A factor not anti-isomorphic to itself

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Abstract

We construct a factor, acting in a separable Hilbert space, and not anti-isomorphic to itself.

Introduction

We construct a family $Q_{\lambda,p,\gamma}$, $\lambda \in]0, 1[$, $p = 1, 2, \cdots, \gamma \in \mathbf{C}, \gamma^p = 1$, of mutually non-isomorphic factors. For each λ, p, γ, the factor $Q_{\lambda,p,\gamma}$ is anti-isomorphic to $Q_{\lambda,p,\bar{\gamma}}$, so that if $\gamma^2 \neq 1$ then $Q_{\lambda,p,\gamma}$ is not anti-isomorphic to itself. The construction of the Q's and the proof of non-isomorphism rely on:

1) Theorem 4.4.1 of our classification of type III factors [3], which asserts the existence and uniqueness of the discrete decomposition of an arbitrary factor of type III_λ, $\lambda \in]0, 1[$ as the cross product of a factor of type II_∞ by an automorphism multiplying the trace by λ.

2) The existence shown in [5] and reproved here (Part 4) of distinct outer conjugacy classes s_p^γ, $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$, of periodic automorphisms of the hyperfinite II_1-factor R. Henceforward we shall denote the hyperfinite II_1-factor by R.

3) The existence shown in [4], for each $\lambda \in]0, 1[$, of a factor N_λ^1 of type II_∞, with no non-trivial central sequences, but having λ in its fundamental group (i.e., there exists an automorphism θ_λ^1 of N_λ^1 multiplying the trace by λ). The $Q_{\lambda,p,\gamma}$ are then defined as the cross product of $R \otimes N_\lambda^1 = N_\lambda$ by $s_p^\gamma \otimes \theta_\lambda^1$, so that they are factors of type III_λ, depending exactly (by 1) on the outer conjugacy class of $s_p^\gamma \otimes \theta_\lambda^1$ in $\mathrm{Aut}\, N_\lambda$. From the choice of N_λ^1 (without central sequences) it follows that the image of $\mathrm{Aut}\, R$ in $\mathrm{Out}\, N_\lambda$ by the map $\alpha \rightarrow \alpha \otimes 1_{N_\lambda^1}$ is a normal subgroup of $\mathrm{Out}\, N_\lambda$, equal to the image of the closure of inner automorphisms of N_λ in the natural topology of $\mathrm{Aut}\, N_\lambda$.

Also from this choice it follows that the image of $\mathrm{Aut}\, N_\lambda^1$ in $\mathrm{Out}\, N_\lambda$ is contained in the normal subgroup of $\mathrm{Out}\, N_\lambda$ of centrally trivial automorphisms, i.e., automorphisms which act trivially on central sequences.

We then show that the two above subgroups $\overline{\mathrm{Int}\, N_\lambda}$ and $\mathrm{Ct}\, N_\lambda$ have trivial intersection in $\mathrm{Out}\, N_\lambda$, or in other words that

$$\overline{\mathrm{Int}\, N_\lambda} \cap \mathrm{Ct}\, N_\lambda = \mathrm{Int}\, N_\lambda \, .$$

The non-isomorphism of the Q's follows then from the uniqueness (in Out N_λ) of the decomposition $s_p^\gamma \otimes \theta_\lambda^1 = (s_p^\gamma \otimes 1)(1 \otimes \theta_\lambda^1)$. Though the existence of the Q's ruins the hope of describing all factors by the usual group measure space construction, there still remains the problem of constructibility of factors from the group measure space construction and an additional 2-cocycle as described in [12].

The construction of the s_p^γ done in Part 4 is different from [5]. The construction makes it obvious that the canonical abelian maximal regular subalgebra of R is left globally invariant, a fact from which it is clear that the $Q_{\lambda,p,\gamma}$ are obtained by the group measure space construction with additional 2-cocycle.

Contents

I. Preliminaries on full factors with non-trivial fundamental group

In all this section we recall the construction of a full factor[1] of type II_∞ with λ in its fundamental group, done in [4, Theorem 2.10]. First we take all the notations of [10, p. 192–195] to get some properties of the Pukanszky factors P_λ. We take P_λ as constructed in [10, p. 192] with $p/q = \lambda \in]0, 1[$. Let φ_λ be the functional $X \in P_\lambda \rightarrow (X\eta_0, \eta_0)$ in the notations of [10].

Recall that with the notations of [10] one has a group $\mathcal{G}$ containing two subgroups $\mathcal{G}_1$ and G and that $\mathcal{G}$ has a unitary representation in $L^2(\Omega, \mu)$ such that for $g \in \mathcal{G}$, $f \in L^\infty(\Omega, \mu)$,

$$U_g^{-1} f U_g = f^g\ , \quad f^g(\xi) = f(\xi g^{-1}) \qquad \text{for all } \xi \in \Omega\ .$$

Moreover recall that P_λ acts in a Hilbert space $\tilde{\mathcal{H}}$ direct sum of the $J_g\mathcal{H}$ where $\mathcal{H} = L^2(\Omega, \mu)$; that there is an isomorphism Φ of $L^\infty(\Omega, \mu)$ onto a maximal abelian subalgebra $\mathfrak{A}$ of P_λ and a unitary representation $\tilde{U}$ of $\mathcal{G}$ in $\tilde{\mathcal{H}}$ such that

$$\tilde{U}_g^{-1}\Phi(f)\tilde{U}_g = \Phi(f^g) \qquad \text{for all } g \in \mathcal{G}\ ;$$

and also that the generic element of P_λ has the form $\sum_{g \in \mathcal{G}} \Phi(f_g)\tilde{U}_g = (f_{gh^{-1}} U_{gh^{-1}})$.

In the notations of [3], [12], this means that P_λ is the cross product of $L^\infty(\Omega, \mu)$ by the action of $\mathcal{G}$ on Ω such that:

[1] i.e., its group of inner automorphisms is closed in the u-topology (see Part II).

$$g\cdot\xi = \xi g^{-1} \qquad \text{for all } g\in\mathcal{G}\,, \qquad \text{for all } \xi\in\Omega\,.$$

Also Φ is the canonical isomorphism of $L^\infty(\Omega, \mu)$ in P_λ, and $g \to \tilde{U}_g$ the canonical homomorphism of $\mathcal{G}$ in the unitary group of P_λ, of Proposition 1.4.6 [3], up to a spatial isomorphism. Let $1 = (0, e)$ be the unit of $\mathcal{G}$ (see [10, p. 193]). Then the map E which to $X = \sum \Phi(f_g)\tilde{U}_g$ associates $\Phi(f_1)\tilde{U}_1 = \Phi(f_1)$ is the canonical conditional expectation of P_λ onto $\mathfrak{A}$.

Now $\eta_0 = J_1\,(1)$ where $1 \in L^2(\Omega, \mu)$ has the obvious meaning ([10, p. 194]) and hence, for $X = \sum \Phi(f_g)\tilde{U}_g = (f_{gh^{-1}}U_{gh^{-1}})$, one has:

$$\varphi_\lambda(X) = \langle XJ_1 1, J_1 1\rangle = \int f_1(\xi)d\mu(\xi)\,.$$

Then $\varphi_\lambda = \mu\Phi^{-1}E$ is the state on P_λ canonically associated to the measure μ on Ω. Hence by [3, Lemma 1.4.5], one has:

$$(1.1) \qquad \mathfrak{A} \subset (P_\lambda)_{\varphi_\lambda}\,, \quad \sigma_t^{\varphi_\lambda}(\tilde{U}_g) = \tilde{U}_g\rho_g^{it}$$

where

$$\rho_g(\xi) = d\mu(g\xi)/d\mu(\xi) \qquad \text{for all } \xi\in\Omega\,, \qquad \text{for all } g\in\mathcal{G}\,.$$

(See the computation in [3, p. 161, proof of 1.4.8].)

Now for any $g \in G \subset \mathcal{G}$ one has $d\mu(\xi g^{-1})/d\mu(\xi) = 1$, for all $\xi \in \Omega$, because the action of G on Ω is just a permutation of the components. It follows that

$$(1.2) \qquad \tilde{U}_g \in (P_\lambda)_{\varphi_\lambda}\,, \qquad \text{for all } g\in G\,.$$

From the preceding discussion one sees that the above P_λ is the same as the P_λ defined in [3, p. 207]. Moreover, as in [3, p. 207] one has $d\mu(g\xi)/d\mu(\xi) \in \{\lambda^n, n \in \mathbf{Z}\}$, for all $\xi \in \Omega$, for all $g \in \mathcal{G}$ where $\lambda = p/q$. So P_λ is a factor of type III_λ and φ_λ satisfies

$$(1.3) \qquad \operatorname{Sp}\Delta_{\varphi_\lambda} = S(P_\lambda)\,.$$

Now by [10, Proposition 4.3.19], one has the following inequality, valid for all $X \in P_\lambda$:

$$(1.4) \qquad |\varphi_\lambda(X)|^2 \geqq \varphi_\lambda(X^*X) - 14\,\operatorname{Sup}_i \varphi_\lambda([X, \tilde{U}_{a_i}]^*[X, \tilde{U}_{a_i}])$$

where the a_i are the generators of the free group G. Now (1.4) being true for all elements of P_λ is true in particular for elements of $(P_\lambda)_{\varphi_\lambda}$ which is a factor of type II_1 ([2, p. 1405] and [3, 4.2.6]).

