sequences in M are trivial*) $\Rightarrow$ *M is full.*

Proof. We shall prove that M satisfies (3.1e). By Proposition 2.8 $\|[x_n, \varphi]\| \to_{n\to\infty} 0, \forall \varphi \in M^*$ implies that $(x_n)_{n\in\mathbb{N}}$ is a central sequence so that for some sequences of scalars $(\lambda_n)_{n\in\mathbb{N}}$, $x_n - \lambda_n \to 0$ strongly when $n \to \infty$, $x_n{}^* - \mu_n \to_{n\to\infty} 0$ strongly hence $x_n - \lambda_n \to 0$ $*$ strongly using an auxiliary state to show that $\lambda_n - \bar{\mu}_n \to_{n\to\infty} 0$.

COROLLARY 3.8. *Let M be a factor of type* II_1 *with separable predual then M is full $\Leftrightarrow$ M does not have property Γ.*

Proof. If M has property Γ there are non trivial central sequences on M hence non trivial sequences $(x_n)_{n\in\mathbb{N}}$, such that $\|[x_n, \varphi]\| \to_{n\to\infty} 0$, $\forall \varphi \in M_*$ (Proposition 2.8). Conversely assume that M is not full, let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ we want to show that M_ω does not have any minimal projection (See [7]). As $M_\omega \neq \mathbb{C}$, let $e \in M_\omega$ be a non trivial projection, let τ be the canonical trace on M and τ_ω the corresponding trace on M_ω. Then $\tau_\omega(e) = \lambda \in]0, 1[$ and there exists a representing sequence $(e_n)_{n\in\mathbb{N}}$ for e with e_n projection of M, $\forall n$, and $\tau(e_n) = \lambda$, $\forall n$ (See [7]). Obviously for each $x \in M$ and each central sequence $(x_n)_{n\in\mathbb{N}}$ one has $\tau(xx_n) \sim \tau(x)\,\tau(x_n)$ when $n \to \infty$. Applying this we can choose a subsequence $(e_{k_n})_{n=1,2}$ of $(e_n)_{n=1,2}$ such that:

(a) $\|[e_n, e_{k_n}]\|_2 < 1/n\ \forall n$, (b) $|\tau(e_n e_{k_n}) - \lambda^2| < 1/n$

(c) $\|[e_{k_n}, \varphi]\| \to 0$ when $n \to \omega$, $\forall \varphi \in M_*$.

Then $\rho_\omega((e_n e_{k_n})_{n\in\mathbb{N}})$ is a projection in M_ω which is strictly between 0 and e, hence showing that e cannot be minimal. It follows that an arbitrary maximal abelian subalgebra of M_ω is non atomic and hence that there exists a projection $f \in M_\omega$ with $\tau_\omega(f) = 1/2$. Then $2f - 1$ is a unitary $u \in M_\omega$ which has trace 0. Let (f_n) be a representing sequence for f, with f_n projection $\forall n$. Then $(u_n)_{n\in\mathbb{N}}$, with $u_n = 2f_n - 1$ is a sequence of unitaries in M, $[u_n, v] \to_{n\to\omega} 0$ strongly $\forall v \in M$, and $\tau(u_n) \to_{n\to\omega} 0$. It follows immediately that M has property Γ of von Neumann. In the general case we do not know if M is full $\Leftrightarrow$ M does not have property L of Pukanszky.

PROPOSITION 3.9. *Let M be a factor of type* III_0, *then M is not* *full, in fact, for any semi-finite faithful normal weight φ on M, one has:*

$$\sigma_t^{\varphi} \in \overline{\mathrm{Int}\, M}$$

Proof. There exists (see the proof of Theorem 1.5) an increasing sequence of von Neumann subalgebras $N_k \subset M$ such that:

(1) N_k is semifinite and $N_k' \cap M \subset N_k$

(2) N_k is the range of a (necessarily unique) normal conditional expectation E_k

(3) $\bigcup_{k=1}^{\infty} N_k$ is strongly dense in M.

Let φ be a faithful normal state on N_1, and $\varphi_0 = \varphi \circ E_1$, then $\sigma_t^{\varphi_0}$ leaves N_k globally invariant and is inner on N_k (because $\varphi_0 \circ E_k = \varphi_0$) so that there exists a sequence of unitaries $u_k \in N_k \cap M\varphi_0$ such that

$$\mathrm{Ad}\, u_k(x) \to \sigma_t^{\varphi_0}(x) * \text{strongly} \quad \forall x \in \bigcup_{k=1}^{\infty} N_k$$

It follows easily from $\varphi_0 \circ \mathrm{Ad}\, u_k = \varphi_0$ that $\sigma_t^{\varphi_0} = \lim_{k\to\infty} \mathrm{Ad}\, u_k$ in $\mathrm{Aut}\, M$.

PROPOSITION 3.9. *Let P be a von Neumann algebra acting in a separable Hilbert space $\mathscr{H}$, with cyclic and separating vector ξ_0. Let G_2 be the free group of 2 generators s_1, s_2 and $N = \bigotimes_{s\in G_2} (P, \xi_0)$, π_s* for *$s \in G_2$ being the canonical injection $x \to \cdots 1 \otimes x \otimes 1 \cdots$* of *$P$* in *$N$. Let θ be the representation of G_2 on N such that:*

$$\theta_s \pi_{s'}(x) = \pi_{ss'}(x) \qquad \forall s, s' \in G_2 .$$

Finally let $M = \mathscr{W}^*(G_2, N)$ be the cross product of N by G_2, let I be the canonical injection of N in M; $s \to U_s$ the canonical injection of G_2 in the unitary group of M and E the conditional expectation of M onto $I(N)$.

