A FACTOR NOT ANTI-ISOMORPHIC TO ITSELF

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ABSTRACT

We prove for each $\lambda \in]0, 1[$, the existence of a factor of type III_λ, acting on a separable hilbert space, and not anti-isomorphic to itself. It follows that it is not constructible by the group measure space construction. The proof of non anti-isomorphism is an application of our classification of type III factors.

Introduction

One of the best behaved infinite dimensional generalisation of the simple central algebras over $\mathbb{C}$ are the factors of Murray and von Neumann. As $\mathbb{C}$ is algebraically closed, its Brauer group is trivial, any (finite dimensional) simple central algebra over $\mathbb{C}$ is a matrix algebra and is anti-isomorphic to itself.

We construct a factor Q acting in a separable hilbert space and which is not anti-isomorphic to itself.

As the classical constructions of factors, like the group measure space construction or like the von Neumann algebra of the left regular representation of locally compact groups, give von Neumann algebras anti-isomorphic to themselves, the factor Q cannot be obtained this way.

The construction of Q relies essentially on our theorem of classification of factors of type III_λ by automorphisms θ of factors of type II_∞ N such that $\tau \circ \theta = \lambda\tau$ where τ is the trace on N [**1**]. This theorem drove us to study the automorphism groups of factors of type II_∞, and also (because any II_∞ is the tensor product of a II_1 by a type I_∞) to study the automorphisms of II_1 factors.

The two aspects of the study of automorphism groups of II_1 factors deal with the problems: (1) of conjugacy of two automorphisms: α and $\beta \in \operatorname{Aut} N$ are conjugate when $\beta = \sigma\alpha\sigma^{-1}$ for some $\sigma \in \operatorname{Aut} N$; (2) of outer conjugacy of two automorphisms: α and $\beta \in \operatorname{Aut} N$ are outer conjugate when β is conjugate to the product of α by an inner automorphism of N†.

The problem (1) can be considered as an aspect of non abelian ergodic theory.

The problem (2) is the one which is directly related to the classification of type III_λ factors $\lambda \in]0, 1[$.

We began our investigations on problems (1), (2) with the simplest case: periodic automorphisms of the hyperfinite factor R of type II_1. In general, when M is an

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† i.e. $\varepsilon(\alpha)$ and $\varepsilon(\beta)$ are in the same conjugacy class in $\operatorname{Out} N = \operatorname{Aut} N/\operatorname{Int} N$, with ε the canonical surjection $\operatorname{Aut} N \xrightarrow{\varepsilon} \operatorname{Out} N$.

[BULL. LONDON MATH. SOC., 7 (1975), 171–174]

arbitrary factor, one defines for any $\alpha \in \operatorname{Aut} M$, two invariants $p_0(\alpha)$, $\gamma(\alpha)$ in the following way:

$$T(\alpha) = \{n \in \mathbb{Z}, \alpha^n \in \operatorname{Int} M\} = p_0(\alpha)\mathbb{Z} \quad \text{and} \quad p_0(\alpha) \in \mathbb{N}.$$

$$\gamma(\alpha) = \gamma \in \mathbb{C}, (\alpha^{p_0(\alpha)} = \operatorname{Ad} u, u \text{ unitary in } M) \Rightarrow \alpha(u) = \gamma u.$$

As M is a factor those definitions are unambiguous. Moreover a simple computation shows that p_0, γ are outer invariants of α, i.e. are not affected if α is replaced by any β outer conjugate to α, and that $\gamma(\alpha)$ is always a $p_0(\alpha)$th root of 1 in $\mathbb{C}$.

