ENTROPY FOR AUTOMORPHISMS OF II_1 VON NEUMANN ALGEBRAS

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In this paper we study the following problem: prove that for $n \neq m$ the n-shift of the hyperfinite II_1 factor is not conjugate to the m-shift. For each n the n-shift is the automorphism corresponding to the translation of 1 in the infinite tensor product $R_1 = \otimes_{\nu \in \mathbf{Z}}(N, \tau)_\nu$ where N is the algebra of $n \times n$ matrices and τ the normalised trace on N. As the above R_1 does not depend on n, up to isomorphism, because it is the unique hyperfinite factor of type II_1, the above problem makes sense.

There is an obvious analogy between this problem and the problem of conjugacy of Bernoulli shifts in ergodic theory. In fact both are special cases of the conjugacy problem for automorphisms corresponding to the translation of 1 in $\otimes_{\nu \in \mathbf{Z}}(N, \tau)_\nu$ where τ is a normalised trace on the finite dimensional von Neumann algebra N.

So it is natural to try and extend the notion of entropy, from ergodic theory to the non commutative frame. It is known since a long time that given a von Neumann algebra M with semi-finite faithful normal trace τ, and considering a state ϕ on M, with $\phi = \tau(\varrho\,\cdot)$, the quantity $S(\phi) = \tau(\eta(\varrho))$ where $\eta(x) = -x \log x$, measures the amount of randomness in ϕ for the purposes of quantum mechanics. Also Wigner and Yanase defined the purely non commutative notion of skew information between ϱ and a selfadjoint observable K:

$$I_{\frac{1}{2}}(\varrho, K) = \tfrac{1}{2}\tau([\varrho^{1/2}, K]^2).$$

Dyson extended this definition and introduced, for $0 < p < 1$,

$$I_p(\varrho, K) = \tfrac{1}{2}\tau([\varrho^p, K][\varrho^{1-p}, K]) = \tau(\varrho^{1-p}K\varrho^p K) - \tau(\varrho K^2)$$

They conjectured that I_p was a concave function of ϱ for each fixed K. This conjecture was proven by E. Lieb with K not necessarily selfadjoint and $I_p = \tau(\varrho^{1-p}K^*\varrho^p K) - \tau(\varrho K^* K)$.

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In fact, it follows easily from Lieb's theorem, which was shown for type I factors, but whose proof also goes through for finite von Neumann algebras, that

$$I(\varrho, K) = \lim_{p\to 0} \frac{1}{p} I_p(\varrho, K) = \tau(\varrho K^*[\log \varrho, K])$$

is a concave function of ϱ for each K [5].

G. Lindblad defines in [7] the relative entropy of a state ϱ_1 given the state ϱ_2 by the formula

$$S(\varrho_1 | \varrho_2) = \tau(\varrho_1 (\log \varrho_1 - \log \varrho_2)).$$

He proved [7, Theorem 1], [8] that S is a jointly convex function of ϱ_1, ϱ_2, a fact which also follows from the equality

$$S(\varrho_1 | \varrho_2) = -I\left(\begin{bmatrix} \varrho_1 & 0 \\ 0 & \varrho_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right).$$

Because of such progress in non commutative information theory it was reasonable to expect one can get a correct definition of the entropy of automorphisms in the non abelian frame.

The main difficulty comes from the fact that two finite dimensional subalgebras of a non abelian von Neumann algebra can fail to generate a finite dimensional algebra, indeed abelian ones will do, see e.g. [1, Theorem 3(ii)]. Thus there is no immediate analogue for the operation $P \vee Q$ between partitions of abelian von Neumann algebras. So we looked for a replacement, not of the above $\vee$, but of the quantity $H(P_1 \vee P_2 \dots \vee P_n)$.

We construct for each n a function $H(P_1, \dots, P_n)$, where the P_j are arbitrary finite dimensional von Neumann subalgebras of a given von Neumann algebra R with finite trace τ, and H is symmetric in its arguments.