(a) *For each $s \in G_2$, U_s is in the centraliser of the state $\psi \in M_*$ $\psi(x) = \omega_{\eta_0}(I^{-1}(E(x)))$, $\forall x \in M$ where $\eta_0 = \bigotimes_{s\in G_2} \xi_0$.*

(b) *$\forall x \in M$ one has $\| x - \psi(x)\|_\psi \leqslant 28 \sum_{j=1}^{2} \|[x, U_{s_j}]\|_\psi$.*

(c) *The modular operator $\Delta_{\psi,M}$* of *ψ relative to M is, up to multiplicity, the infinite tensor product of the $\Delta_{\xi_0,P}$ acting in $\bigotimes_{\nu\in G_2}(\mathscr{H}, \xi_0)$*

Proof. (a) By construction ω_{η_0} is θ_s-invariant for each $s \in G_2$ hence ([4] Proposition 1.3) ψ is Ad U_s invariant for each $s \in G_2$ and $U_s \in M_\psi$.

LEMMA 3.10. *Let* $\mathscr{K} = \otimes_{s\in G_2} (\mathscr{H}, \xi_0)$; $\eta_0 = \otimes_{s\in G_2} \xi_0$, *then* $\forall x \in N$

$$\|\langle x\eta_0 , \eta_0\rangle \eta_0 - x\eta_0\| \leqslant 14 \sum_{j=1,2} \|(\theta_{s_j}(x) - x)\eta_0\|$$

Proof. Let $\mathscr{B}$ be an orthonormal basis of $\mathscr{H}$ containing ξ_0 and $\mathscr{B}^{(G_2)}$ be the set of all maps g from G_2 to $\mathscr{B}$ which except on a finite subset of G_2 satisfy $g(s) = \xi_0$. For each $g \in \mathscr{B}^{(G_2)}$ put $\xi_g = \otimes_{s\in G_2} g(s)$ and note that $(\xi_g)_{g\in\mathscr{B}^{(G_2)}}$ is an orthonormal basis for $\mathscr{K}$. For $g \in \mathscr{B}^{(G_2)}$ and $s \in G_2$, put g_s, $g_s(t) = g(s^{-1}t)\ \forall t \in G_2$. Then $g \to g_s$ is a bijection of $\mathscr{B}^{(G_2)}$ onto $\mathscr{B}^{(G_2)}$ and it defines a unitary V_s in $\mathscr{K}$; $V_s\xi_g = \xi_{g_s}$ $\forall g \in \mathscr{B}^{(G_2)}$. It is easy to check that $\theta_s(x) = V_s x V_s^*$, $s \in G_2$, as well as $V_s\eta_0 = \eta_0$. Now the action of G_2 on $\mathscr{B}^{(G_2)}$ is free except on $g = \xi_0$, for, assume $g \in \mathscr{B}^{(G_2)}$, $g(s_0) \neq \xi_0$, $g_s = g$ with s_0, $s \in G_2$ then $g(s^{-k}s_0) \neq \xi_0$ $\forall k = 1, 2\ldots$, which if $s \neq 1$ contradicts ($g(t) = \xi_0$ except on a finite subset of G_2). Let $g \in \mathscr{B}^{(G_2)}$, $g \neq \xi_0$. As $s_1 \neq s_2 \Rightarrow g_{s_1} \neq g_{s_2} \Rightarrow \xi_{g_{s_1}} \perp \xi_{g_{s_2}}$ we have $f \in l^2(G_2)$ where $f(s) = \langle x\eta_0 , \xi_{g_s}\rangle$, $\forall s$. Then Lemma 4.3.20 in [12] yields:

$$\sum_{s\in G_2} |\langle x\eta_0 , \xi_{g_s}\rangle|^2 \leqslant (14)^2 \sum_{s\in G_2} \sum_{j=1,2} |\langle V_{s_j} x\eta_0 , \xi_{g_s}\rangle - \langle x\eta_0 , \xi_{g_s}\rangle|^2$$

Adding the inequalities corresponding to each orbit of G_2 in $\mathscr{B}^{(G_2)}$ yields:

$$\sum_{g\in\mathscr{B}^{(G_2)}, g\neq\xi_0} |\langle x\eta_0 , \xi_g\rangle|^2 \leqslant (14)^2 \sum_{j=1,2; g\neq\xi_0} |\langle (V_{s_j} x\eta_0) - x\eta_0 , \xi_g\rangle|^2$$

Hence

$$\| x\eta_0 - \langle x\eta_0 , \eta_0\rangle \eta_0\|^2 \leqslant (14)^2 \sum_{j=1,2} \|(\theta_{s_j}(x) - x)\eta_0\|^2.$$

Proof of (b). To avoid cumbersome notations we put $I(x) = x$ $\forall x \in N$. Let $y \in M$, then y is a sum of a strongly convergent sequence, where $x_s \in N$, $y = \sum_{s\in G_2} x_s U_s$ and $\| y \|_\psi^2 = \sum \| \theta_s(x_s)\eta_0\|^2 = \sum \| x_s\eta_0\|^2$ (We have used the θ-invariance of ω_{η_0}). Then

$$\|[y, U_{s_j}]\|_\psi^2 = \| U_{s_j}^{-1} y U_{s_j} - y \|_\psi^2 = \left\| \sum_s \theta_{s_j}(x_s) U_{s_j^{-1}ss_j} - \sum_t x_t U_t \right\|_\psi^2$$

$$= \sum \|(x_s - \theta_{s_j}(x_{s_j^{-1}ss_j}))\eta_0\|^2.$$

For $s \in G_2$ we put $f(s) = \| x_s\eta_0 \|$.