As in [4] we let $N_0^\lambda = (P_\lambda)_{\varphi_\lambda}$, and, since $\tilde{U}_{a_i} \in N_0^\lambda$ and φ_λ is the canonical trace of the factor N_0^λ we get:

LEMMA 1.5. *For each $\lambda \in]0, 1[$, N_0^λ is a factor of type* II_1 *which contains two unitaries U_1, U_2 such that, with τ_λ the canonical trace, one has:*

$$\|x - \tau_\lambda(x)\|_2^2 \leqq 14\,\operatorname{Sup}_i \|[x, U_i]\|_2^2\,.$$ [2]

[2] In a finite factor, $\|\ \|_2$ is the canonical L^2 norm.

Proof. One has

$$\begin{aligned}\tau_\lambda((x - \tau_\lambda(x))^*(x - \tau_\lambda(x)))\\ &= \tau_\lambda(x^*x) - \tau_\lambda(x^*)\tau_\lambda(x) - \tau_\lambda(x^*)\tau_\lambda(x) + \tau_\lambda(x^*)\tau_\lambda(x)\\ &= \tau_\lambda(x^*x) - |\tau_\lambda(x)|^2 = \varphi_\lambda(x^*x) - |\varphi_\lambda(x)|^2\\ &\leqq 14\,\mathrm{Sup}_i\,\varphi_\lambda([x, U_i]^*[x, U_i]) = 14\,\mathrm{Sup}_i\,\|[x, U_i]\|_2^2 .\end{aligned}$$

Notation 1.6. Throughout we let N_λ^1 be the tensor product of N_0^λ by a type I_∞ factor $\mathcal{L}(\mathcal{H})$, and θ_λ^1 be an automorphism of N_λ^1 such that the couple $(N_\lambda^1, \theta_\lambda^1)$ is a discrete decomposition of P_λ. (See [3, Theorem 4.4.1]; in fact $(\varphi_\lambda \otimes \mathrm{Trace})$ is a generalised trace on $P_\lambda \otimes \mathcal{L}(\mathcal{H})$ whose centraliser is obviously N_λ^1).

Definition 1.7. Let M be a von Neumann algebra; then a centralising sequence $(x_n)_{n\in\mathbf{N}}$ in M is a uniformly bounded sequence of elements of M such that $\|[x_n, \psi]\| \to 0$, for all $\psi \in M_*$ (i.e., for all ψ, there exist $\varepsilon_n \to 0$ with $|\psi(x_n y - y x_n)| \leqq \varepsilon_n \|y\|$, for all $y \in M$, for all $n \in \mathbf{N}$). Now let R be the hyperfinite factor of type II_1 with trace τ. Let $\mathcal{H} = l^2(Z)$, write $x = (x_{ij})_{i,j\in\mathbf{Z}}$, $x_{ij} \in \mathbf{C}$ for the generic element of $\mathcal{L}(\mathcal{H})$, and define states on $\mathcal{L}(\mathcal{H})$ by:

$$\rho_0(x) = \frac{7}{9}\sum\nolimits_{\mathbf{Z}} 2^{-3|j|}x_{jj} ,$$

$$\rho_1(x) = \frac{1}{3}\sum\nolimits_{\mathbf{Z}} 2^{-|j|}x_{jj} ,$$

$$\rho_2(x) = \frac{3}{5}\sum\nolimits_{\mathbf{Z}} 2^{-2|j|}x_{jj} ,$$

each of them faithful and normal on $\mathcal{L}(\mathcal{H})$. We want to obtain:

PROPOSITION 1.8. *Let $N_\lambda = R \otimes N_0^\lambda \otimes \mathcal{L}(\mathcal{H})$. Let $(x_n)_{n\in\mathbf{N}}$ be a centralising sequence in N_λ, then*:

$$\|x_n - (1 \otimes \tau_\lambda \otimes \rho_0)(x_n)\|^\#_{\tau\otimes\tau_\lambda\otimes\rho_0} \longrightarrow 0 \quad \text{when } n \longrightarrow \infty .^3$$

In other words any centralising sequence in N_λ is equivalent to a sequence in $R \otimes 1$.

The proof is already contained in [4] but we want to make one of the lemmas more explicit.

LEMMA 1.9. *Let $\mathcal{H} = l^2(Z)$, $\mathcal{L}(\mathcal{H})$, ρ_0, ρ_1, ρ_2 be as above and let $(\lambda_j)_{j\in Z}$ be such that $\lambda_j = 2^{-j}$, $j \geqq 0$, $\lambda_j = 2^{j+1/2}$, $j < 0$ and $b_1 = \sum_{j\in\mathbf{Z}} \lambda_j e_{jj}$,*[4] *while b_2 is the unitary corresponding to the shift. Then*

$$(\|[x, b_j]\|_{\rho_j} \leqq \varepsilon, \text{ for all } j = 1, 2) \Longrightarrow \|x - \rho_0(x)\|_{\rho_0} \leqq 14\varepsilon .$$

[3] For any state ψ on a von Neumann algebra M, any $x \in M$, one takes $\|x\| = \psi(x^*x)^{1/2}$, $\|x\|_\psi^\# = \psi(x^*x + xx^*)^{1/2}$.

[4] (e_{ij}) is the canonical system of matrix units in $\mathcal{L}(\mathcal{H})$.

Proof. For $x = (x_{ij}) \in \mathcal{L}(\mathcal{H})$, the j^{th} diagonal element of the matrix of x^*x is $\sum_i |x_{ij}|^2$, so that we have:

$$\| x - x_{00} \|_{\rho_0}^2 = \frac{7}{9} \sum_{i\neq j} | x_{ij} |^2 2^{-3|j|} + \frac{7}{9} \sum_j | x_{jj} - x_{00} |^2 2^{-3|j|} ,$$

$$A = \sum_{i\neq j} | x_{ij} |^2 2^{-3|j|} , \qquad B = \sum_j | x_{jj} - x_{00} |^2 2^{-3|j|}$$

$$\| [x, b_1] \|_{\rho_1}^2 = \frac{1}{3} \sum | x_{ij}(\lambda_i - \lambda_j) |^2 2^{-|j|} \geqq \frac{1}{3 \times 14} \sum_{i\neq j} | x_{ij} |^2 2^{-3|j|}$$

because

$$| \lambda_i - \lambda_j |^2 = \left| \frac{\lambda_i}{\lambda_j} - 1 \right|^2 \lambda_j^2 \geqq (2^{-1/2} - 1)^2 2^{-2|j|} \qquad \text{for } i \neq j .$$

$$\| [x, b_2] \|_{\rho_2}^2 = \| b_2^* x b_2 - x \|_{\rho_2}^2 \geqq \frac{3}{5} \sum | x_{j+1,j+1} - x_{jj} |^2 2^{-2|j|} .$$

So that, with $\| [x, b_2] \|_{\rho_2} \leqq \varepsilon$, one has $| x_{j+1,j+1} - x_{jj} | \leqq 2^{|j|}\varepsilon(5/3)^{1/2}$ for all j, hence $| x_{n,n} - x_{0,0} | \leqq 2^{|n|} 2\varepsilon (5/3)^{1/2}$ and $B \leqq 14\varepsilon^2$.

So $\| [x, b_j] \|_{\rho_j} \leqq \varepsilon$, $j = 1, 2$, implies $\| x - x_{00} \|_{\rho_0}^2 \leqq (7/9)(3 \times 19)\varepsilon^2 \leqq (7\varepsilon)^2$ and hence $\| x - \rho_0(x) \|_{\rho_0} \leqq 7\varepsilon + 7\varepsilon = 14\varepsilon$.

Proof of 1.8. Any centralising sequence on any von Neumann algebra is a central sequence ([4, Prop. 2.8, $\beta) \Rightarrow \gamma)$]). Then by [4, Lemma 2.11] we first have that:

$$\| x_n - (1 \otimes \rho_0)(x_n) \|_\varphi \longrightarrow 0 \quad \text{when } n \longrightarrow \infty$$

and where $\varphi = \tau \otimes \tau_\lambda \otimes \rho_0$ is the canonical state on N_λ. As $(x_n^*)_{n\in N}$ is also a centralising sequence, we get:

$$\| x_n - (1 \otimes \rho_0)(x_n) \|_\varphi^\# \longrightarrow 0 \quad \text{when } n \longrightarrow \infty .$$

Then $((1 \otimes \rho_0)(x_n))_{n\in N}$ is also a centralising sequence on $R \otimes N_0^\lambda$ and by Lemma 1.5 and [4, Lemma 2.11] we get:

$$\| (1 \otimes \rho_0)x_n - (1 \otimes \tau_\lambda \otimes \rho_0)(x_n) \|_\varphi^\# \longrightarrow 0 \quad \text{when } n \longrightarrow \infty . \qquad \text{Q.E.D.}$$

II. The group of approximately inner automorphisms

Let N be an arbitrary factor, with separable predual. We put on Aut N the topology of pointwise norm convergence in N_* called the u-topology in [8] (cf. [1] and [8]).

