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Structure Theory for Type III Factors

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Motivated by the study of the regular representation of nonunimodular locally compact groups J. Dixmier, in 1952, introduced quasi-Hilbert algebras. M. Tomita proved in 1967 that any von Neumann algebra M arises from a modular Hilbert algebra.[1] As shown then by F. Combes and M. Takesaki, each weight[2] φ on M gives rise to a modular Hilbert algebra giving back M, and in particular to a positive operator Δ_φ (the modular operator) and a one-parameter group of automorphisms of M: σ^φ (the modular automorphism group). See [7].

The formulas

(A) $$r_\infty(M) = \bigcap \mathrm{Sp}\, \Delta_\varphi,^3$$

(B) $$\rho(M) = \{e^{-2\pi/T}, \exists\, \varphi \text{ with } \sigma^\varphi_T = 1\}$$

relating the above objects to the Powers, Araki, Woods, Krieger classification [1] if M is an infinite tensor product of factors of type I drove us to study the two invariants:

$$S(M) = \bigcap \mathrm{Sp}\, \Delta_\varphi, \qquad T(M) = \{T \in \boldsymbol{R}, \exists\, \varphi, \sigma^\varphi_T = 1\}$$

for arbitrary type III factors.

It was essential, in this respect, to determine how the modular automorphism group σ^φ depends on the choice of φ. The answer [2] is summarised in:

(1°) For any weight φ on the von Neumann algebra M there exists a unique uni-

[1]Hence a quasi-Hilbert algebra.

[2]We mean a faithful semifinite normal weight.

[3]V. Ya. Golodets reached a formula very close to (A); see [2] for bibliography.

tary cocycle[4] $t \to \mu_t$ (μ denotes $(D\psi{:}D\varphi)$ because it is a "Radon-Nikodym" derivative) such that:

$$\sigma_t^\psi(x) = \mu_t\, \sigma_t^\varphi(x)\mu_t^* \qquad \forall\, x \in M, \forall\, t \in \boldsymbol{R}$$

and

$$\psi(x) = \varphi(x\mu_{-i}) \qquad \forall\, x \in M_+.^5$$

(2°) For any unitary cocycle $t \to \mu_t$ there exists a unique weight ψ on M such that $(D\psi{:}D\varphi)_t = \mu_t\ \forall\, t \in \boldsymbol{R}$.

Hence there exists an abstract kernel δ, homomorphism from $\boldsymbol{R}$ to the center of Out M = Aut M/Int M which characterises the automorphism groups σ^φ by the commutativity of the diagram:

$$\begin{array}{ccc} & & \text{Aut } M \\ & \overset{\sigma^\varphi}{\nearrow} & \downarrow \\ \boldsymbol{R} & & \\ & \underset{\delta}{\searrow} & \\ & & \text{Out } M \end{array}$$

Moreover $T(M)$ = Ker δ and $S(M) \cap \boldsymbol{R}_+^*$ is the spectrum (in the sense of [2]) of δ provided M is a factor. It follows that both are groups,[6] that when $S(M) \neq \{0, 1\}$ $T(M)$ is the orthogonal of $S(M) \cap \boldsymbol{R}_+^*$, while when $S(M) = \{0, 1\}$, it can be any denumerable subgroup of $\boldsymbol{R}$.

Also T and S become easy to compute; for instance, when M is the factor arising from an ergodic action of a group, they are related by formulas (A), (B) to the invariants r and ρ introduced in ergodic theory by W. Krieger.

In particular this showed that property L_λ of Power is not equivalent to L'_λ of Araki [1], so that in general S and r_∞ are different invariants. But let us enter in the details of the classification: Any factor of type III belongs to one of the following three classes:

III_λ $\lambda \in \,]0, 1[$ meaning $S(M) = \{\lambda^n, n \in \boldsymbol{Z}\}^-$,

III_0 i.e., $S(M) = \{0, 1\}$, and

III_1 i.e., $S(M) = [0, +\infty[$.