This function will satisfy the following requirements:

(A) $H(N_1, \dots, N_k) \leq H(P_1, \dots, P_k)$ when $N_j \subset P_j \quad j = 1, \dots, k$

(B) $H(N_1, \dots, N_k, N_{k+1}, \dots, N_p) \leq H(N_1, \dots, N_k) + H(N_{k+1}, \dots, N_p)$

(C) $P_1, \dots, P_n \subset P \Rightarrow H(P_1, \dots, P_n, P_{n+1}, \dots, P_m) \leq H(P, P_{n+1}, \dots, P_m)$

(D) For any family of minimal projections of N, $(e_\alpha)_{\alpha \in I}$ such that $\Sigma e_\alpha = 1$ one has $H(N) = \Sigma_{\alpha \in I}\, \eta\tau(e_\alpha)$

(E) If $(N_1 \cup \dots \cup N_k)''$ is generated by pairwise commuting von Neumann subalgebras P_j of N_j then

$$H(N_1, \dots, N_k) = H((N_1 \cup \dots \cup N_k)'')$$

The other properties of H connect it with a relative entropy function $H(N | P)$ analogous to the relative entropy of ergodic theory:

(F) $H(N_1, ..., N_k) \leqslant H(P_1, ..., P_k) + \Sigma_j H(N_j|P_j)$

(G) $H(N|Q) \leqslant H(N|P) + H(P|Q)$

(H) $H(N|P)$ is increasing in N and decreasing in P

(I) Let N_1, N_2 commute, then $H(N_2|N_1) = H((N_1 \cup N_2)''|N_1) = H((N_1 \cup N_2)'') - H(N_1)$.

Moreover we shall prove that H is strongly continuous, more precisely given n and $\varepsilon > 0$, we find $\delta > 0$ with

$$(\dim N = n, N \overset{\delta}{\subset} P) \Rightarrow H(N|P) < \varepsilon.$$

Equipped with such a function H it is then easy to define for an arbitrary τ-preserving automorphism θ of the couple (R, τ) the quantities:

$$H(N, \theta) = \lim_{k \to \infty} \frac{1}{k} H(N, \theta(N), \theta^2(N), \dots, \theta^k(N))$$

$$H(\theta) = \sup_{\substack{N \text{ finite} \\ \text{dimensional}}} H(N, \theta).$$

Moreover the Kolmogoroff-Sinai theorem is true, provided R is hyperfinite (it cannot even be stated otherwise) and takes the following form:

THEOREM 1. *Let N_k be an increasing sequence of finite dimensional von Neumann subalgebras of R with $\bigcup N_k$ weakly dense in R, then*

$$H(\theta) = \sup_{k \in \mathbf{N}} H(N_k, \theta) \quad \text{for any } \tau\text{-preserving } \theta.$$

Using property D and property E one gets

THEOREM 2. *Let S_n be the n-shift of the hyperfine II_1 factor then* $H(S_n) = \log n$.

This proves in particular that those shifts are non conjugate for different n's. We shall in Theorem 4 generalize Theorem 2 to Bernoulli shifts θ of R defined by positive numbers $\lambda_1, ..., \lambda_n$ with sum 1. For such a θ, $H(\theta) = -\Sigma\lambda_j \log \lambda_j$.

1. Preliminaries

Here we fix our notation and remind the reader of some known facts of non commutative information theory, then we prove an important inequality ((8)) to be used later. Throughout R will be a finite von Neumann algebra with faithful normal finite trace τ. When N is a von Neumann subalgebra of R we let E_N by the unique faithful normal conditional expectation of R onto N which is τ-preserving.