Clearly $f \in l^2(G_2)$ and

$$\sum | f(s_j^{-1}ss_j) - f(s)|^2 = \sum | \| x_s\eta_0 \| - \| x_{s_j^{-1}ss_j}\eta_0 \| |^2$$

$$= \sum | \| x_s\eta_0 \| - \| \theta_{s_j}(x_{s_j^{-1}ss_j}) \eta_0 \| |^2$$

$$\leqslant \sum \| x_s\eta_0 - \theta_{s_j}(x_{s_j^{-1}ss_j}) \eta_0 \|^2 = \|[y, U_{s_j}]\|^2$$

hence Lemma 4.3.3 of [12] yields:

$$\sum_{s\neq 1} \| x_s\eta_0 \|^2 \leqslant (14)^2 \sum_{j=1,2} \|[y, U_{s_j}]\|_\psi^2$$

which means that $\forall y \in M$ one has:

$$\| y - E(y)\|_\psi \leqslant 14 \sum_{j=1,2} \|[y, U_{s_j}]\|_\psi$$

Moreover one has:

$$\| x_1 - \theta_{s_j}(x_1)\|_{\omega_{\eta_0}} \leqslant \|[y, U_{s_j}]\|_\psi$$

hence by Lemma 3.4

$$\| x_1\eta_0 - \omega_{\eta_0}(x_1) \eta_0 \| \leqslant 14 \sum_{j=1,2} \|[y, U_{s_j}]\|_\psi$$

which implies

$$\| E(y) - \psi(E(y))\|_\psi \leqslant 14 \sum_{j=1,2} \|[y, U_{s_j}]\|_\psi$$

and as $\psi \circ E = \psi$ we get

$$\| y - \psi(y)\|_\psi \leqslant 28 \sum_{j=1,2} \|[y, U_{s_j}]\|_\psi .$$

Proof of (c). Let M act in a Hilbert space $\mathscr{H}_\psi$ and $\xi_\psi \in \mathscr{H}_\psi$ be cyclic and separating for M with $\omega_{\xi_\psi} = \psi$. Put

$$\mathscr{H}_1 = \overline{I(N)\,\xi_\psi}\,, \qquad \mathscr{H}_s = U_s\mathscr{H}_1 \qquad \forall s \in G_2\,.$$

Then for $s \neq s'$, $s, s' \in G_2$, $\mathscr{H}_s$ is orthogonal to $\mathscr{H}_{s'}$, moreover

$$\mathscr{H}_\psi = \bigoplus_{s\in G_2} \mathscr{H}_s$$

As $I(N)$ is globally invariant under σ_t^{ψ}, $\forall t \in \mathbb{R}$, we see that $\mathscr{K}_1$ is invariant under Δ_ψ and that the restriction of Δ_ψ of $\mathscr{K}_1$ is unitarily equivalent to $\Delta_{\eta_0,N}$ using the unitary equivalence of the triplets $(\mathscr{K}, N, \eta_0)$ and $(\mathscr{K}_1, I(N), \xi_\psi)$. As U_s commutes with Δ_ψ, $\forall s \in G_2$ (Use a)), we see that, up to multiplicity, Δ_ψ is equivalent to $\Delta_{\eta_0,N} = \bigotimes_{\nu\in G_2} \Delta_{\xi_0,P}$.

COROLLARY 3.10. *There exist full factors of type* I, II_1, II_∞, III_λ, $\lambda \neq 0$.

Proof. Obviously from 3.9(b) the von Neumann algebras constructed in 3.9 are full factors. Moreover as M_ψ contains U_{s_1} and U_{s_2} it is a factor hence it follows from [4] Corollary 3.2.5b) that for each $\lambda \in]0, 1]$ there exists a full factor of type III_λ. The cases II_1, II_∞ follow from section 2 and the other cases are trivial.

IV. Full Factors with Almost Periodic States

In all this section, M is a full factor with separable predual. To Compute $Sd(M)$ we shall use the following :

THEOREM 4.1. *Let Γ be a denumerable subgroup of $\mathbb{R}_+^*$, φ, an Γ-almost periodic weight on M, then*:

$$\mathrm{Sd}(M) = \Gamma(\sigma^{\varphi,\Gamma}) = \bigcap_{e \text{ projection} \in M_\varphi,\ e \neq 0} \text{point spectrum } \Delta_{\varphi e}$$

This formula is to compare to [4] 3.2.1. However, it is not true in general, for non full factors. The fundamental lemma is:

LEMMA 2. *Let M and Γ as in Theorem* 3.1, β, G, $\tilde{\beta}$ *as in* 1.1, *and* φ_1, φ_2 *be Γ-almost periodic weights on M. Let G act on the unitary group $\mathscr{U}(M)$ by means of $\sigma^{\varphi,\Gamma}$.*

Then there exists a cocycle $v \in Z^1(G, \mathscr{U}(M))$, strongly continuous in $s \in G$ such that $\sigma_s^{\varphi_2,\Gamma} = \mathrm{Ad}\, v_s \cdot \sigma_s^{\varphi_1,\Gamma}\ \forall s \in G$.

Proof. Let $s \in G$, $t_n \in \mathbb{R}$ be such that $\tilde{\beta}(t_n) \to s$. Then $\sigma_{\tilde{\beta}(t_n)}^{\varphi_1,\Gamma} \to \sigma_s^{\varphi_1,\Gamma}$ in the topology on Aut M of pointwise norm convergence in M_*. Hence $\sigma_s^{\varphi_2,\Gamma}(\sigma_s^{\varphi_1,\Gamma})^{-1}$ is the limit in this topology of $\sigma_{t_n}^{\varphi_2,\Gamma}(\sigma_{t_n}^{\varphi_1,\Gamma})^{-1} = \mathrm{Ad}\, u_{t_n}$ where $u_t = (D\varphi_2 : D\varphi_1)_t$ (See [4]). But by Theorem 3.1, the group of inner automorphisms of M is closed, so that $\forall s \in G$, $\sigma_s^{\varphi_2,\Gamma}(\sigma_s^{\varphi_1,\Gamma})^{-1} \in \mathrm{Int}\, M$. For each $s \in G$, let F_s be the set of unitaries in M such that $\sigma_s^{\varphi_2,\Gamma} = \mathrm{Ad}\, v\sigma_s^{\varphi_1,\Gamma}$. We know that F_s is non empty for any s, hence there exists a Borel map $s \to w_s$ from G to $\mathcal{U}(M)$ (with the strong topology) such that $w_s \in F_s$, $\forall s \in G$ (See [6]). For s, s' in G one gets $w_s \sigma_s^{\varphi_1, \Gamma}(w_{s'}) w_{s+s'}^* \in$ Center of M, hence there exists a Borel map γ from G^2 to $T_1 = \{z \in \mathbb{C}, |z| = 1\}$ such that:

