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On Hyperfinite Factors of Type III_0 and Krieger's Factors

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A factor M is the cross product of an abelian von Neumann algebra by a single automorphism iff there exists an increasing sequence of normal conditional expectations of M onto finite-dimensional subalgebras N_k with $(\bigcup N_k)^- = M$. Assuming the uniqueness of the hyperfinite factor of type II_∞, we prove then that any hyperfinite factor of type III_0 is the cross product of an abelian von Neumann algebra by a single automorphism.

The problem of classifying the factors which are cross products of an abelian von Neumann algebra by a single automorphism has been shown by W. Krieger [7] to be the same as the problem of classifying ergodic flows. This fact which is a striking isomorphism between the classification theory of some factors and ergodic theory makes interesting to try to characterise the class of Krieger's factors thus obtained.[1]

Here first we give (Theor. 1) a characterisation of Krieger's factors by a property apparently stronger than hyperfiniteness.

Then in Part II, Theorem II.1 we apply this to prove, assuming the uniqueness of the hyperfinite factor of type II_∞, that any hyperfinite factor of type III_0 is a Krieger's factor. Recall that a factor M is hyperfinite iff it is the weak closure $(\bigcup_1^\infty N_k)^-$ of an increasing sequence of finite dimensional subalgebras N_k.

I

THEOREM 1. *For a factor M in a separable Hilbert space*[2] *the following conditions are equivalent.*

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[1] By definition M is a Krieger's factor iff it is the cross product of an abelian von Neumann algebra by a single automorphism.

[2] We assume that M is not finite dimensional.

(a) *M is a Krieger's factor (i.e., is the cross product of an abelian von Neumann algebra by a single automorphism).*

(b) *there exists an increasing sequence E_k of normal conditional expectations of M onto finite dimensional subalgebras N_k with*

$$\left(\bigcup_{k=1}^{\infty} N_k\right)^- = M \qquad (\textit{weak closure}).$$

Proof of (b) $\Rightarrow$ (a).

LEMMA 2. *Let M be a von Neumann algebra $(E_k)_{k=1,2,\ldots}$ be an increasing sequence of normal conditional expectations on von Neumann subalgebras $N_k = E_k(M)$ then:*

(a) *There exists a faithful normal state φ on M such that N_k is globally invariant under σ_t^φ, for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$.*

(b) *($\bigcup_1^\infty N_k$ is weakly dense in M) $\Leftrightarrow$ ($E_k(x) \to_{k\to\infty} x$ strongly, $\forall x \in M$).*

Proof. (a) Take $\varphi = \psi \circ E_1$ where ψ is a faithful normal state on N_1, then $\varphi \circ E_k = \psi \circ E_1 \circ E_k = \psi \circ E_1 = \varphi$. Then apply [8, p. 309, Theorem 8].

(b) Let $\mathscr{H}_\varphi$, ζ_φ be the Gelfand Segal construction of φ, then by [8, p. 315] we have $E_k(x)\zeta_\varphi = e_k(x\zeta_\varphi)$, $\forall x \in M$, where e_k is the orthogonal projection of $\mathscr{H}_\varphi$ onto $(N_k\zeta_\varphi)^-$. The conclusion follows.

LEMMA 3. *Let M be a von Neumann algebra, $(E_k)_{k=1,2,\ldots}$ as in Lemma 2, with N_k of type* I *for all k, and $(\bigcup_{k=1}^\infty N_k)^- = M$. Then there exists another sequence $(F_k)_{k=1,2,\ldots}$ such that*

(a) *$(F_k)_{k=1,\ldots}$ is an increasing sequence of normal conditional expectations of M onto type* I *von Neumann subalgebras P_k.*

(b) *$(\bigcup_1^\infty P_k)^- = M$.*

(c) *For k odd, one has center of $P_k \subset$ center of P_{k+1}.*

(d) *For k even, one has $P_k' \cap P_{k+1} \subset P_k$.*

Proof. Let φ be a faithful normal state on M satisfying condition 2 (a) with respect to $(N_k)_{k=1,2,\ldots}$.