For $\alpha \in \text{Aut } N$ a basis of neighborhoods is given by

$$\mathcal{V}_{\alpha,\varphi_1\cdots\varphi_k,\varepsilon} = \{\beta \in \text{Aut } N, \| \varphi_j \circ \beta - \varphi_j \circ \alpha \| < \varepsilon, \text{ for all } j\} ,$$

where $\varepsilon > 0$ and $\varphi_j \in N_*$, $j = 1, \cdots, k$.

Gifted with the u-topology Aut N is a topological group ([1], [8]) which is Polish as a topological space. The u-topology is in general stronger than the topology of simple * strong convergence in N, as can be seen by direct computation or as in [1], [8].

When N is of type II_1 it coincides with the topology of simple strong convergence (if $\alpha_n(k) \to \alpha(k)$ strongly, for all $k \in N$ then $\tau(k\,.) = \varphi$ satisfies $\varphi\alpha_n^{-1} \to \varphi\alpha^{-1}$, for all $k \in N$ and τ the trace on N) (see [8]). Moreover the normal subgroup $\mathrm{Int}\, N$ is in general not closed (see [4, Theorem 3.1]); its closure $\overline{\mathrm{Int}\, N}$ is a normal subgroup of $\mathrm{Aut}\, N$.

THEOREM 2.1. *Let* $\lambda \in]0, 1[$ *and* $N_\lambda = R \otimes N_\lambda^1$ *as defined in Section* 1. *Then* $\alpha \in \overline{\mathrm{Int}\, N_\lambda}$ *if and only if there exists a unitary* $X \in N_\lambda$ *and an* $\alpha_0 \in \mathrm{Aut}\, R$ *with*

$$\alpha = \mathrm{Ad}\, X(\alpha_0 \otimes 1_{N_\lambda^1}) .$$

Throughout the proof we let φ_0 be the state $\tau_\lambda \otimes \rho_0$ on N_λ^1 already considered in Section 1. We let $\varphi = \tau \otimes \varphi_0$ where τ is the trace on R. Also we assume that N_λ acts in the Hilbert space $\mathcal{H}_\varphi$ of the Gelfand-Segal construction of φ, and that

$$\langle x\xi_\varphi, \xi_\varphi\rangle = \varphi(x) , \qquad \text{for all } x \in N_\lambda . \tag{2.2}$$

We note that $R \otimes 1$ is contained in the centraliser of φ in N_λ because τ is a trace on R; in particular:

$$\tilde{y}\xi_\varphi = y\xi_\varphi, \text{ for all } y \in R \otimes 1 , \quad \text{where } \tilde{x} = J_\varphi x^* J_\varphi, \text{ for all } x \in N_\lambda . \tag{2.3}$$

$$\varphi(wxw^*) = \varphi(x), \text{ for all } x \in N_\lambda, \text{ for all } w \text{ unitary in } R \otimes 1 . \tag{2.4}$$

LEMMA 2.5. *There exists a basis* $(\mathcal{V}_n)_{n \in N}$ *of neighborhoods of the identity in* $\mathrm{Aut}\, N_\lambda$, *such that* $\mathcal{V}_{n+1} \subset \mathcal{V}_n$, *for all* n *and that*: (u, u' *unitaries in* N_λ, $(\mathrm{Ad}\, u')^{-1}(\mathrm{Ad}\, u) \in \mathcal{V}_n$) $\Rightarrow$ (*there exists a unitary* $X \in N_\lambda$ *and* $w \in R \otimes 1$ *with* $u'^{-1}u = Xw$ *and* $\|(X - 1)\xi_\varphi\| \leqq 1/2^n$).

Proof. Using (1.8), one can, for each $\varepsilon > 0$, find a finite number of elements of $(N_\lambda)_*$: $\psi_1^\varepsilon, \cdots, \psi_{q_\varepsilon}^\varepsilon$ and an $\eta_\varepsilon > 0$ such that

$$\begin{aligned} &(x \in N_\lambda, \|x\| \leqq 1, \|[x, \psi_j^\varepsilon]\| \leqq \eta_\varepsilon, \text{ for all } j) \\ &\quad\Longrightarrow (\|(x - (1 \otimes \varphi_0)(x))\xi_\varphi\| \leqq \varepsilon \quad \text{and} \\ &\quad\quad \|(x^* - (1 \otimes \varphi_0)x^*)\xi_\varphi\| \leqq \varepsilon) . \end{aligned} \tag{2.6}$$

Let $(\mathcal{W}_n)_{n \in N}$ be a basis of neighborhoods of the identity in $\mathrm{Aut}\, N_\lambda$. Choose $\varepsilon_n > 0$ such that $\varepsilon_n + 2(2\varepsilon_n)^{1/2} \leqq 1/2^n$. We define $(\mathcal{V}_n)_{n \in N}$ by:

$$\mathcal{V}_n = \mathcal{V}_{n-1} \cap \mathcal{W}_n \cap \{\alpha \in \mathrm{Aut}\, N_\lambda, \|\psi_j^{\varepsilon_n} \circ \alpha - \psi_j^{\varepsilon_n}\| \leqq \eta_{\varepsilon_n}, \text{ for all } j\} . \tag{2.7}$$

We have to show that any unitary $U \in N_\lambda$ such that $\mathrm{Ad}\, U \in \mathcal{V}_n$ can be written $U = XW$, $\|(X - 1)\xi_\varphi\| \leqq 1/2^n$, W unitary in $R \otimes 1$. Put $y = (1 \otimes \varphi_0)U \in R \otimes 1$. Then the hypothesis $\mathrm{Ad}\, U \in \mathcal{V}_n$ implies that

$$\| [\psi_j^{\varepsilon_n}, U] \| = \| \psi_j^{\varepsilon_n} \circ \operatorname{Ad} U - \psi_j^{\varepsilon_n} \| \leqq \eta_{\varepsilon_n} , \qquad \text{for all } j = 1, \cdots, q_{\varepsilon_n}$$

so that 2.6 implies: $\| (U - y)\xi_\varphi \| \leqq \varepsilon_n$, $\| (U^* - y^*)\xi_\varphi \| \leqq \varepsilon_n$. We want to replace y by a unitary in $R \otimes 1$. We have:

$$\begin{aligned} \| (y^*y - 1)\xi_\varphi \| = \| (\tilde{y}y^* - 1)\xi_\varphi \| &\leqq \| \tilde{y} \| \, \| (y^* - U^*)\xi_\varphi \| + \| (\tilde{y}U^* - 1)\xi_\varphi \| \\ &\leqq \varepsilon_n + \| U^*(\tilde{y} - U)\xi_\varphi \| \leqq 2\varepsilon_n . \end{aligned}$$

$\| (|y| - 1)\xi_\varphi \| \leqq \varphi(|y^*y - 1|)^{1/2}$ because $(|y| - 1)^2 \leqq \big| |y|^2 - 1 \big| = |y^*y - 1|$. But then:

$$\| (|y| - 1)\xi_\phi \| \leqq \| (y^*y - 1)\xi_\phi \|^{1/2} \leqq (2\varepsilon_n)^{1/2} .$$

Let $y = w_0 |y|$ be the polar decomposition of y. Then w_0 is a partial isometry belonging to the II_1 factor $R \otimes 1$, so there exists a unitary $w = w_0 + w_1$, where $w_1 \in R \otimes 1$, $w_1^* w_1 = 1 - w_0^* w_0$. As $y^*y \leqq w_0^* w_0$, one has

$$\| w_1 \xi_\varphi \|^2 = \varphi(1 - w_0^* w_0) \leqq \varphi(1 - y^*y) \leqq 2\varepsilon_n$$

so that $\| (w_0 - w)\xi_\varphi \| \leqq (2\varepsilon_n)^{1/2}$ and:

$$\begin{aligned} \| (y - w)\xi_\varphi \| &\leqq \| (y - w_0)\xi_\varphi \| + (2\varepsilon_n)^{1/2} \\ &\leqq \| (|y| - 1)\xi_\varphi \| + (2\varepsilon_n)^{1/2} \leqq 2(2\varepsilon_n)^{1/2} . \end{aligned}$$

As y and w belong to the centraliser of φ, we have also:

$$\| (y^* - w^*)\xi_\varphi \| \leqq 2(2\varepsilon_n)^{1/2} , \quad \| (U^* - w^*)\xi_\varphi \| \leqq \varepsilon_n + 2(2\varepsilon_n)^{1/2} \leqq \frac{1}{2^n} ;$$

then $X = Uw^*$ satisfies

$$\| (X - 1)\xi_\varphi \| = \| (U^*X - U^*)\xi_\varphi \| \leqq \frac{1}{2^n} . \qquad \text{Q.E.D.}$$

Proof of 2.1. a) Let $(\mathfrak{U}_n)_{n \in N}$ be a basis of neighborhoods of α in $\operatorname{Aut} N_\lambda$, such that:

(2.8) $\qquad \mathfrak{U}_{n+1} \subset \mathfrak{U}_n \quad \text{for all } n \in N^{\cdot}, \qquad \mathfrak{U}_n^{-1}\mathfrak{U}_n \subset \mathfrak{V}_n \quad \text{for all } n \in N$

where $(\mathfrak{V}_n)_{n \in N}$ is as in Lemma 2.5. As $\alpha \in \overline{\operatorname{Int} N_\lambda}$ we can find for each n, a unitary $u_n \in N_\lambda$ such that $\operatorname{Ad} u_n \in \mathfrak{U}_n$. Then $(\operatorname{Ad} u_n)^{-1} \operatorname{Ad} u_{n+1} \in \mathfrak{U}_n^{-1}\mathfrak{U}_n \subset \mathfrak{V}_n$.