(1) $w_{s+s'} = \gamma(s, s') w_s \sigma_s^{\varphi_1, \Gamma}(w_{s'}) \qquad \forall (s, s') \in G^2$

(2) $\gamma(s, t) \gamma(r + s, t)^{-1} \gamma(r, s + t) \gamma(r, s)^{-1} = 1, \forall r, s, t \in G$

We shall now show that $\gamma(s, s') = \gamma(s', s)$, $\forall s. s' \in G$. To see this let $\mathcal{H}_{\varphi_1}$, Δ_{φ_1} correspond to φ_1, as usual, and let $u_t = (D\varphi_2 : D\varphi_1)_t$, for $t \in \mathbb{R}$. For $t_1, t_2 \in \mathbb{R}$ one has:

$$u_{t_1} \Delta_{\varphi_1}^{it_1} u_{t_2} \Delta_{\varphi_1}^{it_2} = u_{t_1} \sigma_{t_1}^{\varphi_1}(u_{t_2}) \Delta_{\varphi_1}^{i(t_1+t_2)} = u_{t_1+t_2} \Delta_{\varphi_1}^{i(t_1+t_2)}$$

$$u_{t_2} \Delta_{\varphi_1}^{it_2} u_{t_1} \Delta_{\varphi_1}^{it_1} = u_{t_2+t_1} \Delta_{\varphi_1}^{i(t_1+t_2)} = u_{t_1} \Delta_{\varphi_1}^{it_1} u_{t_2} \Delta_{\varphi_1}^{it_2}$$

so that the $u_t \Delta_\varphi^{it}$ generate an abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H}_{\varphi_1})$. Let $s \in G$, $t_n \in \mathbb{R}$ be such that $s_n = \tilde{\beta}(t_n) \to s$ when $n \to \infty$. Then Ad $u_{t_n} \to$ Ad v_s for the topology of norm pointwise convergence in M_* so that (Theorem 3.1b)) there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n \in T_1$ such that $\lambda_n u_{t_n} \to v_s$ $*$ strongly when $n \to \infty$. It follows that, with $\Delta_{\varphi_1} = \sum_{\lambda \in \Gamma} \lambda E_\lambda$, $\Delta_{\varphi_1}^{(s)} = \sum_{\lambda \in \Gamma} (s, \lambda) E_\lambda$, one has:

$$v_s \Delta_{\varphi_1}^{(s)} = \lim_{n \to \infty} \lambda_n u_{t_n} \Delta_{\varphi_1}^{(s_n)} = \lim_{n \to \infty} \lambda_n u_{t_n} \Delta_{\varphi_1}^{it_n} \in \mathcal{A}$$

Hence

$$v_s \Delta_{\varphi_1}^{(s)} v_{s'} \Delta_{\varphi_1}^{(s')} = v_{s'} \Delta_{\varphi_1}^{(s')} v_s \Delta_{\varphi_1}^{(s)}$$

for any $s, s' \in G$ and

$$\gamma(s, s') = \gamma(s', s), \qquad \forall s, s \in G'.$$

Now this means that the extension of T_1 by G corresponding to γ is Abelian and hence splits ([10]). It follows that one can choose the v_s forming a 1-cocycle, hence 4.2 follows.

Proof of Theorem 4.1. Let ψ be another almost periodic weight on M, and let Γ be a denumerable subgroup of $\mathbb{R}_+^*$ containing point Spect. Δ_φ and p. Sp. Δ_ψ; then φ and ψ are Γ-almost periodic and Lemma 3.2 shows that $\sigma^{\varphi, \Gamma} \sim \sigma^{\psi, \Gamma}$ in the sense of [4] def. 2.3.3. Then by Theorem 2.2.4 of [4] one has $\Gamma(\sigma^{\varphi, \Gamma}) = \Gamma(\sigma^{\psi, \Gamma})$ hence Theorem 3.1 follows from formula 1.

Remark 4.3. Let M, Γ, φ_1, φ_2 and $v \in Z^1(G, \mathcal{U}(M))$ be as in Lemma 4.2, and let $u_t = (D\varphi_2 : D\varphi_1)_t$, $\forall t \in \mathbb{R}$. Then there exists $\lambda \in \mathbb{R}_+^*$ such that $v_{\beta(t)} = \lambda^{it} u_t$. In particular $(D\lambda\varphi_2 : D\varphi_1)$ then extends to G, but it is not true in general that $(D\varphi_2 : D\varphi_1)_t$ itself extends to G.

COROLLARY 4.4. *Let Λ be an arbitrary denumerable subgroup of $\mathbb{R}_+^*$ then there exists a (full) factor M acting in a separable Hilbert space such that*

$$\mathrm{Sd}(M) = \Lambda.$$

Proof. In fact we shall construct explicitly a map $\Lambda \to M(\Lambda)$. Let Λ be given, put $(P_\Lambda, \varphi_\Lambda) = \bigotimes_{\lambda \in \Lambda} (R_\lambda, \varphi_\lambda)$ where R_λ is the Powers factor of type III_λ and φ_λ is the canonical product state on R_λ.