For factors M of type III_λ, $\lambda \in \,]0, 1[$, we have:

There exist maximal subalgebras[7] of type II_∞ of M. Let N be a maximal II_∞ subalgebra of M; then it is a factor and M is generated by N and a unitary U in the normaliser of N such that $\tau(UxU^*) = \lambda\tau(x)$, $\forall\, x \in N_+$, $\forall\, \tau$ normal trace on N. Let $M = N_1(U_1) = N_2(U_2)$ be two decompositions of M as above; then there is an inner automorphism ψ of M such that $\psi(N_1) = N_2, \psi(U_1) = U_2$. This decomposition allows us to translate most of the problems on M in terms of N which is a simpler object. For instance any normal state φ on M is unitarily equivalent to a state $\psi \circ E$

[4]For each $t \in \boldsymbol{R}$, μ_t is a unitary in M, the map $t \to \mu_t$ is continuous and satisfies $\mu_{t_1+t_2} = \mu_{t_1}\sigma^\varphi_{t_1}(\mu_{t_2})$ $\forall\, t_1, t_2 \in \boldsymbol{R}$.

[5]For a precise statement see [2]

[6]For S this property was proven in collaboration with van Daele.

[7]Here by "subalgebra" we mean a von Neumann subalgebra range of a normal conditional expection of M.

where ϕ is a normal state on N and E is the unique expectation of M onto N.

The cross product $N(\theta)$ of a factor N of type II_∞ by an automorphism θ multiplying traces by λ is a factor of type III_λ, and any factor of type III_λ is obtained this way, with $N_1(\theta_1)$ isomorphic to $N_2(\theta_2)$, if and only if there exists[8] an isomorphism π of N_1 on N_2 such that $\pi\theta_1\pi^{-1} = \theta_2$. There are factors N of type II_∞ and automorphisms θ_1, θ_2 of N multiplying the traces by the same number $\lambda \in]0, 1[$ though they do not belong to the same conjugacy class in Aut N.

Factors of type III_0 appear as a limiting case of the type II_∞. Let M be of type III_0; then it is the cross product of a von Neumann algebra of type II_∞ by the infinite coproduct of groups of two elements. Moreover any subalgebra[9] of type II_∞ of M is the first element of an increasing sequence N_j of von Neumann subalgebras of type II_∞ with $\overline{\cup N_j} = M$. Let N be a von Neumann algebra of type II_∞, and $\theta \in \mathrm{Aut}\ N$ be a strict contraction with respect to some trace (see [2]) and strictly ergodic on the center C of N. Then the cross product $N(\theta)$ is a factor of type III_0. Any factor of type III_0 arises this way and $N_1(\theta_1)$ is isomorphic to $N_2(\theta_2)$ iff there are nonzero projections $e_j \in C_j$ such that the automorphisms θ_j, e_j induced by θ_j on e_j in the sense of Kakutani are the same.

Using this and the previous work of W. Krieger on automorphisms which are not of infinite product type, one gets a hyperfinite factor which is not an infinite tensor product of type I factors [2]. Now starting from a discrete decomposition $M = N(\theta)$ as above of a factor of type III_0 and building the flow on θ under the function $d\tau/d\tau\circ\theta$,[10] one obtains a one-parameter group $(\alpha_\lambda)_{\lambda\in \boldsymbol{R}_+^*}$ of automorphisms of a von Neumann algebra p of type II_∞ yielding the decomposition of M as a continuous cross product given by M. Takesaki [8] which this time is unique.

His duality technique allowed him to prove the following final results [8]:

Factors of type III_1. Let N be a factor of type II_∞, $(\theta_t)_{t\in\boldsymbol{R}}$ a one-parameter group of automorphisms of N, with $\tau\circ\theta_t = e^{-t}\tau$ for any normal trace τ; then the continuous cross product of N by $(\theta_t)_{t\in\boldsymbol{R}}$ is a factor of type III_1. Any factor of type III_1 arises this way and the decomposition is unique (as for factors of type III_λ).

However the appearance of continuous cross products complicates the study of M. For instance, though on factors of type III_λ, $\lambda \neq 1$, any normal state has a centraliser containing a maximal abelian subalgebra of M, it fails for factors of type III_1. Also, using the closure of the range of the modular homomorphism δ in Aut $M/\overline{\mathrm{Int}\ M}$ as a Galois group of M, one can exhibit factors of type III_1 which admit no decompositions as cross products of semifinite von Neumann algebras by discrete abelian groups [4].