By Lemma 2.5, there exists for each n, a unitary $w_n \in R \otimes 1$ and a unitary $X_n \in N_\lambda$ such that

(2.9) $$u_n^{-1} u_{n+1} = X_n w_n , \qquad \| (X_n - 1)\xi_\varphi \| \leqq \frac{1}{2^n} .$$

By induction one shows, using (2.9), that for each $n \in N$:

(2.10) $$\begin{aligned} u_n &= u_1 Z_1 Z_2 \cdots Z_{n-1} w_1 w_2 \cdots w_{n-1} & &\text{where } Z_0 = 1 , \\ Z_j &= w_1 w_2 \cdots w_{j-1} X_j w_{j-1}^* \cdots w_1^* & &\text{for all } j \geqq 1 . \end{aligned}$$

In fact if (2.10) is true for n, then by (2.9):

$$u_{n+1} = u_n X_n w_n = u_1 Z_1 \cdots Z_{n-1} w_1 w_2 \cdots w_{n-1} X_n w_n$$
$$= u_1 Z_1 \cdots Z_{n-1} Z_n w_1 w_2 \cdots w_{n-1} w_n \,.$$

Using (2.4) and the second part of (2.9) we see that $\|(Z_n - 1)\xi_\varphi\| \leqq 1/2^n$ for all n, and hence that the sequence of unitaries:

$$Y_n = Z_1 \cdots Z_n \in N_\lambda \,, \quad \text{satisfies } \|(Y_{n+1} - Y_n)\xi_\varphi\| \leqq \frac{1}{2^{n+1}} \quad \text{for all } n \,,$$

and hence (ξ_φ is cyclic for N_λ) converges strongly to some isometry Y belonging to N_λ.

We shall see that the surjectivity of α implies that Y is unitary. We have $\operatorname{Ad} u_n \to \alpha$ in $\operatorname{Aut} N_\lambda$ and hence in particular:

$$u_n x u_n^* \longrightarrow \alpha(x) \,, \qquad \text{for all } x \in N_\lambda \,.$$

Now, by (2.10), we have $u_n = u_1 Y_{n-1} w_1 \cdots w_{n-1} = u_1 Y_{n-1} V_{n-1}$ with $V_n = w_1 \cdots w_n \in R \otimes 1$. So we get: $u_1 Y_n V_n x V_n^* Y_n^* u_1^* \to \alpha(x)$ strongly for all $x \in N_\lambda$.

Put $e = \alpha^{-1}(u_1(1 - YY^*)u_1^*)$. Then $e \geqq 0$ and:

$$Y_n V_n e V_n^* Y_n^* \longrightarrow u_1^* \alpha(e) u_1 = 1 - YY^* \quad \text{(strongly)} \,.$$

As $Y_n \to Y$ strongly, and as the product is strongly continuous on bounded subsets of $\mathcal{L}(\mathcal{H}_\varphi)$, we have:

$$(Y_n V_n e V_n^* Y_n^*) Y_n \longrightarrow (1 - YY^*) Y = 0 \,.$$

So $Y_n V_n e V_n^* \to 0$ strongly, $V_n e V_n^* \xi_\varphi \to 0$ in $\mathcal{H}_\varphi$ and $\varphi(e) = \varphi(V_n e V_n^*) = 0$ (using (2.4)).

We have shown that Y is unitary, hence that $Y_n \to Y_*$ strongly and that $\operatorname{Ad} Y_n \to \operatorname{Ad} Y$ in $\operatorname{Aut} N_\lambda$. Put $X = u_1 Y$; then $\operatorname{Ad} Y_{n-1}^* u_1^* \to \operatorname{Ad} X^*$ when $n \to \infty$, and $\operatorname{Ad} u_n = \operatorname{Ad} u_1 Y_{n-1} V_{n-1} \to \alpha$ when $n \to \infty$.[5] So

$$\operatorname{Ad}(Y_{n-1}^* u_1^*) \operatorname{Ad}(u_1 Y_{n-1} V_{n-1}) \longrightarrow \operatorname{Ad} X^* \circ \alpha \quad \text{when } n \longrightarrow \infty \,.$$

Now $\operatorname{Ad} V_n \to \operatorname{Ad} X^* \circ \alpha$ in $\operatorname{Aut} N_\lambda$, so that in particular for all $x \in 1 \otimes N_\lambda^1$, one has:

$$\operatorname{Ad} X^* \circ \alpha(x) = \lim_{n\to\infty} V_n x V_n^* = x \quad (\text{because } V_n \in R \otimes 1) \,.$$

This shows that $\operatorname{Ad} X^* \circ \alpha$ leaves $R \otimes 1$ globally invariant and is equal to the product $\alpha_0 \otimes 1$ of its restriction α_0 to R by the identity on N_λ^1. Q.E.D.

b) We now let $\alpha_0 \in \operatorname{Aut} R$ and prove that $\alpha_0 \otimes 1 \in \overline{\operatorname{Int} N_\lambda}$. Let K_n be an increasing sequence of finite dimensional subfactors of R, generating R, and (Lemma 3.11) let $u \in R$ be such that $\alpha_0^{-1}(x) = u_n^* x u_n$, for all $x \in K_n$. Let

[5] In both cases for the u-topology of $\operatorname{Aut} N_\lambda$.

$k \in K_{n_0}$, and $n \geqq n_0$; then with $\psi = \tau(k.) \in R_*$ one has

$$\psi \circ \mathrm{Ad}\, u_n = \tau(u_n^* k u_n.) = \tau(\alpha_0^{-1}(k).) = \psi \circ \alpha_0 .$$

Then $\| \psi \circ \mathrm{Ad}\, u_n - \psi \circ \alpha_0 \| \longrightarrow_{n\to\infty} 0$ for all $\psi \in R_*$ and

$$\| (\psi_1 \otimes \psi_2) \circ (\mathrm{Ad}\, u_n) \otimes 1 - (\psi_1 \otimes \psi_2) \circ (\alpha_0 \otimes 1) \| \longrightarrow_{n\to\infty} 0 ,$$
$$\text{for all } \psi_1 \in R_*,\ \psi_2 \in (N_\lambda^1)_* ,$$

so that $\mathrm{Ad}\,(u_n \otimes 1)$ converges to $\alpha_0 \otimes 1$ in $\mathrm{Aut}\, N_\lambda$. Q.E.D.

III. The group of centrally trivial automorphisms

Let N be an arbitrary factor.

Definition 3.1. An automorphism $\alpha \in \mathrm{Aut}\, N$ is centrally trivial if and only if for any centralising sequence $(x_n)_{n\in N}$, $x_n \in N$,[6] one has $\alpha(x_n) - x_n \longrightarrow_{n\to\infty} 0$ $_*$ strongly.

The set $\mathrm{Ct}\,(N)$ of centrally trivial automorphisms is a subgroup of $\mathrm{Aut}\, N$ because if $\alpha, \beta \in \mathrm{Ct}\,(N)$ and $(x_n)_{n\in N}$ is a centralising sequence, we have:

$$\alpha^{-1}\beta(x_n) - x_n = \alpha^{-1}(\beta(x_n) - x_n) + \alpha^{-1}(x_n - \alpha(x_n)) \qquad \text{for all } n \in N ,$$

so that, as α^{-1} is $_*$ strongly continuous, we get $\alpha^{-1}\beta(x_n) - x_n \longrightarrow 0$ $_*$ strongly.

$\mathrm{Ct}\,(N)$ is a normal subgroup of $\mathrm{Aut}\, N$, because for any $\sigma \in \mathrm{Aut}\, N$ and any centralising sequence $(x_n)_{n\in N}$, the sequence $(\sigma(x_n))_{n\in N}$ is also centralising (so for $\alpha \in \mathrm{Ct}\, N$ one gets $\alpha\sigma(x_n) - \sigma(x_n) \longrightarrow 0$ $_*$ strongly and $\sigma^{-1}\alpha\sigma(x_n) - x_n \longrightarrow 0$ is strongly).

One can check that $\mathrm{Ct}\, N$ is in fact the kernel of the homomorphism $\theta \longrightarrow \theta_\omega$ defined in [4, Theorem 2.9]. Moreover $\mathrm{Ct}\,(N)$ contains $\mathrm{Int}\, N$ because for any centralising sequence $(x_n)_{n\in N}$ and any unitary $u \in N$ one has $ux_nu^* - x_n \longrightarrow 0$ $_*$ strongly (using [4, Prop. 2.8, $\beta) \Rightarrow \gamma)$]).