Each φ_λ is almost periodic with $\mathrm{Sp}\, \Delta_{\varphi_\lambda} = \{\lambda^n, n \in \mathbb{Z}\}$, hence it is easy to conclude that φ_Λ is almost periodic with

point spectrum $\Delta\varphi_\Lambda = \Lambda$.

Now let M_Λ be the full factor corresponding to the couple P_Λ, φ_Λ by Proposition 3.9 with $\omega_{\xi_0} = \varphi_\Lambda$. Let also ψ_Λ be the corresponding faithful normal state on M_Λ.

By Proposition 3.9, Δ_{ψ_Λ} is a diagonal operator so that ψ_Λ is almost periodic. By Proposition 3.9(c) one has point spectrum $\psi_\Lambda = \Lambda$. Finally by Proposition 3.9 the relative commutant of the centraliser M_{ψ_Λ} of ψ_Λ in M_Λ is reduced to $\mathbb{C}$ hence M_{ψ_Λ} is a factor. Hence it follows from [4] 2.2.2(b) that $\Gamma(\sigma^{\psi_\Lambda, \Gamma}) = \mathrm{Sp}(\sigma^{\psi_\Lambda, \Gamma}) = \Lambda$ and from Theorem 4.1 that $\mathrm{Sd}(M_\Lambda) = \Lambda$.

COROLLARY 4.5. *The Borel space of isomorphism classes of factors of type* III_1 *acting in a separable Hilbert space is not countably separated.*

Proof. Let $\mathcal{B}$ be the Borel space obtained dividing $\mathbb{R}$ by the relation $t_1 \sim t_2$ iff $\mathbb{Q}t_1 + \mathbb{Q} = \mathbb{Q}t_2 + \mathbb{Q}$. Then $\mathcal{B}$ is not countably separated. Put $\Gamma_t = \{e^\alpha, \alpha \in \mathbb{Q}t + \mathbb{Q}\}$. We shall admit that the map $t \to M_{\Gamma_t}$ is Borel. Now if $t_1 \not\sim t_2$ the factor $M_{\Gamma_{t_1}}$ is not isomorphic to $M_{\Gamma_{t_2}}$ for $\mathrm{Sd}(M_{\Gamma_t}) = \Gamma_t$. If $t_1 \sim t_2$ by [9]. theorem 4.1 p. 111 the couples $(P_{\Gamma_{t_1}}, \varphi_{\Gamma_{t_1}})$, $(P_{\Gamma_{t_2}}, \varphi_{\Gamma_{t_2}})$ are isomorphic so that $M_{\Gamma_{t_1}}$ is isomorphic to $M_{\Gamma_{t_2}}$. Hence $t \to M_{\Gamma_t}$ defines an injection of $\mathcal{B}$ into the Borel space of isomorphism classes of factors of type III_1.

COROLLARY 4.6. *There are type* III_1 *factors for which*

$$\text{Center of Out } M \neq \delta_M(\mathbb{R})$$

Proof. Let Γ be a dense subgroup of $\mathbb{R}_+^*$, M a full factor of type III_1 and φ a Γ-almost periodic weight on M. As M is full, Out M = Aut M/Int M is hausdorff. Put $\delta_M(s) = \lim_{t\to s} \delta_M(t)$ for all $s \in G$ (where $\tilde{\beta}$ is noted as identity).

Then $\delta_M(G) \subset$ Center of Out M is a compact subgroup of Out M so that Lemma 3.4 prevents the injective map $t \in \mathbb{R} \to \delta_M(t) \in \delta_M(G)$ to be surjective.

THEOREM 4.7. *Let M be a full factor with separable predual, with* $\mathrm{Sd}(M) = \Gamma \neq \mathbb{R}$.

(1) *There exists an almost periodic weight* φ *such that*

$\mathrm{Sd}(M)$ = point spectrum Δ_φ

(2) *Let* φ_1 *and* φ_2 *be two* Γ*-almost periodic weights on M such that* $\varphi_1(1) = \varphi_2(1) = +\infty$ *then there exists a unitary* $u \in M$ *and an* $\alpha \in \mathbb{R}_+^*$ *such that* $\varphi_2 = \alpha\varphi_1(u \cdot u^*)$.

In the proof we shall show the following analogue of Theorem 4.2.6 [4].

LEMMA 4.8. *Let M be a full factor with* $Sd(M) = \Gamma \neq \mathbb{R}_+^*$, *let* φ *be an almost periodic weight on M then the following conditions are equivalent.*

(a) φ *is a* Γ*-almost periodic weight.*

(b) *Point spectrum* $\Delta_\varphi = \mathrm{Sd}(M)$.

(c) $M_\varphi' \cap M = \mathbb{C}$.

(d) M_φ *is a factor.*

(e) ($M_\varphi \subset M_\psi$, ψ *faithful semi-finite normal weight*) $\Rightarrow \psi = \alpha\varphi$ *for some* $\alpha > 0$.

Proof. (a) $\Leftrightarrow$ (b) is clear. (b) $\Rightarrow$ (d) One has $\mathrm{Sp}(\sigma^{\varphi,\Gamma}) = \Gamma(\sigma^{\varphi,\Gamma})$ hence by Theorem 2.4.1 of [4], M_φ is a factor. (d) $\Rightarrow$ (c) follows from the inclusion $M \cap M_\varphi' \subset M_\varphi$.

(c) $\Rightarrow$ (e) By hypothesis the $u_t = (D\psi : D\varphi)_t$ belong to $M_\varphi' \cap M = \mathbb{C}$ hence ψ is proportional to φ (compare with [4] Theorem 4.2.1b)).

(e) $\Rightarrow$ (d) Take $h \in [1/2, 1]$, $h \in$ Center of M_φ then $\psi = \varphi(h\cdot)$ has a centralizer containing M_φ hence $h = \alpha$ for some $\alpha \in \mathbb{R}_+^*$ so that M_φ is a factor.