We put $P_1 = N_1$, $P_2 = (\text{center of } N_1)' \cap N_2$, $P_3 = N_{2,\ldots}$, $P_{2k} = (\text{center of } N_k)' \cap N_{k+1}$, $P_{2k+1} = N_{k+1}$.

By construction P_n, $n = 1, 2, \ldots$ is globally invariant under σ_t^φ, $\forall t \in \mathbb{R}$ so that [8, p. 309, Theor.] there exists a unique conditional expectation F_n, $\varphi \circ F_n = \varphi$, $F_n(M) = P_n$ and one checks conditions (a) and (b).

Condition (c) is clearly satisfied. Finally for all $k = 1, 2, \ldots$, the commutant of P_{2k} in N_{k+1} is generated by the center of N_k and the center of N_{k+1}, because N_{k+1} is normal. It follows that $P'_{2k} \cap P_{2k+1} \subset P_{2k}$.

LEMMA 4. *Let $Q_1 \subset Q_2$ be two type* I *von Neumann algebras with center of $Q_1 \subset$ center Q_2, φ be a faithful normal state on Q_2 with $\sigma_t^\varphi Q_1 = Q_1$, $\forall t \in \mathbb{R}$. Then for any maximal abelian subalgebra $\mathcal{O}_1$ of Q_1, included in $Q_{2,\varphi}$ there exists a maximal abelian subalgebra $\mathcal{O}_2$ of Q_2 containing $\mathcal{O}_1$, contained in $Q_{2,\varphi}$, and such that any unitary $u \in Q_1$ satisfying $u\mathcal{O}_1 u^* = \mathcal{O}_1$ also satisfies $u\mathcal{O}_2 u^* = \mathcal{O}_2$.*

Proof. Using desintegration of Q_1, we can reduce to the case when Q_1 is a factor of type I.

We hence can suppose that $Q_2 = P \otimes Q_1$, identifying Q_1 with $\mathbb{C} \otimes Q_1 \subset P \otimes Q_1$. Then the two states $\varphi/P \otimes \varphi/Q_1$ and φ have the same modular automorphism groups: $(\sigma^\varphi)_P \otimes (\sigma^\varphi)_{Q_1}$ and the same restriction to the center of $P \otimes Q_1$. It follows that $\varphi = \varphi/P \otimes \varphi/Q_1$ [8, p. 310].

Choose $\mathcal{O}_2 = \mathcal{O} \otimes \mathcal{O}_1$ where $\mathcal{O}$ is a maximal abelian subalgebra of P such that $\mathcal{O} \subset P_{\varphi/P}$. Then $\mathcal{O}_2$ is maximal abelian in $P \otimes Q_1$ and any unitary u of $\mathbb{C} \otimes Q_1$ such that $u(\mathbb{C} \otimes \mathcal{O}_1)u^* = \mathbb{C} \otimes \mathcal{O}_1$ also satisfies $u\mathcal{O}_2 u^* = \mathcal{O}_2$.

LEMMA 5. *Let M be a continuous factor, $(E_k)_{k=1,2,\ldots}$ be an increasing sequence of normal conditional expectations on von Neumann subalgebras of type* I, *$N_k = E_k(M)$, with $(\bigcup_{k=1}^\infty N_k)^- = M$. Then M is the crossed product of an abelian von Neumann algebra by a single automorphism.*

Proof. Using Lemma 3, we replace $(E_k)_{k=1\ldots}$ by $(F_k)_{k=1,2,\ldots}$ satisfying 3 (a, b, c, d). Also we let φ be a faithful normal state on M such that $\sigma_t^\varphi(P_k) = P_k$, $\forall t \in \mathbb{R}$, $k = 1, 2, \ldots$. Let $\mathcal{O}_1$ be a maximal abelian subalgebra of P_1, $\mathcal{O}_1 \subset P_{1,\varphi} = P_1 \cap M_\varphi$. By Lemma 4 we find a maximal abelian subalgebra $\mathcal{O}_2$ of P_2 such that $\mathcal{O}_1 \subset \mathcal{O}_2 \subset (P_2)_\varphi$ and that the normaliser $\mathcal{G}_1$ of $\mathcal{O}_1$ in P_1 is contained in the normaliser $\mathcal{G}_2$ of $\mathcal{O}_2$ in P_2.