THEOREM 3.2.[7] *Let $N_\lambda = R \otimes N_\lambda^1$ where R is the hyperfinite factor of type* II_1 *and N_λ^1 is as in sections* I, II.

a) *Any $\alpha \in \mathrm{Ct}\, N_\lambda$ which preserves the trace on N_λ is equal to* $\mathrm{Ad}\, X(1 \otimes \beta)$ *for some $\beta \in \mathrm{Aut}\, N_\lambda^1$, X unitary in N_λ;*

b) $$\overline{\mathrm{Int}\, N_\lambda} \cap \mathrm{Ct}\, N_\lambda = \mathrm{Int}\, N_\lambda .$$

LEMMA 3.3. *Let P_1, P_2 be factors of type* II_1 *and put $P = P_1 \otimes P_2$. Let $\alpha \in \mathrm{Aut}\, P$ be such that*

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[6] i.e., $\| x_n \|$ uniformly bounded and $\| [x_n, \psi] \| \to_{n\to\infty} 0$, for all $\psi \in N_*$.

[7] Statement b) can be proved in a simpler way using [5, Lemma 3.4] (see remark 3.14 below).

$$\| \alpha(u \otimes 1) - u \otimes 1 \|_2 \leqq \frac{1}{2}, \quad \textit{for all } u \textit{ unitary in } P_1 .$$

Then there exists $\alpha_2 \in \operatorname{Aut} P_2$ *and* X *unitary in* P *with*:

$$\alpha = \operatorname{Ad} X(1 \otimes \alpha_2) .$$

Proof. Let τ be the trace on P and $\mathcal{H}_\tau = L^2(P, \tau)$ be the Hilbert space of the Gelfand-Segal representation of P with respect to τ. We let η be the canonical injection of P in $\mathcal{H}_\tau$. The unitary group $\mathcal{U}_1$ of $P_1 \otimes 1$ has the following representation in $\mathcal{H}_\tau$:

(3.4) $$\phi_u(\eta(x)) = \eta(ux\alpha(u^*)) \qquad \text{for all } u \in \mathcal{U}_1, \text{ for all } x \in P .$$

The equality (3.4) defines for each $u \in \mathcal{U}_1$ a unitary ϕ_u of $\mathcal{H}_\tau$, because right multiplication by unitaries result in unitaries. Let C be the closed convex hull in $\mathcal{H}_\tau$ of $\{\phi_u \eta(1), u \in \mathcal{U}_1\}$. Then C is ϕ invariant and by the hypothesis of the lemma, one has $\| \xi - \eta(1) \| \leqq 1/2$, for all $\xi \in C$.

As $\| \eta(1) \| = 1$, we see that the orthogonal projection ξ_0 of 0 on C is not 0, and is a fixed point for ϕ. Also as the image under η of the unit ball of P is a weakly closed convex subset of $\mathcal{H}_\tau$ containing the $\phi_u \eta(1)$, we have $\xi_0 = \eta(y)$ for some $y \neq 0$, $y \in P$. The equality $\phi_u \eta(y) = \eta(y)$, for all $u \in \mathcal{U}_1$ implies:

(3.5) $$uy\alpha(u^*) = y \qquad \text{for all } u \in \mathcal{U}_1 .$$

Let $y = w\rho$ be the polar decomposition of y; then for each $u \in \mathcal{U}_1$, $(uw\alpha(u^*))(\alpha(u)\rho\alpha(u^*))$ is a polar decomposition of y, so that

(3.6) $$uw\alpha(u^*) = w, \text{ for all } u \in \mathcal{U}_1 , \qquad uww^*u^* = ww^*, \text{ for all } u \in \mathcal{U}_1 .$$

It follows that $e = ww^*$ is a projection belonging to the commutant of $P_1 \otimes 1$ in P, and hence of the form $1 \otimes e_2$, $e_2 \in P_2$. Also, one can linearise (3.6):

(3.7) $$(x \otimes 1)w = w\alpha(x \otimes 1) \qquad \text{for all } x \in P_1 .$$

Let now v be a unitary of P such that $ev = w$. (For instance take w_1: $1 - w^*w \to 1 - ww^*$ and $v = w + w_1$.) As e commutes with $x \otimes 1$, for all $x \in P_1$, one gets

(3.8) $$e(x \otimes 1)v = ev\alpha(x \otimes 1) \qquad \text{for all } x \in P_1 .$$

Put $\beta(y) = v\alpha(y)v^*$, for all $y \in P$; then:

(3.9) $$e(x \otimes 1) = e\beta(x \otimes 1) \qquad \text{for all } x \in P_1 .$$

As e commutes with $x \otimes 1$, $x = x^* \in P_1$ we see that, in this case, $e\beta(x \otimes 1)$ is self adjoint so that e commutes with $\beta(x \otimes 1)$ for all $x \in P_1$. It follows that $\beta^{-1}(e) \in 1 \otimes P_2$ and that we can find a unitary $X \in 1 \otimes P_2$ such that:

$$\beta^{-1}(e) = XeX^* .$$

Put $\alpha' = \operatorname{Ad} X^* \circ \beta^{-1}$. Then we have for $x \in P_1$:

$$\beta^{-1}(e)\beta^{-1}(x \otimes 1) = \beta^{-1}(e)(x \otimes 1) \qquad (\text{using } (3.9)) . \tag{3.10}$$

$$XeX^*\beta^{-1}(x \otimes 1) = XeX^*(x \otimes 1) .$$

$$e\alpha'(x \otimes 1) = eX^*(x \otimes 1)X = e(x \otimes 1) \qquad (\text{because } X \in 1 \otimes P_2)$$

while $\alpha'(e) = X^*\beta^{-1}(e)X = e$.

Now α' leaves the reduced von Neumann algebra P_e globally invariant. Moreover $P_e = (P_1 \otimes P_2)_{1\otimes e_2} = P_1 \otimes (P_2)_{e_2}$ (see [6, p. 16]) and (3.10) means $\alpha'(x \otimes e_2) = x \otimes e_2$, for all $x \in P_1$, so that there exists an $\alpha'_2 \in \operatorname{Aut}(P_2)_{e_2}$ such that:

$$\alpha' \text{ restricted to } P_1 \otimes (P_2)_{e_2} = 1 \otimes \alpha'_2 .$$

Let now $\alpha''_2 \in \operatorname{Aut} P_2$ be such that $\alpha''_2(e_2) = e_2$ and that

$$\alpha''_2 \text{ restricted to } (P_2)_{e_2} = \alpha'_2 .$$

Then the automorphism $1 \otimes \alpha''_2$ coincides with α', when restricted to P_e. It follows from [3, 1.5.2] that α' is equal to $1 \otimes \alpha''_2$ modulo Int P. Hence β^{-1} is equal to $1 \otimes \alpha''_2$ modulo Int P and β is equal to $1 \otimes (\alpha''_2)^{-1}$ modulo Int P.

Q.E.D.

LEMMA 3.11.[8] *Let N be a factor, τ a semi-finite faithful normal trace on N, α a τ-preserving automorphism of N and F a type* I *subfactor of N with τ/N semi-finite. Then there exists a unitary $V \in N$ such that:*

$$\alpha(x) = VxV^* \qquad \textit{for all } x \in F .$$

Proof. Let $(e_{ij})_{i,j\in\{1,\cdots,n\}}$ be a system of matrix units in F where $n \in \{1, \cdots, \infty\}$ and the e generate F. We have $\tau\alpha(e_{11}) = \tau(e_{11}) < \infty$ by hypothesis, so that e_{11} is equivalent to $\alpha(e_{11})$ relative to N. Let u be a partial isometry belonging to N, having initial support e_{11} and final support $\alpha(e_{11})$. Put $V = \sum_{j=1}^{n} \alpha(e_{j1})ue_{1j}$. Then, as each $\alpha(e_{j1})ue_{1j}$ has e_{jj} as initial support and $\alpha(e_{jj})$ as final support, V is unitary. Moreover we have, for $k, l \in \{1, \cdots, n\}$ that

$$Ve_{kl}V^* = \alpha(e_{k1})ue_{1k}e_{kl}e_{l1}u^*\alpha(e_{1l}) = \alpha(e_{k1}e_{11}e_{1l}) = \alpha(e_{kl}) . \qquad \text{Q.E.D.}$$

Proof of 3.2. a) Let $(K_n)_{n\in N}$ be an increasing sequence of finite dimensional subfactors of R generating R and $R_n = K'_n \cap R$ be the relative commutant of K_n in R. Put $L_n = R_n \otimes 1 \subset R \otimes N_0^\lambda$, where $N_\lambda^1 = N_0^\lambda \otimes \mathcal{L}(\mathcal{H})$. As $1 \otimes \mathcal{L}(\mathcal{H})$ is a subfactor of N_λ satisfying the conditions of (3.11) we can modify α by an inner automorphism and assume that $\alpha = \alpha_0 \otimes 1$ for some $\alpha_0 \in \operatorname{Aut}(R \otimes N_0^\lambda)$. If x_n is an arbitrary centralising sequence in a factor P then $x_n \otimes 1$ is centralising in $P \otimes Q$ for any factor Q so that $\alpha_0 \in \operatorname{Ct}(R \otimes N_0^\lambda)$.

[8] This lemma is classical; the proof is given for the sake of completeness.

There exists an n_0 such that:

$$\text{For all } x \in L_{n_0},\ \|x\| \leqq 1, \text{ one has } \|\alpha_0(x) - x\|_2 \leqq \frac{1}{2}.^{9}$$

In fact, otherwise there exists a uniformly bounded sequence

$$x_n,\ \|x_n\| \leqq 1,\ x_n \in L_n,\ \|\alpha_0(x_n) - x_n\|_2 > \frac{1}{2}$$

and $(x_n)_{n \in N}$ is a central (ising) sequence in $R \otimes N_0^\lambda$ because for each m and $n \geqq m$, x_n commutes with $K_m \otimes N_0^\lambda$.