(d) $\Rightarrow$ (a) follows from Proposition 2.2.2(b) in [4] and Theorem 4.1 above.

LEMMA 4.9. *Let M be a factor, φ be an Γ-almost periodic weight on M. Let B be the operator of multiplication by the function $\gamma \rightarrow \beta(\gamma)$ in $l^2(\Gamma)$, and $\omega = \mathrm{tr}(B\cdot)$ the corresponding weight in $\mathscr{L}(l^2(\Gamma))$* (Tr *is the usual trace*). *Then $M \otimes \mathscr{L}(l^2(\Gamma))$ is isomorphic to the cross product of the centraliser $(M \otimes \mathscr{L}(l^2(\Gamma)))_{\varphi\otimes\omega}$ by an action of the group Γ (with discrete topology).*

Proof. The weight $\varphi \otimes \omega$ is Γ-almost periodic on $P = M \otimes \mathscr{L}(l^2(\Gamma))$ hence $P_{\varphi\otimes\omega}$ is the range of a normal conditional expectation E from P. Moreover the inclusion $P'_{\varphi\otimes\omega} \subset P_{\varphi\otimes\omega}$ follows from an immediate modification of [4] Lemma 4.2.3.

For $\gamma \in \Gamma$ let u_γ be the unitary in $l^2(\Gamma)$ corresponding to translation of γ. Clearly $\gamma \rightarrow U_\gamma = 1 \otimes u_\gamma$ is an homomorphism of Γ in the unitary group of P such that: $\sigma_t^{\varphi\otimes\omega,\Gamma}(U_\gamma) = (t, \gamma)\, U_\gamma$, $\forall t \in G$. It follows that Ad U_γ leaves $P_{\varphi\otimes\omega}$ globally invariant, thus defining an automorphism V_γ of this von Neumann algebra. Moreover using [4] Part. 2 and the discreteness of Γ we see that $P_{\varphi\otimes\omega}$ and the U_γ generate the von Neumann algebra P.

Let τ be the restriction of $\varphi \otimes \omega$ to $P_{\varphi\otimes\omega}$; it is faithful semifinite normal trace and $\tau \circ V_\gamma = \beta(\gamma)\tau$ (Use [4] Lemma 1.4.5(b)) so that for any $\gamma \neq 1$ the automorphism V_γ is outer and satisfies $p(V_\gamma) = 0$ with the notations of [4] Proposition 1.5.1.

Now the conclusion follows from [4] Remark 4.1.3(d).

LEMMA 4.10. *Let Λ be a discrete Abelian group acting by automorphisms $x \rightarrow g \cdot x$ on a von Neumann algebra N. Assume that the center C of N is diffuse and that the action of Λ on C is ergodic. Then $P = W^*(\Lambda, N)$ is not a full factor and has property L of Pukanszky.*

Proof. The action of Λ on C is weakly equivalent to a free action of $(\mathbb{Z}/2)^{(\mathbb{N})}$ on C (result due to W. Krieger). Let φ be an arbitrary faithful normal state on C. Then for each $n = 1, 2...$ there exists a unitary $u_n \in C$ such that: $\varphi(u_n) = 0$ and

$$S_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots)}u_n = u_n \qquad \forall \epsilon_j = 0, 1 \qquad j = 1,\ldots, n.$$

Identifying N with its canonical image in $P = W^*(\Lambda, N)$, we note E the canonical conditional expectation of P onto N and $\lambda \rightarrow U_\lambda$ the canonical homomorphism of Λ in the unitary group of P.

For $\lambda \in \Lambda$ the restriction of Ad U_λ to C belongs to the full group of the S_ϵ, $\epsilon \in (\mathbb{Z}/2)^{(\mathbb{N})}$ so that there exists a family of projections $(e_\epsilon^\lambda)_{\epsilon\in(Z/2)^{(\mathbb{N})}}$ in C such that $U_\lambda x U_\lambda^* = \sum S_\epsilon(e_\epsilon^\lambda x)$. Let then $e_n^\lambda = \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_n,0,\ldots)} S_\epsilon(e_\epsilon^\lambda)$. When $n \rightarrow \infty$, e_n^λ tends to 1 strongly and as

$U_\lambda u_n U_\lambda^* e_n^\lambda = u_n e_n^\lambda$, $U_\lambda u_n U_\lambda^* - u_n = u_n(e_n^\lambda - 1) + U_\lambda u_n U_\lambda^*(1 - e_n^\lambda)$ tends to 0 strongly. Moreover for each n, $u_n \in P_\psi$ where $\psi = \varphi \cdot E$. Since $[u_n, xU_\lambda] \to_{n\to\infty} 0$ $*$ strongly for any $x \in N$, we see that $\|[u_n, y\psi]\|_{n\to\infty} \to 0$ for each y in the linear span of the NU_λ, $\lambda \in \Lambda$ in P.

As the set of such $y\psi$ is norm dense in M_*, and as $\psi(u_n) = 0$, $\forall n$, we conclude that P does not satisfy condition (d) in 3.1. Moreover the sequence $(u_n)_{n\in\mathbb{N}}$ is a central sequence in P (use the proposition 2.8) hence P has property L of Pukanszky.

Proof of (1) *in Theorem* 4.7. Let φ be an almost periodic weight on M, with Λ = group generated by point spectrum of φ. Assume that the center of M_φ is diffuse. Let $\psi = \varphi \otimes \omega$ be as in Lemma 4.9, on $P = M \otimes \mathscr{L}(l^2(\Gamma))$ and for $\lambda \in \Lambda$ let E_λ be the projection in $\mathscr{L}(l^2(\Lambda))$ corresponding to multiplication by the characteristic function of $\{\lambda\}$. Then $\psi_{1\otimes E_\lambda}$ is isomorphic to $\beta(\lambda)\varphi$ and hence the center of its centraliser is diffuse.