We put $\mathcal{O}_3 = \mathcal{O}_2$ and we note that, as $P_2' \cap P_3 \subset P_2$, we have for all projections $e \in \mathcal{O}_2$ which are abelian in P_2, that:

$$(\mathcal{O}_2)_e = (P_2)_e \text{ is maximal abelian in } (P_3)_e .$$

Now as by construction $\mathcal{O}_2$ is the range of a normal conditional expectation from P_2 (because $\mathcal{O}_2 \subset (P_2)_\varphi$), there exists a partition of unity $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_2$, of abelian projections of P_2. It follows that $\mathcal{A}_3 = \mathcal{A}_2$ is maximal abelian in P_3. We can hence construct inductively a sequence $(\mathcal{A}_k)_{k=1,2,\ldots}$ of abelian subalgebras of M such that:

(1) $\mathcal{A}_k \subset M_\varphi$, $k = 1, 2,\ldots$;

(2) $\mathcal{A}_k$ is an abelian maximal subalgebra of P_k, $k = 1, 2,\ldots$;

(3) $\mathcal{A}_k \subset \mathcal{A}_{k+1}$, $k = 1,\ldots$, and the normaliser $\mathcal{G}_k$ of $\mathcal{A}_k$ in P_k is for all $k = 1,\ldots$, contained in $\mathcal{G}_{k+1}$.

Put $\mathcal{A} = (\bigcup_{k=1}^{\infty} \mathcal{A}_k)^-$. Then $\mathcal{A}$ is a maximal abelian subalgebra of M because $x \in \mathcal{A}' \cap M$ implies $F_k(x) \in \mathcal{A}_k' \cap P_k = \mathcal{A}_k$, $\forall k \in \mathbb{N}$ (use $F_k(ax) = aF_k(x)$, $F_k(xa) = F_k(x)a$, $\forall a \in \mathcal{A}_k$). Also $\mathcal{A}$ is the range of a normal conditional expectation E because $\mathcal{A} \subset M_\varphi$ (so that $\sigma_t^\varphi \mathcal{A} = \mathcal{A}$ $\forall t \in \mathbb{R}$).

Put $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$. Then $\mathcal{G}$ is contained in the normaliser of $\mathcal{A}$ and as $\mathcal{A}_k$ and $\mathcal{G}_k$ generate P_k for all k(P_k is type I and $\mathcal{G}_k$ is the normaliser of $\mathcal{A}_k$ in P_k) we see that $\mathcal{A}$ and $\mathcal{G}$ generate M.

For each k the group $\mathcal{E}_k$ of automorphisms of $\mathcal{A}$ of the form $\mathrm{Ad}u/\mathcal{A}$, $u \in \mathcal{G}_k$ has the following property: There exists in $\mathcal{A}$ a partition of unity formed of projections $(e_n^k)_{n\in\mathbb{N}}$ such that the restriction of any $g \in \mathcal{E}_k$ to $e_n^k g^{-1}(e_n^k)$ is the identity.

(To prove it choose $e_n^k \in \mathcal{A}_k$, e_n^k abelian for P_k. For $u \in \mathcal{G}_k$, $n \in \mathbb{N}$ $e_n^k u e_n^k$ is a partial isometry $v \in \mathcal{A}$ with initial and final supports equal to $e_n^k u^* e_n^k u$ and such that $u(e_n^k u^* e_n^k u) = v$.)