The letter η designates the function $x \in [0, +\infty[\to -x \log x$. We recall that

$$\log \text{ is an operator increasing function on } [0, +\infty[, \text{ [11] Prop. 2.5.8} \tag{1}$$

$$\eta \text{ is an operator concave function on } [0, +\infty[, \text{ [9].} \tag{2}$$

$$\eta(xy) = \eta(x)y + x\eta(y) \text{ for commuting operators } x, y. \tag{3}$$

We put, for $x, y \in R_+$, $x \leqslant \lambda y$ for some $\lambda > 0$,

$$S(x|y) = \tau(x (\log x - \log y)),$$

and it follows, as shown in the introduction, that

$$S(x|y) \text{ is a jointly convex function of } (x, y). \tag{4}$$

It implies in particular the well known Peyerls Bogoliubov's inequality

$$\tau(x)\,(\log \tau(x) - \log \tau(y)) \leqslant \tau(x(\log x - \log y)). \tag{5}$$

Note also that

$$S(\lambda x|\lambda y) = \lambda S(x|y), \quad \lambda > 0. \tag{6}$$

Using (4) and (6) the following inequality is then easy to show:

$$y_i, x_i \in R_+,\ y_i \leqslant x_i,\ \sum_{i=1}^{n} x_i = 1 \quad \text{implies} \quad \sum_{i=1}^{n} \tau(y_i(\log x_i - \log y_i)) \leqslant \tau\eta\Big(\sum_{i=1}^{n} y_i\Big). \tag{7}$$

(Because $\sum_{i=1}^{n} S(y_i|x_i) \geqslant S(\sum y_i|\sum x_i) = -\tau\eta(\sum_{i=1}^{n} y_i)$).

We now derive from (7) an inequality which will play a fundamental role in the sequel, and is used in the proof of property C.

Let I, J be two finite sets, $(x_{ij})_{i\in I,\, j\in J}$ be a family of elements of R_+ such that $\sum_{I\times J} x_{ij} = 1$ then

$$\sum_{i,j} \tau\eta(x_{ij}) \leqslant \sum_{i} \tau\eta(x_i^1) + \sum_{j} \tau\eta(x_j^2) \tag{8}$$

where $x_i^1 = \sum_j x_{ij}$, $x_j^2 = \sum_i x_{ij}$.

To prove (8), we just have to show that

$$\sum_{i,j} \tau\eta(x_{ij}) - \sum_{j} \tau\eta(x_j^2) = \sum_{i,j} \tau(x_{ij}(\log x_j^2 - \log x_{ij}))$$

and then apply (7). But the above equality follows from

$$\sum_{i} \tau(x_{ij} \log x_j^2) = -\tau\eta(x_j^2) \quad \text{for any} \quad j \in J.$$

2. Definition of H and proof of properties (A) to (I) of the joint entropy

Let R and τ be as in section 1. For each $k \in \mathbf{N}$, we let S_k be the set of all families $(x_{i_1, i_2, \ldots, i_k})_{i_j \in \mathbf{N}}$ of positive elements of R, zero except for a finite number of indices, and satisfying:

$$\sum_{i_1, \ldots, i_j, \ldots, i_k} x_{i_1, i_2, \ldots, i_k} = 1.$$

For $x \in S_k$, $l \in \{1, \ldots, k\}$, and $i_l \in \mathbf{N}$ we put

$$x^l_{i_l} = \sum_{i_1, \ldots i_{l-1}, i_{l+1}, \ldots, i_k} x_{i_1, i_2, \ldots, i_k}.$$

Definition 1. Let $N_1, \ldots, N_k$ be finite dimensional von Neumann subalgebras of R then:

$$H(N_1, \ldots, N_k) = \operatorname*{Sup}_{x \in S_k} \sum_{i_1, \ldots, i_k} \eta\tau(x_{i_1, \ldots, i_k}) - \sum_l \sum_{i_l} \tau\eta E_{N_l}(x^l_{i_l}).$$

It is clear that H is positive and symmetric, but it is not clear that it has a finite value. This will in fact follow from properties (B), (D):

Property (A). $H(N_1, \ldots, N_k) \leqslant H(P_1, \ldots, P_k)$ when $N_j \subset P_j \forall j$.