As the $(1 \otimes E_\lambda)_{\lambda\in\Lambda}$ form a partition of unity in the centraliser of ψ it follows that the center of this centraliser is diffuse. But using Lemmas 4.8 and 4.9, it contradicts the fact that M is full. Now let $e \in M_\varphi$ be an atom in the center of M_φ, then the weight φ_e on M_e satisfies condition (d) of Lemma 4.8. Now Theorem 4.7 being trivial for factors of type II we shall assume that M is of type III, hence that M_e is isomorphic to M. Then the corresponding weight on M satisfies condition (b) of 4.8 hence (1) of 4.7.

Proof of (2) *in Theorem* 4.7. Let $\alpha \in \mathbb{R}_+^*$ be such that $u_t = (D\varphi_2, D\alpha\varphi_1)_t$ extends to the dual group G of Γ. Let $Q = M \otimes F_2$ be the von Neumann algebra of 2×2 matrices over M, and φ, $\varphi(\sum x_{ij} \otimes e_{ij}) = \alpha\varphi_1(x_{11}) + \varphi_2(x_{22})$ be the corresponding weight on Q.

By Proposition 1.1 we see that φ is Γ-almost periodic on Q, and as $\mathrm{Sd}(Q) = \Gamma$ that the centraliser Q_φ of φ is factor (Lemma 4.8). In particular the two infinite projections $1 \otimes e_{11}$, $1 \otimes e_{22}$ of Q_φ are equivalent and consequently there exists a unitary $u \in M$, with $u^* \otimes e_{21} \in Q_\varphi$ and it follows (as in [4] p. 221) that $\varphi_2 = \alpha\varphi_{1,u}$.

COROLLARY 4.11. *Let M be a full factor with separable predual then*

$$\overline{\mathrm{Sd}(M)} = S(M).$$

Proof. If $\mathrm{Sd}(M) = \mathbb{R}_+^*$ the conclusion follows from 1.7, so we can assume that $\mathrm{Sd}(M) = \Gamma \neq \mathbb{R}_+^*$. Let φ be a Γ-almost periodic weight on M (Theorem 4.7) then M_φ is a factor (Lemma 4.8) hence by [4] 2.2.2(b) we have $S(M) = \mathrm{Sp}\,\Delta_\varphi$. But as Δ_φ is diagonal its spectrum is the closure of its spectrum and we get 4.11.

COROLLARY 4.12. *Let M be a full factor with separable predual with* $\mathrm{Sd}(M) = \Gamma \neq \mathbb{R}_+^*$. *Then if M is not finite it is the cross product of a factor N of type II_∞ by an action $\gamma \to \theta_\gamma$ of Γ on N such that*

$$\tau \circ \theta_\gamma = \beta(\gamma)\tau \qquad \forall \gamma \in \Gamma.$$

Moreover in such a description the isomorphism class of N as well as the conjugacy class in Out N *of the θ_γ are uniquely determined by M.*

Proof. Starting from a Γ-almost periodic weight φ on M such that $\varphi(1) = +\infty$ we consider $\psi = \varphi \otimes \omega$ on $P = M \otimes \mathscr{L}(l^2(\Gamma))$ as in Lemma 4.9. Then ψ is Γ-almost periodic and $\psi(1) = +\infty$ so that P_ψ is (use 4.8) a factor of type II_∞. So that the existence of N and θ follows from Lemma 4.9.

Now assume $M = W^*(\Gamma, N)$ where Γ acts on the type II_∞ factor by θ: $\tau \circ \theta_\gamma = \beta(\gamma)\tau$, $\forall \gamma \in \Gamma$. Let N be identified to a von Neumann subalgebra of M, E be the corresponding conditional expectation and $\varphi = \tau \circ E$. Then it follows from [4] and Proposition 1.1 that φ is Γ-almost periodic on M with $\varphi(1) = +\infty$, hence the uniqueness statement (4.7(b)) implies the last conclusion of 4.12.

V. FULL FACTORS WITHOUT ALMOST PERIODIC STATES

Our aim is to prove the existence of such factors.

DEFINITION 5.1. Let M be a full factor of type III_1, we note $\tau(M)$ the weakest topology on $\mathbb{R}$ for which the modular homomorphism $\mathbb{R} \xrightarrow{\delta} \mathrm{Out}\, M$ is continuous.

We shall from now on assume that M has a separable predual. Then Out M is a metrisable topological group hence $\tau(M)$ is a metrisable group topology on $\mathbb{R}$, weaker than the usual one. Also $\tau(M)$ is entirely determined by the knowledge of which sequences $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{R}$ are $\tau(M)$ converging to 0.

THEOREM 5.2. *Let ρ be an arbitrary injective separable unitary representation of $\mathbb{R}$ then there exists a full factor M of type III_1 acting in a separable Hilbert space such that $\tau(M)$ = weakest topology on $\mathbb{R}$ for which ρ is strongly continuous.*

Proof. We can assume that there exists a finite measure μ on $\mathbb{R}_+^*$ with $\int \lambda \, d\mu(\lambda) < \infty$ such that for each t, $\rho(t)$ is the multiplication by λ^{it} in $L^2(\mathbb{R}_+^*, d\mu)$. Let $P = L^\infty(\mathbb{R}_+^*, \mu) \otimes F_2$, φ the unique state on P proportional to the functional

$$f = \sum f_{ij} \otimes e_{ij} \to \int f_{11}(\lambda)\, d\mu(\lambda) + \int \lambda f_{22}(\lambda)\, d\mu(\lambda)$$

By [4] 1.2.3(b) we have, for $f = \sum f_{ij} \otimes e_{ij}$ and $t \in \mathbb{R}$,

$$\sigma_t^{\varphi}(f) = f_{11} \otimes e_{11} + \rho(t) f_{21} \otimes e_{21} + \overline{\rho(t)} f_{12} \otimes e_{12} + f_{22} \otimes e_{22}$$