Now this property shows that each $\mathcal{E}_k$ is of type I in the sense of H. Dye. Then $\mathcal{E} = \bigcup_1^{\infty} \mathcal{E}_k$ is an hyperfinite group acting ergodically on the nonatomic abelian von Neumann algebra $\mathcal{A}$ (recall that M is a continuous factor). By a result of Krieger generalising the measure preserving case treated by Dye, one knows then that there exists a single ergodic automorphism T of $\mathcal{A}$ whose full group $[T]$ is equal to the full group of $\mathcal{E}$. By [1, 1.5.5(c)] there exists a unitary $X \in M$, $X\mathcal{A}X^* = \mathcal{A}$ such that $\mathrm{Ad}\, X/\mathcal{A} = T$ and the property $\mathcal{E} \subset [T]$ easily shows that $\mathcal{A}$ and X generate M. So the conditions of [1, 4.1.2] are fulfilled and M is isomorphic to $W^*(T, \mathcal{A})$.

Proof of (a) $\Rightarrow$ (b). We let Ω, μ be a finite measure space, T be an ergodic automorphism of Ω, μ. We assume that Ω, μ is nonatomic.

We denote by M the cross product of $L^\infty(\Omega, \mu)$ by the action of $\mathbb{Z}$ corresponding to the transpose of T^{-1}. (As in [1, 1.4.7.)

Also $L^\infty(\Omega, \mu)$ is identified to a maximal abelian subalgebra $\mathcal{A}$ of M and is hence the range of a unique conditional expectation E from M.

Now we need to define certain partial isometries in M, and we want to extend to partial transformations of Ω, μ the canonical representation of the full group of T in the unitary group of M. Recall that to T is associated an equivalence relation in Ω, corresponding to the partition into orbits and that a bimeasurable transformation S of Ω belongs to $[T]$ if and only if $S(s) \sim (s)\ \forall s \in \Omega$. By a partial transformation S of Ω, μ we mean a bimeasurable bijection of a measurable subset D_S of Ω onto $R_S \subset \Omega$. We write $S \in [T]$ to mean that $S(s) \sim s\ \forall s \in D_S$.

As T is aperiodic on Ω, μ, there exists, given $S \in [T]$, a unique partition $(E_n)_{n\in\mathbb{Z}}$ of D_S such that

$$S(s) = T^n(s) \qquad \forall s \in E_n .$$

Now to T there corresponds a unitary $U \in M$ such that

$$U^k f U^{-k} = f \circ T^{-k} \qquad \forall f \in L^\infty(\Omega, \mu).$$

For each S as above we set

$$\bar{S} = \sum U^n \chi_{E_n} \qquad \text{where} \quad \chi_{E_n}(s) = \begin{cases} 0 & \text{if} \quad s \notin E_n , \\ 1 & \text{if} \quad s \in E_n . \end{cases}$$

One checks that if for two partial transformations S_1, S_2 the product S_2S_1 is defined in the obvious way, with $D_{S_2S_1} = S_1^{-1}(D_{S_2} \cap R_{S_1})$, one has $\overline{S_2S_1} = \bar{S}_2\bar{S}_1$, $\bar{S}^* = \overline{S^{-1}}$.

We now give a simple criterion for an $\bar{S}$ to be an eigenvector for the modular automorphism group $\sigma^{\nu\circ E}$, ν measure equivalent to μ.

LEMMA 6. *Let Ω, μ, T, $[T]$ be as above, as well as M, E,... . Let ν be a positive finite measure on Ω equivalent to μ. Also let S be a partial transformation of Ω such that for some $\lambda > 0$,*

$$S \in [T], \qquad d\nu S(x)/d\nu(x) = \lambda \qquad \forall x \in D_S .$$

Then $\bar{S}$ is an eigenvector for $\sigma^{\nu\circ E}$ for the eigenvalue λ.