Proof. For any $y \in R_+$, and $j \in \{1, .., k\}$ we have by Jensen's inequality, which follows from (2) and [2],

$$\eta(E_{N_j}(y)) = \eta(E_{N_j} E_{P_j}(y)) \geqslant E_{N_j} \eta(E_{P_j}(y)).$$

Hence $\tau\eta E_{N_j}(y) \geqslant \tau\eta E_{P_j}(y)$, which makes the proof obvious.

Property (B). $H(N_1, \ldots, N_k, N_{k+1}, \ldots, N_p) \leqslant H(N_1, \ldots, N_k) + H(N_{k+1}, \ldots, N_p)$

Proof. Let $x \in S_p$, and define $x' \in S_k$, $x'' \in S_{p-k}$ by

$$x'_{i_1, \ldots, i_k} = \sum_{i_{k+1}, \ldots, i_p} x_{i_1, \ldots, i_p}, \quad x''_{j_1, \ldots, j_{p-k}} = \sum_{i_1, \ldots, i_k} x_{i_1, \ldots, i_k, j_1, \ldots, j_{p-k}}.$$

We have $x'^l_{i_l} = x^l_{i_l}$, $l \in \{1, \ldots, k\}$, $x''^l_{j_l} = x^{l+k}_{j_l}$, $l \in \{1, \ldots, p-k\}$.

We hence just have to prove that

$$\sum_{i_1, \ldots, i_p} \eta\tau(x_{i_1, \ldots i_p}) \leqslant \sum_{i_1, \ldots, i_k} \eta\tau(x'_{i_1, \ldots, i_k}) + \sum_{j_1, \ldots, j_{p-k}} \eta\tau(x''_{j_1, \ldots, j_{p-k}}).$$

To do this let m be the Lebesgue measure on [0, 1], $\mathcal{P}$ be a partition of [0, 1] whose generic atom $a_{i_1, \ldots, i_p}$ has measure $\tau(x_{i_1, \ldots, i_p})$. Let $\mathcal{P}'$ (resp. $\mathcal{P}''$) be the partition whose atoms are union of atoms of $\mathcal{P}$ with fixed k first indices (resp. $p-k$ last), then clearly the above inequality follows from $h(\mathcal{P}' \vee \mathcal{P}'') \leqslant h(\mathcal{P}') + h(\mathcal{P}'')$ where h is the classical entropy of ergodic theory.

Property (C). $P_1, \ldots, P_n \subset P \Rightarrow H(P_1, \ldots, P_n, P_{n+1}, \ldots, P_m) \leqslant H(P, P_{n+1}, \ldots, P_m)$

Proof. Using property A we can assume that $P_j = P$, $j = 1, \ldots, n$. We let $x \in S_m$, then we build an $X \in S_{m-n+1}$ such that, with $E_l = E_{P_l}$, $l \in \{1, \ldots, m\}$, we have

$$\sum \eta\tau(x_{i_1,\ldots,i_m}) - \sum_l \sum_{i_l} \tau\eta E_l(x^l_{i_l}) \leqslant \sum \eta\tau(X_{j_1,\ldots,j_{m-n+1}}) - \sum_l \sum_{j_l} \tau\eta(E_{l+n-1}(X^l_{j_l})). \tag{9}$$

To get X, let ϕ be a bijection of $\mathbf{N}$ onto $\mathbf{N}^n$ and put:

$$X_{j_1,\ldots,j_{m-n+1}} = x_{\phi(j_1)_1,\ldots,\phi(j_1)_n,j_2,\ldots,j_{m-n+1}}.$$

Clearly, for $l > 1$, $X^l_{j_l} = x^{n+l-1}_{j_l}$ and also:

$$X^1_j = \sum_{j_2,\ldots,j_{m-n+1}} x_{\phi(j)_1,\ldots,\phi(j)_n,j_2,\ldots,j_{m-n+1}} = x'_{\phi(j)_1,\ldots,\phi(j)_n}$$

where

$$x'_{i_1,\ldots,i_n} = \sum_{i_{n+1},\ldots,i_m} x_{i_1,\ldots,i_m}.$$

So that the inequality (9) is equivalent to

$$\sum_{1 \leqslant l \leqslant n} \sum_{i_l} \tau\eta E_l(x^l_{i_l}) \geqslant \sum_j \tau\eta(E_n x'_{\phi(j)_1,\ldots,\phi(j)_n}). \tag{10}$$