(where $\rho(t)(\lambda) = \lambda^{it}$, $\forall \lambda \in \mathbb{R}_+^*$). Hence we conclude that for sequences $(t_n)_{n \in \mathbb{N}}$, $t_n \in \mathbb{R}$ one has: $\sigma_{t_n}^{\psi} \to 1$ in Aut $P \Leftrightarrow \rho(t_n) \to 1$ strongly. Let P act in $\mathscr{H}$, and ξ_0 be cyclic and separating with $\omega_{\xi_0} = \varphi$. We now adopt the notations of Proposition 3.9 and let M be the corresponding factor. By (3.9c) we have for any sequence $(t_n)_{n \in \mathbb{N}}$, $t_n \in \mathbb{R}$

$$(\sigma_{t_n}^{\psi} \to 1 \textit{ in } \operatorname{Aut} M) \Leftrightarrow (\Delta_{\psi,M}^{it_n} \to 1 \text{ strongly}) \Leftrightarrow (\Delta_{\varphi,P}^{it_n} \to 1 \text{ strongly})$$

hence $\sigma_{t_n}^{\psi} \to 1$ in Aut $M \Leftrightarrow \rho(t_n) \to 1$ strongly. Now assume that $\delta_M(t_n) \to 1$ when $n \to \infty$. Let u_n, $n = 1, 2 \ldots$ be unitaries in M such that Ad $u_n \circ \sigma_{t_n}^{\psi} \to 1$ in Aut M with u topology. Then

$$\operatorname{Ad} u_n \circ \sigma_{t_n}^{\psi}(U_{s_j}) \to U_{s_j}$$

strongly when $n \to \infty$ hence $[u_n^*, U_{s_j}]$ tends to zero strongly when $n \to \infty$. Also $\sigma_{-t_n}^{\psi} \circ \operatorname{Ad} u_n^*(U_{s_j}) \to U_{s_j}$ strongly so that

$$\| \operatorname{Ad} u_n^* U_{s_j} - U_{s_j} \|_{\psi} \to 0 \text{ when } n \to \infty \quad \text{and} \quad [u_n, U_{s_j}] \to 0 * \text{strongly}.$$

Applying Proposition (3.9b) we get a sequence λ_n of complex numbers of modulus 1 such that $u_n - \lambda_n \to 0$ * strongly. Then for any $x \in M$ we have:

$$\sigma_{t_n}^{\psi}(x) = u_n^{-1}(\operatorname{Ad} u_n \circ \sigma_{t_n}^{\psi}(x))\, u_n = \lambda_n u_n^*(\operatorname{Ad} u_n \circ \sigma_{t_n}^{\psi}(x))\, \bar{\lambda}_n u_n$$

which tends to x when $n \to \infty$ because $\lambda_n u_n^* \to 1$ strongly, and $\bar{\lambda}_n u_n \to 1$ strongly. Using this we see that $\sigma_{t_n}^{\psi} \to 1$ in Aut M. It follows that $\delta_M(t_n) \to 1$ when $n \to \infty \Leftrightarrow \rho(t_n) \to_{n \to \infty} 1$ strongly.

COROLLARY 5.3. *There exists a factor acting in a separable Hilbert space and which possesses no almost periodic state or weight.*

Proof. Take ρ to be the regular representation of $\mathbb{R}$ in 5.2, then let M be a full factor such that $\tau(M)$ = weakest topology on $\mathbb{R}$ making ρ strongly continuous = usual topology of $\mathbb{R}$.

In particular the completion of $\mathbb{R}$ with τ topology (more precisely the two-sided corresponding uniform structure) is $\mathbb{R}$. If there were any almost periodic weight φ on M this completion would be $G = \hat{\Gamma}$ where $\Gamma = \operatorname{Sd} M$, according to Section IV.

COROLLARY 5.4. *There exists a finite measure space X, μ and an ergodic group $\mathcal{G}$ of non singular transformations of X, μ such that for any $\nu \sim \mu$ the set of values $d\nu(g, t)/d\nu t$, $g \in \mathcal{G}$, $t \in X$ is not denumerable.*

Proof. All the factors constructed in the Proof of 5.2 can be obtained by the group measure space construction from a triplet X, μ, $\mathcal{G}$.

COROLLARY 5.5. *There are factors of type* III_1 *acting in a separable Hilbert space and which are isomorphic to no cross product of a semifinite von Neumann algebra by an Abelian* discrete *group.*

Proof. Let M be a full factor without almost periodic state. Assume $M = W^*(\Lambda, N)$ where N is a semifinite von Neumann algebra and Λ an abelian group. Then by Lemma 4.10 the center C of N has an atom and the action of Λ on C being ergodic, C is purely atomic. So for any pair of faithful semifinite and normal traces on N the map $t = (D\tau_2 : D\tau_1)_t$ extends to the Bohr compactification of $\mathbb{R}$. Hence it follows from Proposition 1.1 that $\tau \circ E$ is an almost periodic weight on M for any choice of τ, a contradiction.

COROLLARY 5.6. *Let G be a locally compact Abelian group, then the following two conditions are equivalent*

(1) *Any factor of type* III *has a decomposition Semi-finite $\otimes$ G*

(2) *G contains a closed subgroup isomorphic to $\mathbb{R}$.*

Proof. (2) $\Rightarrow$ (1) is an easy consequence of [13]. Assume that G does not satisfy the condition (2) above, then by classical structure theorems G contains an open compact subgroup K. Moreover, it is an easy exercise, using for instance [13] and conditional expectations, that the cross product of a semifinite von Neumann algebra by an Abelian compact group is still semifinite. As a full factor without almost periodic state has no decomposition semifinite $\otimes$ discrete Abelian, it does not belong to the class semifinite $\otimes$ G.

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