As the hyperfinite II_1 factor R is isomorphic to $R \otimes R$, and also to R^0 the opposite factor, one can define operations on outer conjugacy classes c of elements of Aut R, denoted by $c_1 \otimes c_2$ and c^0. Our main result for periodic automorphisms of R is that for each $n \in \mathbb{N}$ the set $\operatorname{Br}(\mathbb{Z}/n, R)$ of outer conjugacy classes of automorphisms $\alpha \in \operatorname{Aut} R$, such that $p_0(\alpha) = n$, gifted with the product $c_1 \otimes c_2$ and the inverse operation $c \to c^0$ is a group. Also we show that γ is an isomorphism of this group on the group of nth roots of one in $\mathbb{C}$. In the theory of simple algebras with centre a field K, each algebra of a class is expressed as the tensor product of a matrix algebra over K by a division algebra, belonging to the same class of Brauer equivalence. Here the situation is very similar. For each class $c \in \operatorname{Br}(\mathbb{Z}/n, R)$ there exists an $\alpha \in \mathbb{C}$, whose period (i.e. the smallest $m \geqslant 1$ with $\alpha^m = 1$) is minimal among periods of elements of c. Moreover such an α is unique up to conjugacy† and any periodic automorphism $\beta \in c$ is the tensor product of a minimal $\alpha \in c$ by an inner automorphism. This allows to solve problem (1) for periodic automorphisms of R. Any periodic automorphism of R is conjugate to the tensor product $s_n^{\gamma} \otimes u$, where u is an inner automorphism, and where for each $n \in \mathbb{N}$, $\gamma \in \mathbb{C}$, $\gamma^n = 1$, s_n^{γ} is an automorphism of R constructed in the following way:

Let K be the algebra on $\mathbb{C}$ of $n \times n$ matrices and for each q let Π_q be an isomorphism of K onto a subfactor K_q of R such that the K_q commute pairwise and generate R. Let θ be the unilateral shift on R, such that $\theta\Pi_q = \Pi_{q+1}$ $\forall q \in \mathbb{N}$. Let (e_{ij}) be a system of matrix units in K, and $e_{ij}{}^q = \Pi_q(e_{ij})$ $\forall q \in \mathbb{N}$.

Put for $\gamma \in \mathbb{C}$, $\gamma^n = 1$,

$$u_\gamma = \sum_{j=1}^{n} \gamma^j e_{jj}^1, \; v_\gamma = e_{n,1}^1 \theta(u_\gamma^*) + \sum_{j=1}^{n-1} e_{j,j+1}^1.$$

Then one shows that the infinite product $\operatorname{Ad} v_\gamma \operatorname{Ad}\theta(v_\gamma) \ldots \operatorname{Ad}\theta^m(v_\gamma)$ makes sense in the topological group‡ $\operatorname{Aut} R$ and defines an automorphism s_n^{γ} of R such that $(s_n^{\gamma})^n = \operatorname{Ad} u_\gamma$, $s_n^{\gamma}(u_\gamma) = \gamma u_\gamma$. It follows from the above classification of periodic automorphisms that the cross product of R by any cyclic finite group is hyperfinite, because each s_n^{γ} has an increasing sequence of globally invariant finite dimensional subalgebras $F_m = (K_1 \cup \ldots \cup K_m \cup \theta^m\{(u_\gamma)\})''$, with union strongly dense in R.

† Such an α is called minimal.

‡ With strong pointwise convergence.

Coming back to the anti-isomorphism problem, we see that the automorphism $(s_n{}^{\gamma})^0$ opposite to $s_n{}^{\gamma}$† is conjugate to $s_n{}^{\bar{\gamma}}$, so that if $\gamma \neq \bar{\gamma}$, $s_n{}^{\gamma}$ is not even outer conjugate to its opposite. The simplest case is with $n = 3$, $\gamma = \exp i2\pi/3$ so that the corresponding automorphism s has period 9.

The idea of the construction of the factor Q not anti-isomorphic to itself is to build a factor of type II_∞, N_1 and an automorphism θ_1 of N_1 multiplying the trace τ_1 by some $\lambda \in]0, 1[$ such that θ_1 is outer conjugate to its opposite $\theta_1{}^0$ but that, on $R \otimes N_1$, $s \otimes \theta_1$ is no longer outer conjugate to its opposite $s^0 \otimes \theta_1{}^0$ on $R^0 \otimes N_1{}^0$.

The difficulty is to be able to deduce the outer conjugacy of $s \otimes 1$ with $s^0 \otimes 1$ as automorphisms of $R \otimes N_1$, $R^0 \otimes N_1{}^0$ from the outer conjugacy of $s \otimes \theta_1$ with $s^0 \otimes \theta_1{}^0$.