Proof. $\rho_n = dT^{-n}\nu/d\nu$ is a positive operator affiliated to $L^\infty(\Omega, \mu)$, then [1, 1.4.5(b)] one has:

$$\sigma_t^{\nu\circ E}(U^n f) = U^n f \rho_n^{it} \qquad \forall n \in \mathbb{Z}, \qquad \forall f \in L^\infty(\Omega, \mu)$$

we have $\bar{S} = \sum U^n \chi_{E_n}$ and we just need to remark that $\chi_{E_n}\rho_n^{it} = \lambda^{it}\chi_{E_n}$, $\forall n \in \mathbb{Z}$, $\forall t \in \mathbb{R}$ because for each $x \in E_n$,

$$\rho_n(x) = d\nu(T^n x)/d\nu x = d\nu(Sx)/d\nu x = \lambda.$$

LEMMA 7. *Let Ω, μ, T as above, then there exists a finite measure ν equivalent to μ, and a sequence (S_k) of transformations of period 2 of Ω, pairwise commuting such that*:

(1) *The full group of the $(S_k)_{k\in\mathbb{N}}$ is equal to $[T]$.*

(2) *The ratios $d\nu\, S_k(x)/d\nu(x)$ are in finite number for each k.*

Proof. First choose a finite measure $\mu_1 \sim \mu$ and an aperiodic μ_1 preserving transformation $S \in [T]$ (using [5, p. 172, Prop. 2.9]). Then modify the proof of Proposition 2.2 of [6] to get a sequence of commuting transformations $(S_k)_{k\in\mathbb{N}}$ of period 2 satisfying 7.1 with $dS_k\mu_1/d\mu_1$ bounded for all k, and a sequence $(A_k)_{k\in\mathbb{N}}$ of subsets of Ω, $A_1 = \Omega$ such that, for all k the sets A_{k+1}, S_kA_{k+1} form a partition of A_k. Finally modify the proof of Lemma 3.1 of [5] to get ν equivalent to μ_1 satisfying Lemma 7, (2).

LEMMA 8. *Choose ν and $(S_k)_{k=1,2,\ldots}$ as in Lemma 7 also let $(B_k)_{k=1,2,\ldots}$ be a sequence of measurable subsets of Ω, μ. Then, for each n, the von Neumann subalgebra of M generated by the $\sigma_t^{\nu\circ E}(\bar{S}_k)$, χ_{B_k} for $k = 1, 2,\ldots, n$, $t \in \mathbb{R}$ is finite dimensional.*

Proof. Let n be given and $\mathscr{P}$ be a finite measurable partition of Ω, μ such that (1) Each B_k, $k = 1,\ldots, n$ is a union of atoms of $\mathscr{P}$;

(2) For each atom A of $\mathscr{P}$ and each $\epsilon = (\epsilon_1 \cdots \epsilon_n) \in \{0, 1\}^n$ the Radon–Nikodym derivative $d\nu S_\epsilon(x)/d\nu(x)$[3] is constant on A;

(3) For each atom A of $\mathscr{P}$ and $\epsilon \in \{0, 1\}^n$, $S_\epsilon(A)$ is an atom of $\mathscr{P}$. Then consider the set $\mathscr{G}$ of partial transformations of Ω, μ of the form $S_{\epsilon_1\epsilon_2\cdots\epsilon_n}$ restricted to A, $\epsilon_j = 0, 1$, A atom of $\mathscr{P}$. Clearly the product of a pair of elements of $\mathscr{G}$ is either in $\mathscr{G}$ or has an empty domain. It follows that the linear span of $\mathscr{G}$ in M is a finite dimensional $*$ subalgebra of M, invariant under $\sigma_t^{\nu\circ E}$ $\forall t \in \mathbb{R}$ by Lemma 6, and containing the χ_{B_k}, $k = 1,\ldots, n$ as well as the $\bar{S}_1,\ldots, \bar{S}_k,\ldots, \bar{S}_n$.

End of the proof of (a) $\Rightarrow$ (b). We choose ν and $(S_k)_{k=1,2,\ldots}$ as in Lemma 7. Also we let $(B_k)_{k=1,2,\ldots}$ be a sequence of measurable subsets of Ω, μ which generate all measurable subsets of Ω, μ modulo μ. For each n, let M_n be the finite-dimensional subalgebra of M generated by the $\sigma_t^{\nu\circ E}(S_k)$, χ_{B_k}, $k = 1, 2,\ldots, n$, $t \in \mathbb{R}$. By [8] there exists a unique conditional expectation E_n of M onto M_n such that $(\nu \circ E) \circ E_n = \nu \circ E$. As $M_n \subset M_{n+1}$, $\forall n$, we see that the sequence E_n is increasing.