Now by (3.3), up to inner automorphisms, α_0 is of the form $1_{R_{n_0}} \otimes \alpha_2$ where α_2 is an automorphism of $K_{n_0} \otimes N_0^\lambda$. By Lemma 3.11, α_2 is, up to inner automorphisms, of the form $1_{K_{n_0}} \otimes \beta$ and we get the desired result.

b) It is a general fact for factors of type II_∞ that any $\alpha \in \overline{\mathrm{Int}\, N}$ preserves the trace; however, here one can use (2.1). By (2.1) and 3.2 a) we can find an automorphism α_1 of R, a unitary X_1 of N_λ, an automorphism α_2 of N_λ^1, and a unitary X_2 of N_λ such that:

$$\alpha = \mathrm{Ad}\, X_1(\alpha_1 \otimes 1) = \mathrm{Ad}\, X_2(1 \otimes \alpha_2)\ . \tag{3.13}$$

Then $\alpha_1 \otimes \alpha_2^{-1}$ is an inner automorphism of N_λ, so that by [7, Cor. 6], both α_1 and α_2 are inner; hence α is inner.

Remark 3.14. Lemma 3.3 is not necessary to prove 3.2 b) which is the only statement of Theorem 3.2 that is used in Part 5. In fact Lemma 3.4 of [5] shows that any outer automorphism α_0 of R fails to belong to $\mathrm{Ct}\, R$ so that any outer automorphism $\alpha = \mathrm{Ad}\, X(\alpha_0 \otimes 1)$ of N_λ with $\alpha_0 \in \mathrm{Aut}\, R$ fails to belong to $\mathrm{Ct}\, N_\lambda$. Hence by (2.1) we get $\overline{\mathrm{Int}\, N_\lambda} \cap \mathrm{Ct}\, N_\lambda = \mathrm{Int}\, N_\lambda$.

Remark 3.15. Let β be an arbitrary automorphism of N_λ^1; then $1_R \otimes \beta$, as an automorphism of N_λ, is centrally trivial. In fact for any centralising sequence $(x_n)_{n \in N}$ on N_λ there exists, by Prop. 1.8, a sequence $(y_n)_{n \in N}$, $y_n \in R$ such that $x_n - y_n \otimes 1 \to 0$ $*$ strongly, so that $(1 \otimes \beta)(x_n) - x_n \to 0$ $*$ strongly. It is not clear that any automorphism $\alpha \in \mathrm{Ct}\, N_\lambda$ is equal to some $1_R \otimes \beta$ modulo inner. It would be the case if the fundamental group of N_λ^1 was $\mathbf{R}_+^*$, by Theorem 3.2 a).

IV. Some periodic automorphisms of the II_1-hyperfinite factor

First we shall associate to each automorphism α of a factor M a pair $p_0(\alpha)$, $\gamma(\alpha)$, that we call the outer invariants of α. As usual we let $\mathrm{Int}\, M$ be the group of inner automorphisms of M. We define $p_0(\alpha)$ as being the

[9] $R \otimes N_0^\lambda$ is of type II_1; $\|\ \|_2$ is its trace norm.

integer $\geqq 0$ such that:

(4.1) $$\alpha^n \in \operatorname{Int} M \quad \text{if and only if} \quad n \in p_0(\alpha)Z .$$

When no nonzero power of α is inner, we have $p_0(\alpha) = 0$ and we say that α is aperiodic. In any case $p_0(\alpha)$ is called the outer period of α. We define $\gamma(\alpha)$ as being the complex number of modulus 1 such that:

(4.2) $$\big(u \in M_u,\ \alpha^{p_0(\alpha)}(x) = uxu^*,\ \text{for all } x \in M\big) \Longrightarrow \alpha(u) = \gamma(\alpha)u .$$[10]

This definition makes sense because $\alpha^{p_0(\alpha)}$ is an inner automorphism so that the set of u's satisfying $\alpha^{p_0(\alpha)} = \operatorname{Ad} u$ is not empty; moreover, for any such u, one has:

$$\alpha\alpha^{p_0(\alpha)}\alpha^{-1}(x) = \alpha(u)x\alpha(u^*) , \qquad \text{for all } x \in M$$

so that $\alpha(u)u^*$ belongs to the center of M, and is a scalar γ independent of the choice of u such that $\alpha^{p_0(\alpha)} = \operatorname{Ad} u$.

PROPOSITION 4.3. *Let M, α, p_0 and γ be as above:*

a) *For each α, $\gamma(\alpha)$ is a $p_0(\alpha)^{\text{th}}$ root of* 1 *in* C.

b) *Let w be a unitary in M, and $\beta = \operatorname{Ad} w \circ \alpha$, then*

$$p_0(\beta) = p_0(\alpha) , \quad \gamma(\alpha) = \gamma(\beta) .$$

c) *Let N be another factor and take $\beta \in \operatorname{Aut} N \otimes M$, $\beta = 1 \otimes \alpha$; then*

$$p_0(\beta) = p_0(\alpha) , \quad \gamma(\beta) = \gamma(\alpha) .$$

Proof. a) We have with the notations above: $\alpha^{p_0(\alpha)}(u) = uuu^* = u$ for any unitary u as in (4.2), and hence:

$$\gamma(\alpha)^{p_0(\alpha)}u = u \text{ so that } \gamma(\alpha)^{p_0(\alpha)} = 1 .$$

b) For $n \in N$, we have:

(4.4) $$\beta^n = \operatorname{Ad}\big(w\alpha(w) \cdots \alpha^{n-1}(w)\big)\alpha^n$$

as can be seen using an inductive argument. In particular β^n is inner if and only if α^n is inner, which proves that $p_0(\alpha) = p_0(\beta)$.

Now if $p_0(\alpha) = 0 = p_0(\beta)$, both $\gamma(\alpha)$ and $\gamma(\beta)$ are equal to 1 and the same occurs if $p_0(\alpha) = p_0(\beta) = 1$. Put $p = p_0(\alpha) = p_0(\beta) > 1$. Let u be a unitary in M such that $\alpha^p(x) = uxu^*$, for all $x \in M$. By (4.4) we then have:

$$\beta^p(x) = w\alpha(w) \cdots \alpha^{p-1}(w)\alpha^p(x)\alpha^{p-1}(w^*) \cdots \alpha(w^*)w^* , \qquad \text{for all } x \in M .$$

Then the unitary $U = w\alpha(w) \cdots \alpha^{p-1}(w)u$ satisfies:

$$\beta^p(x) = UxU^* \qquad \text{for all } x \in M .$$

We then have to compute $\beta(U) = w\alpha(U)w^*$. We get:

[10] M_u is the unitary group of M.

$$\beta(U) = w\alpha(w\alpha(w) \cdots \alpha^{p-1}(w)u)w^* = w\alpha(w) \cdots \alpha^{p-1}(w)\alpha^p(w)\alpha(u)w^*$$
$$= (w\alpha(w) \cdots \alpha^{p-1}(w))(uwu^*\gamma(\alpha)uw^*) = \gamma(\alpha)((w\alpha(w) \cdots \alpha^{p-1}(w))u)$$

where we have used $\alpha^p = \mathrm{Ad}\, u$ and $\alpha(u) = \gamma(\alpha)u$. We have shown that $\beta(U) = \gamma(\alpha)U$ so that $\gamma(\beta) = \gamma(\alpha)$.

c) First if α^n is inner, so is $1 \otimes \alpha^n$, and conversely if $\beta^n = \mathrm{Ad}\, V$ then $V \in 1 \otimes M = (N \otimes 1)' \cap N \otimes M$ so that α^n is inner. We have $p_0(\alpha) = p_0(\beta) = p$. Let $u \in M_u$ satisfy $\alpha^p = \mathrm{Ad}\, u$. Then $(1 \otimes \alpha)^p = \mathrm{Ad}\, 1 \otimes u$, and:

$$(1 \otimes \alpha)(1 \otimes u) = 1 \otimes \alpha(u) = \gamma(\alpha)(1 \otimes u) .$$

Then $\gamma(\beta) = \gamma(\alpha)$. Q.E.D.

We now construct automorphisms s_p^γ, $p \in N$, $p \geqq 2$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$, of the hyperfinite factor of type II_1: R, such that

$$p_0(s_p^\gamma) = p , \qquad \gamma(s_p^\gamma) = \gamma .$$

A detailed study of those automorphisms will be done in [5], but here we prefer to define them in a different way, using essentially [11].

The numbers p and γ are fixed throughout. Let Z/p be the additive group of integers modulo p. Let X_p be the compact group $\prod_0^\infty Z/p$; each element s of X_p corresponds to a sequence $s = (s_j)_{j \in N}$, $s_j \in Z/p$, for all $j \in N$. Let m_p be the Haar measure of X_p, with $m_p(1) = 1$. Let χ_p be the countable subgroup of X_p defined by:

$$s \in \chi_p \text{ if and only if there exists } j_0 \in N,\ s_j = 0 \text{ for all } j \geqq j_0 .$$

The group χ_p with discrete topology, acts on the abelian von Neumann algebra $L^\infty(X_p, m_p)$ in the following way:

(4.5) for all $t \in \chi_p$, $(t \cdot a)(s) = a(s - t)$ for all $a \in L^\infty(X_p, m_p)$, for all $s \in X_p$.