But $E_l = E_n$ for $l \in \{1, \ldots, n\}$ and with $y_{i_1,\ldots,i_n} = E_n(x'_{i_1,\ldots,i_n})$ we get

$$\sum_{i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_n} y_{i_1,\ldots,i_n} = E_n(x^l_{i_l}).$$

Hence the inequality (10) follows from (8).

Property (D). Let $(e_\alpha)_{\alpha \in I}$ be a family of minimal projections of N such that $\Sigma_{\alpha \in I}\, e_\alpha = 1$. Then $H(N) = \Sigma_{\alpha \in I}\, \eta\tau(e_\alpha)$.

Proof. As $\eta(e_\alpha) = 0$ we have

$$\sum_I \eta\tau(e_\alpha) - \sum_I \tau\eta(E_N(e_\alpha)) = \sum_I \eta\tau(e_\alpha).$$

Hence to prove property D we just have to check the following

$$\text{for } x \in S_1 \quad \text{one has} \quad \sum_{i \in N} \eta\tau(x_i) - \tau\eta(E_N(x_i)) \leqslant \sum_{\alpha \in I} \eta\tau(e_\alpha). \tag{11}$$

As $\tau(E_N(x_i)) = \tau(x_i)$, one can assume to prove (11), that all the x_i's lie in N.

We have for each $a \in N^+$, that $\eta\tau(a) - \tau\eta(a) = \tau(a\,(\log a - \log \tau(a))) = S(a\,|\,\tau(a))$. Hence by (9) and (6) one has

$$a, b \in N^+,\ \eta\tau(a+b) - \tau\eta(a+b) \leqslant (\eta\tau(a) - \tau\eta(a)) + (\eta\tau(b) - \tau\eta(b)). \tag{12}$$

In fact (12) is also an easy consequence of the Peierls Bogoluibov inequality (5).

Using (12) and the spectral decomposition of each x_i, we see that to prove (11) we can assume that each x_i is the product of a positive scalar λ_i by a minimal projection f_i of N.

Let c be a minimal projection in the center of N. For $i \in N$ we have $x_i c = x_i$ or $x_i c = 0$, and for $i \in I$, $e_i c = 0$ or $e_i c = e_i$. Let $A_c = \{i \in \mathrm{N},\ x_i c \neq 0\}$, $J_c = \{\alpha \in I,\ e_\alpha c \neq 0\}$. To prove (11) it is clearly enough to show that:

$$\sum_{i \in A_c} \eta\tau(x_i) - \tau\eta(x_i) = \sum_{\alpha \in J_c} \eta\tau(e_\alpha).$$

Now $x_i = \lambda_i f_i$ so that by (3), $\eta(x_i) = \eta(\lambda_i) f_i$ and $\eta\tau(x_i) - \tau(\eta(\lambda_i) f_i) = \lambda_i \eta\tau(f_i)$. As the f_i, $i \in A_c$, are all equivalent,

$$\sum_{i \in A_c} \eta\tau(x_i) - \tau\eta(x_i) = -\Big(\sum_{i \in A_c} \lambda_i \tau(f_i)\Big) \log \tau(f) = -\tau(c) \log \tau(f)$$

where f is an arbitrary minimal projection of N such that $f \leqslant c$. The same computation shows that $\sum_{\alpha \in J_c} \eta(\tau(e_\alpha)) = -\tau(c) \log \tau(f)$ and proves equality (13).

Property (E). If $(N_1 \cup \ldots \cup N_k)''$ is generated by pairwise commuting subalgebras P_j of N_j then $H(N_1, \ldots, N_k) = H((N_1 \cup N_2 \ldots \cup N_k)'')$.