Insofar as factors of type III_λ, $\lambda \in]0, 1[$, are very simple to analyse (for instance they exhibit strongly their nonnormalcy: the relative commutant of a maximal II_∞ subalgebra is reduced to the scalars), it is often helpful, when trying to prove a general property of type III factors to begin by the case III_λ, $\lambda \in]0, 1[$, and then

[8]The theory of factors of type III_λ as presented in [2] was complete in the spring of 1972; the improvement on [2] on uniqueness was obtained in collaboration with M. Takesaki [5].

[9]Here by "subalgebra" we mean a von Neumann subalgebra range of a normal conditional expectation of M.

[10]τ is a trace contracted by θ; for details see [2].

consider the case III_1 as a limit when $\lambda \to 1$. The final proof is independent of the classification (cf. [2] for the nonnormalcy of the III's).

One of the important effects of Tomita's theory is to yield the right generalisations to type III of notions which existed only for semifinite algebras. For instance, the only Hilbert-Schmidt operator on a factor of type III being 0, it seemed difficult to have an interesting generalisation of the cone of positive Hilbert-Schmidt operators. Using the modular operator, one can construct pre-Hilbert space structures s on M for which the completion of M_+ is a self-dual cone; the cone obtained is then independent of s,[11] and reduces to the Hilbert-Schmidt cone when M is semifinite.[12]

Moreover the class of cones thus obtained is exactly the class of convex self-dual cones V in Hilbert space H satisfying the two following conditions: V is *complex*, i.e., the quotient by its center of the Lie algebra $D(V) = \{\delta,\ \delta \in \mathscr{L}(H),\ e^{t\delta}V = V,\ \forall\, t\}$ is a *complex* Lie algebra for some complex structure I. V is *facially homogeneous*, i.e., for each face F of V the operator $P_F - P_{F^\perp}$ belongs to $D(V)$ where P_F means the orthogonal projection on the linear span of F and $F^\perp$ the face orthogonal to F.

There is a striking analogy between the relations type II and type III and the relations between unimodular and nonunimodular locally compact groups. Compare for instance what is said above for the case III_λ, $\lambda \in]0, 1[$, with the elementary description of a locally compact group G whose module Δ_G has range $\{\lambda^n, n \in \boldsymbol{Z}\}$, as a cross product of a unimodular group (the kernel of Δ_G) by a single automorphism.

If one enlarges the notion of locally compact group to include Mackey's virtual groups, one has still a notion of module Δ of G and of left regular representation of G; it generates a von Neumann algebra $U(G)$. Then the closure of the range of the module is a virtual subgroup of $\boldsymbol{R}_+^*$ and the following is true when G is a *principal* virtual group:[13]

If $\overline{\Delta(G)}$ is a usual subgroup of $\boldsymbol{R}_+^*$, it coincides with $S(U(G))$ (and in particular with $r(G)$, the ratio set of Krieger); otherwise $U(G)$ is of type III_0, and the strictly ergodic action of $\boldsymbol{R}_+^*$ corresponding to $\overline{\Delta(G)}$ is nothing but the flow described above for factors of type III_0.

In [5] we study the virtual modular spectrum $S_V(M)$ for arbitrary factors and show, for instance, by the formula $S_V(M_1 \otimes M_2) = \overline{S_V(M_1) \cdot S_V(M_2)}$ how much it behaves like the closure of the range of a "module" of M.

We cannot end without quoting the beautiful result of W. Krieger on weak equivalence [6]. With the above terminology it shows that the virtual modular spectrum S_V is a complete invariant for the class of factors which arise from ergodic transformations,[14] and can assume as value any virtual subgroup of $\boldsymbol{R}_+^*$.

[11] For a precise statement see [3].

[12] This generalisation was obtained independently by S. L. Woronowicz, H. Araki and myself. See [3] for bibliography.

[13] $U(G)$ is then a factor.

[14] By the group measure space construction.

This shows how important it is to decide whether any hyperfinite factor arises from an ergodic transformation and in particular whether the hyperfinite factor of type II_∞ is unique.

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