We let N_1 and θ_1 correspond to the discrete decomposition [**1**; 4.4.1] of the Pukanszky's factor P_λ of type III_λ. It follows from the choice of N_1 that any bounded sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $R \otimes N_1 = N$, such that $\|x_n \psi - \psi x_n\| \to 0$, $\forall \psi \in N_*$ is equivalent‡ to a sequence $(y_n)_{n \in \mathbb{N}}$ of elements of $R \otimes 1$ [**2**]. One then shows that the normal subgroup $\overline{\mathrm{Int}\, N}$ of $\mathrm{Aut}\, N$ which is the closure of $\mathrm{Int}\, N$ in the topology of norm pointwise convergence in N_* (the pre-dual of N), consists exactly of the automorphisms $\mathrm{Ad}\, X(\alpha \otimes 1)$, X unitary in N, $\alpha \in \mathrm{Aut}\, R$. The group $Ct(N) = \{\alpha \in \mathrm{Aut}\, N,$ $\alpha(x_n) - x_n \to 0 *$ strongly for any bounded sequence $(x_n)_{n \in \mathbb{N}}$, $\|[x_n, \psi]\| \to 0$, $\forall \psi \in N_*\}$ is also a normal subgroup of $\mathrm{Aut}\, N$, and it contains $\mathrm{Int}\, N$. Now

$$Ct(N) \cap \overline{\mathrm{Int}\, N} = \mathrm{Int}\, N$$

because if the automorphism $\alpha \in \mathrm{Aut}\, R$ is outer, one can find a central sequence $(u_n)_{n \in \mathbb{N}}$ of unitaries in R such that $\|\alpha(u_n) - u_n\|_2 \geqslant \frac{1}{2}\ \forall n \in \mathbb{N}$, where $\|\ \|_2$ is the L^2 norm in R, so that $\alpha \otimes 1$ does not transform the sequence $(u_n \otimes 1)_{n \in \mathbb{N}}$ in an equivalent sequence.

As the decomposition $s \otimes \theta_1 = (s \otimes 1)(1 \otimes \theta_1)$ satisfies

$$s \otimes 1 \in \overline{\mathrm{Int}\, N},\ 1 \otimes \theta_1 \in CtN$$

it follows that it determines uniquely the outer conjugacy class in $\mathrm{Aut}\, N$ of $s \otimes 1$ and of $1 \otimes \theta_1$. In particular $p_0(s \otimes 1)$ and $\gamma(s \otimes 1)$ are invariants of the outer conjugacy class of $s \otimes \theta_1$ in $\mathrm{Out}\, N$.

But $(s \otimes \theta_1)^0$ is outer conjugate to $s^0 \otimes \theta_1$ and hence, as

$$\gamma(s^0 \otimes 1) = \exp(-i2\pi/3) \neq \exp(i2\pi/3) = \gamma(s \otimes 1),$$

it is not outer conjugate to $s \otimes \theta_1$.§ It follows that the cross product $W^*(s \otimes \theta_1, N)$ is not isomorphic to $W^*((s \otimes \theta_1)^0, N^0)$, using [**1**; 4.4.1], so that $Q = W^*(s \otimes \theta_1, N)$ is a factor of type III_λ which is not anti-isomorphic to itself. One can check that the factor Q thus obtained contains a maximal abelian subalgebra $\mathscr{A}$ which is regular

† Change the product of R into $(x, y) \to yx$; this gives R^0 and $(s_n{}^{\gamma})^0 = s_n{}^{\gamma}/R^0$.

‡ i.e. $x_n - y_n \to 0$ * strongly.

§ We are using the isomorphism of $R \otimes N_1$ with $R^0 \otimes N_1{}^0$.

(the normaliser $N(\mathscr{A}) = \{u$ unitary in Q, $u\mathscr{A}u^* = \mathscr{A}\}$ generates Q), and which is the range of a unique normal conditional expectation E, hence Q is obtained by the generalised group measure space construction using 2-cocycles as described in [3].

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