Finally the von Neumann subalgebra of M generated by the M_n contains all the $(\chi_{B_k})_{k=1,2,\ldots}$, hence all $L^\infty(\Omega, \mu)$, and all the $(\bar{S}_k)_{k=1,2,\ldots}$ and hence is equal to M.

[3] There $S_{\epsilon_1\epsilon_2\cdots\epsilon_n} = S_1^{\epsilon_1}S_2^{\epsilon_2}\cdots S_n^{\epsilon_n}$.

II. Application to Hyperfinite Factors of Type III_0

Theorem II.1. *Let M be an hyperfinite factor of type* III_0, *$M = W^*(\theta, N)$ a discrete decomposition of M (as in [1, Part V]). Then N is always an hyperfinite von Neumann algebra of type* II_∞ ([2]) *and if only the product hyperfinite factor of type* II_∞ *appears in the central decomposition of N, M is a Krieger's factor.*

In particular if all hyperfinite factors of type II_∞ were isomorphic then any hyperfinite factor of type III_0 would be a Krieger's factor. To prove Theorem 1 we choose on M a lacunary weight φ, $\mathbb{Q}$ almost periodic [3, Theor. 1.5] and with properly infinite centraliser. We put $N = M_\varphi$, $C =$ center of N, E the unique conditional expectation of M onto N, $\mathcal{N}(E)$ the normaliser of E ($\mathcal{N}(E) = \{u$ unitary of M, $uNu^* = N\}$ as in [1, p. 163]. Also we let $\mathcal{G} \subset \operatorname{Aut} N$ be the group of the $\operatorname{Ad} u/N$, $u \in \mathcal{N}(E)$. By [1] 5.3.6, there exists an action $\epsilon \to S_\epsilon$ of $(\mathbb{Z}/2)^{\mathbb{N}}$ on N such that the full group ([1] 1.5.4) of the S_ϵ, $\epsilon \in (\mathbb{Z}/2)^{\mathbb{N}}$ is $\mathcal{G}$. Also we have a decreasing sequence of projections $(f_k)_{k=1,2,\ldots}$ of C such that for all $k \in \mathbb{N}$ the $(S_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}(f_{n+1}))_{\epsilon_j=0,1}$ form a partition of unity in C, with

$$\underbrace{S_{00\cdots01}}_{n}(f_{n+1})) + S_{00\cdots0}(f_{n+1}) = f_n \,.$$

Lemma II.2. *Let τ be the trace on N which is the restriction of φ to N, then there exists a desintegration of (N, τ):*

$$C = L^\infty(X, \mu), \qquad N = \int N_s \, d\mu(s), \qquad \tau = \int \tau_s \, d\mu(s),$$

where X is a standard borel space, $s \to N_s$ is borel from X into the set of subfactors of a fixed factor F_∞ of type I_∞, and τ_s is a semifinite faithful normal trace on N_s, $\forall s \in X$. Moreover one can assume that, for all $\alpha \in \mathcal{G}$, there exists a borel function r_α from X to $\mathbb{Q}_+$ such that for all projections $e \in N$

$$\tau_{\tilde\alpha(t)}((\alpha(e))_{\tilde\alpha(t)}) = r_\alpha(t)\, \tau_t(e_t) \qquad \forall t \in X\,[4].$$

Proof. The first part of the lemma follows from [4]. By [1] 5.3.1 the automorphisms of X, μ of the form $\tilde\alpha$, $\alpha \in \mathcal{G}$ [4] constitute the full group of a single automorphism of X, μ. Hence by [5] 3.1 we can choose μ such that $d\mu\tilde\alpha(s)/d\mu(s) \in \mathbb{Q}$ for all $\alpha \in \mathcal{G}$, all $s \in X$.