This action of χ_p is ergodic and free [10, p. 175], so that the cross product $R = W^*(\chi_p, L^\infty(X_p, m_p))$ is a factor. Moreover m_p is an invariant measure and χ_p is a union of its finite subgroups. Then R is the hyperfinite factor of type II_1. We let I be the canonical isomorphism of $L^\infty(X_p, m_p)$ onto a maximal abelian von Neumann subalgebra $\mathcal{A}$ of R, and let $t \to U_t$ be the homomorphism of χ_p in the unitary group of R, related by the following:

(4.6) $$U_t I(a) U_t^* = I(t \cdot a) \qquad \text{for all } t \in \chi_p, \text{ for all } a \in L^\infty(X_p, m_p) .$$

We define an automorphism Σ of $L^\infty(X_p, m_p)$ by:

$$(\Sigma(a))(s) = a(s - \underline{1}) \qquad \text{for all } a \in L^\infty(X_p, m_p), \text{ for all } s \in X_p$$

where $\underline{1}$ is the element of X_p all of whose coordinates are 1. We have $\Sigma^p = 1$, Σ preserves m_p, and Σ commutes with the action of χ_p on X_p.

We define an automorphism S of R by:

$$(4.7)\qquad S(I(a)) = I(\Sigma(a)) \qquad \text{for all } a \in L^\infty(X_p, m_p)\,,$$
$$S(U_t) = U_t \qquad \text{for all } t \in \chi_p\,.$$

As the $I(a)$ and U_t generate R, and as S is an automorphism of the $_*$ algebra A of finite linear combinations of products $I(a)U_t$, $a \in L^\infty$, $t \in \chi_p$, one checks easily that S defines an automorphism of R [11]. Now let ρ be a mapping from χ_p to the unitary group $\mathcal{A}_u$ of $\mathcal{A}$ such that:

$$(4.8)\qquad \rho_{s+t} = \rho_s U_s \rho_t U_s^* \qquad \text{for all } s, t \in \chi_p\,.$$[11]

Then as in [11], [12] one defines an automorphism τ_ρ of R by the conditions:

$$(4.9)\qquad \tau_p(a) = a,\ \text{for all } a \in \mathcal{A}\,, \qquad \tau_\rho(U_s) = \rho_s U_s,\ \text{for all } s \in \chi_p\,.$$

As $\mathcal{A}$ is abelian, so that the ρ's belong to its center, one sees from (4.8) that τ_ρ defines an automorphism of A and hence of R because it preserves the unique trace τ of R.

From (4.7) and (4.9) we obtain for any $k \in Z$:

$$a = S^k\tau_\rho S^{-k}(a),\ \text{for all } a \in \mathcal{A}\,, \qquad S^k\tau_\rho S^{-k}(U_s) = S^k(\rho_s)U_s,\ \text{for all } s \in \chi_p\,.$$

Therefore we have, with $S^k(\rho)$ mapping χ_p to $\mathcal{A}_u$ by $(S^k(\rho))_s = S^k(\rho_s)$:

$$(4.10)\qquad S^k\tau_\rho S^{-k} = \tau_{S^k(\rho)} \qquad \text{for all } k \in Z\,.$$

For each $n \in N$, we therefore get:

$$(4.11)\qquad (S\tau_\rho)^n = S^n(\tau_{S^{-(n-1)}\rho}) \cdots (\tau_{S^{-1}\rho})\tau_\rho = S^n\tau_{(S^{-(n-1)}\rho \cdots S^{-1}(\rho)\rho)}\,.$$

As for $k = 1, \cdots, p-1$, there is no inner automorphism of R, leaving $\mathcal{A}$ globally invariant and coinciding with S^k on $\mathcal{A}$; by [11] we know that, for any ρ as above, one has

$$(4.12)\qquad p_0(S\tau_\rho) \text{ is a multiple of } p\,.$$

We shall construct a ρ such that $\rho = I(\delta)$ and

$$(4.13)\qquad \left(\Sigma^{-(p-1)}(\delta_t)\Sigma^{-(p-2)}(\delta_t) \cdots \Sigma^{-1}(\delta_t)\delta_t\right)(s) = \gamma^{(s-t)_0}\gamma^{-s_0} = \gamma^{-t_0}$$
$$\text{for all } t \in \chi_p,\ \text{for all } s \in X_p\ \left(\text{with } s = (s_j)_{j \in N}\right).$$

It will follow that $(S\tau_\rho)^p = \operatorname{Ad} U$, where U is the unitary of R, which is the image by I of the function g:

$$s \in X_p \longrightarrow \gamma^{-s_0} = g(s)\,.$$

In fact this follows from (4.11), (4.13) and the equalities

$$\operatorname{Ad} U(a) = a,\ \text{for all } a \in \mathcal{A}\,, \qquad UU_tU^* = U(U_tU^*U_t^*)U_t,\ \text{for all } t \in \chi_p\,.$$

To get δ, we first let f be a map from $Z/p \times Z/p$ to Z/p such that

$$(4.14)\qquad \textstyle\sum_{j=0}^{p-1} f(a+j, b+j) = b - a\,.$$

[11] In notation $\rho \in Z^1(\chi_p, \mathcal{A}_u)$.

Take for instance $f(a, b) = 0$ if $a \neq 0$, and $f(a, b) = b$ if $a = 0$. Then for $s \in X_p$ and $j \in N$ we put $f_j(s) = f(s_j, s_{j+1})$ and

$$\delta_t(s) = \gamma^{(\sum_{j=0}^{\infty}(f_j(s) - f_j(s-t)))} \qquad \text{for all } t \in \chi_p, \text{ for all } s \in X_p .$$

This definition makes sense because, with $t_l = 0$, for all $l \geqq l_0$, one has $(s - t)_l = s_l$, and hence, for any $t \in \chi_p$, the sum $\sum_{j=0}^{\infty}(f_j(s) - f_j(s - t))$ only has a finite number of terms $\neq 0$ and moreover γ^q makes sense for any $q \in Z/p$. Now to check (4.13) we have to prove that for any $s \in X_p$ and $t \in \chi_p$ one has

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{p-1}(f_j(s + \underline{k}) - f_j(s + \underline{k} - t))\right) = -t_0$$

where $\underline{k}$ is the element of X_p all of whose components are equal to k. For each $j \in N$ we have $\sum_{k=0}^{p-1} f_j(s + \underline{k}) = \sum_{k=0}^{p-1} f(s_j + k, s_{j+1} + k) = s_{j+1} - s_j$, using (4.14), and similarly:

$$\sum_{k=0}^{p-1}\left(f_j(s + \underline{k}) - f_j(s + \underline{k} - t)\right) = (s_{j+1} - s_j) - \left((s_{j+1} - t_{j+1}) - (s_j - t_j)\right) = t_{j+1} - t_j .$$

Now only a finite number of the t_j's are different from 0 so:

$$\sum_{j=0}^{\infty}(t_{j+1} - t_j) = -t_0 .$$

THEOREM 4.16. *Let $p, \gamma, R, \cdots$ be as above, and put $s_p^\gamma = S\tau_{I(\delta)}$; then $p_0(s_p^\gamma) = p$ and $\gamma(s_p^\gamma) = \gamma$.*

Proof. We have shown that $(s_p^\gamma)^q$ is outer for $q = 1, \cdots, p - 1$ and that $(s_p^\gamma)^p = \operatorname{Ad} U$ where $U = I(g)$, $g(s) = \gamma^{-s_0}$, for all $s \in X_p$. So $p_0(s_p^\gamma) = p$. Moreover:

$$s_p^\gamma(U) = S(I(g)) = I(\Sigma(g)) = \gamma I(g) = \gamma U . \qquad \text{Q.E.D.}$$

V. The factors $Q_{\lambda,p,\gamma}$, $\lambda \in]0, 1[$, $p \in N$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$

Let $\lambda \in]0, 1[$ and P_λ be the Pukanszky factor of type III_λ. Let $P_\lambda = W^*(\theta_\lambda^1, N_\lambda^1)$ be the discrete decomposition of P_λ as the cross product of a factor of type II_∞: N_λ^1, by an automorphism θ_λ^1 multiplying the trace by λ. By [3, Theorem 4.4.1] we know that the outer conjugacy class of θ_λ^1 is uniquely determined. Let $p \in N$, $p \geqq 2$ and $\gamma \in C$, $\gamma^p = 1$ be given. Then let R and $s_p^\gamma \in \operatorname{Aut} R$ be as constructed in Section IV.

We put $N_\lambda = R \otimes N_\lambda^1$, $\theta_{\lambda,p,\gamma} = s_p^\gamma \otimes \theta_\lambda^1 \in \operatorname{Aut} N_\lambda$. By [3, Theorem 4.4.1], the cross product $W^*(\theta_{\lambda,p,\gamma}, N_\lambda)$ is unaffected by a change in the choice of θ_λ^1. We denote it by $Q_{\lambda,p,\gamma} = W^*(\theta_{\lambda,p,\gamma}, N_\lambda)$.