[4] For each $\alpha \in \mathcal{G}$ we let $\tilde\alpha$ be a point transformation of X, whose inverse transposed to $L^\infty(X, \mu)$ is equal to the restriction of α to C.

Now let $\alpha \in \mathscr{G}$, ρ be a borel function from X to $\mathbb{Q}^+$ such that, identifying it with a positive operator affiliated to C, we have $\tau \circ \alpha = \tau(\rho\ \cdot)$. For all projections e of N, and all $x \in C^+$ we get:

$$\tau(\alpha(e)x) = \tau \circ \alpha(e\alpha^{-1}(x)) = \tau(\rho\, e\alpha^{-1}(x)),$$

$$\int \tau_t((\alpha(e))_t)\, x(t)\, d\mu(t) = \int \tau_s(e_s)\, \rho(s)\, x(\bar{\alpha}(s))\, d\mu(s).$$

Hence

$$\tau_{\bar{\alpha}(s)}(\alpha(e)_{\bar{\alpha}(s)})\, d\mu(\bar{\alpha}(s)) = \tau_s(e_s)\, \rho(s)\, d\mu(s)$$

and Lemma II.2 follows.

LEMMA II.3. *Let f_1 be a projection of C, S be an automorphism of N_{f_1}, $S^2 = 1$, $S \in \mathscr{G}$ (when extended by 1 on $1 - f_1$) and f_2 be a projection of C with $Sf_2 + f_2 = f_1$. Let then P be a von Neumann subalgebra of N_{f_1}, of type I_∞, with the same center as N_{f_1} and having an abelian projection e of central support 1 such that $\tau_t(e_t) \in \mathbb{Q}\ \forall t \in f_1$*[5].

Then there exists a von Neumann subalgebra $P^{(1)}$ of N_{f_1} containing P, satisfying the above conditions for P, and an automorphism $S^{(1)}$ of N_{f_1} equal to S modulo inner automorphisms of N_{f_1} and leaving $P^{(1)}$ globally invariant.

Proof. For the proof we can assume that $f_1 = 1$. Let e be an abelian projection of P with central support 1. Put $a(s) = \tau_s(e_s)$. Then there exists a borel function $s \to b(s)$ from X to $\mathbb{Q}_+$ such that:

$$b(S(s)) = r_S(s)\, b(s) \qquad \forall s \in X, \quad a(s)\, b(s)^{-1} \in \mathbb{N}, \quad \forall s \in X.$$

Then we choose by classical selection theorems a borel map $s \to P_s^{(1)}$ from X to subfactors of type I_∞ of F_∞ in such a way that

$$P_s \subset P_s^{(1)} \subset N_s\,, \qquad \forall s \in X$$

and that for some abelian projection $e^{(1)}$ of central support 1 in $\int P_s^{(1)}\, d\mu(s)$ we have $\tau_s(e_s^{(1)}) = b(s)$. We define $P^{(1)}$ as the integral of the $P_s^{(1)}$, $s \in X$. We consider two subalgebras of N_{1-f_2}, namely $P^{(1)}_{1-f_2}$ and $(SP^{(1)})_{1-f_2} = S(P^{(1)}_{f_2})$. In their central decomposition over $1 - f_2$ appear only factors of type I_∞: $P_s^{(1)}$, $(SP^{(1)})_s$ which for all s are unitarily equivalent inside N_s (for the minimal projections $e_s^{(1)}$ in $P_s^{(1)}$ and $(S(e^{(1)}))_s$ in $(S(P^{(1)}))_s$ satisfy $\tau_s(e_s^{(1)}) = \tau_s(S(e^{(1)})_s)$ for all $s \in X$. It follows that we can find $S^{(1)}$, $S^{(1)2} = 1$, $S^{(1)} = S$ modulo inner automorphisms, and $S^{(1)}P^{(1)} = P^{(1)}$.

[5] f_1 is here identified with a borel subset of X in the obvious way.

LEMMA II.4. *There exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of type I_∞ of N, with the same center as N, with $(\bigcup_{n=1}^{\infty} P_n)^- = N$, and an action $S_\epsilon^{(1)}$, $\epsilon\in(\mathbb{Z}/2)^{\mathbb{N}}$ of $(\mathbb{Z}/2)^{\mathbb{N}}$ on N, equal to S_ϵ up to inner automorphisms of N and such that*:

(1) *for each abelian projection $e\in P_n$, with central support 1, $\tau(e)$ is finite and $\tau_t(e_t)\in\mathbb{Q}$, $\forall t\in X$;*

(2) *each P_n is globally invariant by the $S_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}$, $\epsilon_j = 0, 1$.*

Proof. Let $d(s)$, $s\in X$ be a borel map from X to $\mathbb{Q}^+$, $d(s) > 0$, $\forall s\in X$ such that $\int d(s)\, d\mu(s) < \infty$. Then choose a von Neumann subalgebra P_0 of N with center C, having an abelian projection e with $\tau_t(e_t) = d(t)$, $\forall t\in X$. Also we let $\mathscr{V}_1 \cdots \mathscr{V}_m \cdots$ be a basis of open sets for the strong topology on the unit ball of N.

By Lemma II.3 we choose $P_0^{(1)}$ containing P_0 and invariant under $S_1^{(1)}$. As $S_1^{(1)}$ is now chosen, it is enough, to build P_1, to define $(P_1)_s$ for $s\in f_2$ and then use its required $S_1^{(1)}$ invariance to extend the definition for $s\in 1-f_2$. The condition $P_0\subset P_1$ is insured if $(P_0^{(1)})_s\subset(P_1)_s$, $\forall s\in f_2$ and as there is no other restriction (using the fact that N_s is for each s a factor such that the relative commutant of $(P_0^{(1)})_s$ in it, is hyperfinite), we can build P_1 such that $\mathscr{V}_1\cap P_1\neq\varnothing$.

It is clear that one can go on like this, because to go from step k to step $k+1$ one needs only define the restriction of $S_{00\cdots 1}^{(1)}$ on f_k, as well as the $(P_{k+1})_s$ for $s\in f_{k+1}$ which can be done as above, using Lemma II.3. As we can ensure that $P_k\cap\mathscr{V}_k\neq\varnothing$ for all k we get $(\bigcup_{k=1}^{\infty} P_k)^- = N$ and the lemma is proven.

Proof of Theorem 1. We choose a sequence $(P_k)_{k\in\mathbb{N}}$ and an action $S^{(1)}$ of $(\mathbb{Z}/2)^{\mathbb{N}}$ on N satisfying the conditions of Lemma II.4. We let for each k, N_k be the von Neumann subalgebra of M generated by P_k and the unitaries $u\in\mathscr{N}(E)$ satisfying $\mathrm{Ad}u/N = S^{(1)}_{\epsilon_1\epsilon_2,\ldots,\epsilon_k}$ for some $\epsilon_j = 0, 1$. By [1, 4.1.3] one sees that N_k is the cross product of P_k by the finite group $(\mathbb{Z}/2)^k$ and hence is of type I. Also by [1, 1.4.5] we see that N_k is globally invariant by σ_t^φ, $\forall t\in\mathbb{R}$ (For any of the above unitaries u, one will have $\sigma_t^\varphi(u)\in Cu$, $\forall t\in\mathbb{R}$).

As the restriction of φ to P_k and hence N_k is semifinite it follows from [8, p. 309] that there exists an increasing sequence of conditional expectations E_k of M onto N_k such that $\varphi\circ E_k = \varphi$, $k = 1, 2,\ldots$.

The von Neumann subalgebra of M generated by the N_k contains $(\bigcup_{k=1}^{\infty} P_k)^- = N$ and all the unitaries u of $\mathscr{N}(E)$ so it is equal to M. Theorem II.1 hence follows from Lemma 5 of Part I.

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