THEOREM 5.1. *For $\lambda \in]0, 1[$, $p \in N$, $p \geqq 2$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$, the $Q_{\lambda,p,\gamma}$ are mutally non-isomorphic factors of type* III *with separable predual.*

Proof. Let τ be the trace on R and τ_1 a faithful semi-finite normal trace on N_λ^1; then $\tau \otimes \tau_1$ is a faithful semi-finite normal trace on $N_\lambda = R \otimes N_\lambda^1$ and

$$(\tau \otimes \tau_1) \circ \theta_{\lambda,p,\gamma} = (\tau \circ s_p^\gamma) \otimes (\tau_1 \circ \theta_\lambda^1) = \lambda(\tau \otimes \tau_1) .$$

This, with Theorem 4.4.1 of [3], shows that $Q_{\lambda,p,\gamma}$ is a factor of type III_λ, for all λ, p, γ. Now let $\lambda \in]0, 1[$ and $p, p' \in N$, $\gamma, \gamma' \in C$ be given, with $\gamma^p = 1$ $\gamma'^{p'} = 1$, and let us assume that $Q_{\lambda,p,\gamma}$ is isomorphic to $Q_{\lambda,p',\gamma'}$. Then by [3, Theorem 4.4.1] there exists an automorphism Π [12] of N_λ such that $\Pi\theta_{\lambda,p,\gamma}\Pi^{-1}\theta_{\lambda,p',\gamma'}^{-1} \in \mathrm{Int}\, N_\lambda$; i.e.,

$$\Pi\theta_{\lambda,p,\gamma}\Pi^{-1} = \mathrm{Ad}\, u\theta_{\lambda,p',\gamma'},\ u \text{ unitary of } N_\lambda .$$

Now let $\alpha = \Pi(s_p^\gamma \otimes 1)\Pi^{-1}$, $\beta = \Pi(1 \otimes \theta_\lambda^1)\Pi^{-1}$. As $\overline{\mathrm{Int}\, N_\lambda}$ and $\mathrm{Ct}\, N_\lambda$ are normal subgroups of $\mathrm{Aut}\, N_\lambda$ and as $s_p^\gamma \otimes 1 \in \overline{\mathrm{Int}\, N_\lambda}$ (because $s_p^\gamma \in \overline{\mathrm{Int}\, R}$) and $1 \otimes \theta_\lambda^1 \in \mathrm{Ct}\, N_\lambda$ (see Section 3, remark 3.15) we get:

$$\alpha \in \overline{\mathrm{Int}\, N_\lambda} , \qquad \beta \in \mathrm{Ct}\, N_\lambda .$$

In the same way $\alpha' = s_{p'}^{\gamma'} \otimes 1 \in \overline{\mathrm{Int}\, N_\lambda}$ and $\beta' = \mathrm{Ad}\, u \circ (1 \otimes \theta_\lambda^1)$ belongs to $\mathrm{Ct}\, N_\lambda$.

We have $\beta\alpha = \beta'\alpha'$, so that $(\beta')^{-1}\beta = \alpha'\alpha^{-1}$ belongs to $\mathrm{Ct}\, N_\lambda \cap \overline{\mathrm{Int}\, N_\lambda} = \mathrm{Int}\, N_\lambda$.

We have shown that for some unitary $V \in N_\lambda$ one has $\alpha' = \mathrm{Ad}\, V_0\alpha$. It hence follows from Proposition 4.3 that

$$p_0(\alpha') = p_0(\alpha) , \qquad \gamma(\alpha') = \gamma(\alpha) . \tag{5.2}$$

Now $p_0(\Pi(s_p^\gamma \otimes 1)\Pi^{-1}) = p_0(s_p^\gamma \otimes 1)$ and $\gamma(\Pi(s_p^\gamma \otimes 1)\Pi^{-1}) = \gamma(s_p^\gamma \otimes 1)$ by an obvious computation. Moreover by (4.3) we have

$$p_0(s_p^\gamma \otimes 1) = p_0(s_p^\gamma) , \qquad \gamma(s_p^\gamma \otimes 1) = \gamma(s_p^\gamma) .$$

Hence, using (4.16) we get from (5.2) that

$$p' = p , \qquad \gamma' = \gamma .$$

Q.E.D.

THEOREM 5.3. *For* $\lambda \in]0, 1[$, $p \in N$, $p \geqq 3$, $\gamma \in C$, $\gamma^p = 1$, $\gamma^2 \neq 1$, *the factor* $Q_{\lambda,p,\gamma}$ *is not anti-isomorphic to itself.*

Proof. Let M be an arbitrary von Neumann algebra. The conjugate M^c of M is by definition the algebra whose underlying vector space is the conjugate of M (for $\lambda \in C$, $x \in M$ the product λ by x in M^c is equal to $\bar{\lambda}x$) and whose ring structure is the same as in M. In other words the identity map $x \longrightarrow I(x)$ of M on M^c is a conjugate linear ring isomorphism of M on M^c.

The opposite M^o of M is by definition the algebra whose underlying vector space is the same as for M while the product of x by y is equal to yx

[12] s_p^γ and $s_{p'}^{\gamma'}$ act on the same factor R.

instead of xy. We shall not have to consider M^o but we have obviously an isomorphism $x \to x^*$ of M^c on M^o. (See [9, Section 2.3].)

For $\alpha \in \operatorname{Aut} M$ we denote by α^c the automorphism of M^c such that $\alpha^c I(x) = I(\alpha(x))$ for all $x \in M$. This equality does not mean that α^c and α are in the same conjugacy class, because I is *not* an isomorphism.

With α and M as above, let j be a conjugate linear isomorphism of M on M; then $I \circ j^{-1}$ is a linear isomorphism of M on M^c and $j \circ \alpha \circ j^{-1}$ is an automorphism of M which is in the same conjugacy class as $\alpha^c = (I \circ j^{-1}) j \circ \alpha \circ j^{-1} (I \circ j^{-1})^{-1}$. Moreover with M and α as above, the cross product $W^*(\alpha^c, M^c)$ is isomorphic to $(W^*(\alpha, M))^c$. This can be seen by checking that if π is the canonical isomorphism of M onto a von Neumann subalgebra of $W^*(\alpha, M)$ and X the unitary of $W^*(\alpha, M)$ canonically associated to α (so that $X\pi(y)X^* = \pi(\alpha(y))$, for all $y \in M$), the map

$$I_{W^*(\alpha,M)} \circ \pi \circ I_M^{-1} = \pi'$$

is an isomorphism of M^c onto a von Neumann subalgebra of $(W^*(\alpha, M))^c$ and, with $X' = I_{W^*(\alpha,M)}(X)$, one has

$$X'\pi'(y)X'^* = I_{W^*(\alpha,M)} \circ \pi(\alpha I_M^{-1}(y)) = \pi' \circ \alpha^c(y) \qquad \text{for all } y \in M^c .$$

Now, as P_λ is isomorphic to P_λ^c (because P_λ is obtained by the group measure space construction), we see from [3, 4.4.1] that $(\theta_\lambda^1)^c$ is outer conjugate to θ_λ^1.

Also, let $p \in N$, $p \geqq 2$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$, and R, s_p^γ be as constructed in Section IV. We let j be the conjugate linear isomorphism of R onto R such that

$$\begin{aligned} jI(f) &= I(\bar{f}) && \text{for all } f \in L^\infty(X_p, m_p) , \\ jU_s &= U_s && \text{for all } s \in \chi_p . \end{aligned}$$

Then we get $j \circ S \circ j^{-1} = S$ and $j \circ \tau_\rho \circ j^{-1} = \tau_{\bar\rho}$ for all mappings $\chi_p \to \mathfrak{C}_u$ satisfying the cocycle condition (and where $(\bar\rho)_s = \bar\rho_s$ for all $s \in \chi_p$, the obvious meaning).

It follows that $j s_p^\gamma j^{-1} = s_p^{\bar\gamma}$. Hence we have shown that $s_p^{\bar\gamma}$ is in the same conjugacy class as $(s_p^\gamma)^c$. It follows that $s_p^{\bar\gamma} \otimes \theta_\lambda^1$ is outer conjugate to $(s_p^\gamma)^c \otimes (\theta_\lambda^1)^c$ and hence to $(s_p^\gamma \otimes \theta_\lambda^1)^c$.[13] Therefore $W^*(s_p^{\bar\gamma} \otimes \theta_\lambda^1, N_\lambda)$ is isomorphic to $W^*((s_p^\gamma \otimes \theta_\lambda^1)^c, N_\lambda^c)$ which is isomorphic to the conjugate of $Q_{\lambda,p,\gamma}$. We have shown that $(Q_{\lambda,p,\gamma})^c$ is isomorphic to $Q_{\lambda,p,\bar\gamma}$. Q.E.D.

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[13] because $K = I_{N_\lambda} \circ (I_R \otimes I_{N_\lambda^1})^{-1}$ is an isomorphism of $R^c \otimes (N_\lambda^1)^c$ on $(N_\lambda)^c$ such that $K(s_p^\gamma)^c \otimes (\theta_\lambda^1)^c K^{-1} = (s_p^\gamma \otimes \theta_\lambda^1)^c$.

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