Proof. First assume that the N_j are abelian and commute pairwise. It follows from properties A and C that $H(N_1, \ldots, N_k)$ is smaller than $H((N_1 \cup \ldots \cup N_k)'')$. Let $(e_i^j)_{j \in N}$ be for each $i \in \{1, \ldots, k\}$ the list of minimal projections of N_i, then let $x \in S_k$ be such that $x_{i_1, \ldots, i_k} = e_1^{i_1} e_2^{i_2} \ldots e_k^{i_k}$. By definition 1 we have

$$\sum_{i_1, \ldots i_k} \eta\tau(e_1^{i_1} e_2^{i_2} \ldots e_k^{i_k}) \leqslant H(N_1, \ldots, N_k).$$

But the first term is equal to $H((N_1 \cup \ldots \cup N_k)'')$ by property D. Now we no longer assume the N_j abelian nor commuting. We have by C, $H(N_1, \ldots, N_k) \leqslant H((N_1 \cup \ldots \cup N_k)'')$. To prove that $H(N_1, \ldots, N_k) \geqslant H((N_1 \cup \ldots \cup N_k)'')$ we can replace each N_j by the corresponding P_j, i.e. assume that the N_j are pairwise commuting.

Now let $\mathcal{A}_j \subset N_j$ be maximal abelian in N_j for each j, we have

$$H(N_1, \ldots, N_k) \geqslant H(\mathcal{A}_1, \ldots, \mathcal{A}_k) = H((\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)'').$$

Hence we just have to check that $(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k)''$ is maximal abelian in $(N_1 \cup \ldots \cup N_k)''$. To do it check that any product of minimal projections e_i of N_i, say $e = e_1 \ldots e_k$ will be a minimal projection of $(N_1 \cup \ldots \cup N_k)''$.

Property (F). $H(N_1, N_2, \ldots, N_k) \leqslant H(P_1, \ldots, P_k) + \sum_j H(N_j | P_j)$

where
$$H(N|P) = \sup_{x \in S_1} \sum (\tau\eta E_P(x_i) - \tau\eta E_N(x_i)).$$

Proof. Immediate from definition 1.

Property (G). $H(N|Q) \leqslant H(N|P) + H(P|Q)$.

Proof. Let $x \in S_1$ then:

$$\sum_i (\tau\eta(E_Q(x_i)) - \tau\eta(E_N(x_i)))$$
$$= \sum_i (\tau\eta(E_Q(x_i)) - \tau\eta(E_P(x_i))) + \sum_i (\tau\eta(E_P(x_i)) - \tau\eta(E_N(x_i))) \leqslant H(P|Q) + H(N|P).$$

Property (H). $H(N|P)$ is increasing in N, decreasing in P.

Proof. Same proof as for property A.

Property (I). Let N_1 and N_2 be commuting finite dimensional von Neumann subalgebras of R. Then $H(N_2|N_1) = H((N_1 \cup N_2)''|N_1) = H((N_2 \cup N_1)'') - H(N_1)$.

Proof. We can assume that $R = (N_1 \cup N_2)''$. Then C_1 = Center of N_1 is contained in the center of R and it is easy to check that:

$$H(N_2|N_1) = \sum_{c \text{ atom of } C_1} \tau(c) H((N_2)_c|(N_1)_c),$$
$$H((N_1 \cup N_2)''|N_1) = \sum_{c \text{ atom of } C_1} \tau(c) H(N_1 \cup N_2)''_c|(N_1)_c),$$
$$H((N_1 \cup N_2)'') - H(N_1) = \sum_{c \text{ atom of } C_1} \tau(c)(H((N_1 \cup N_2)''_c) - H((N_1)_c),$$

where in each of the terms like $H((N_2)_c|(N_1)_c)$, the entropy is computed in R_c relative to $\tau/\tau(c)$.

It follows that we can assume N_1 to be a factor, in which case the equalities follow from Lieb-Ruskai second strong subadditivity property ([6]) which shows that

$$H((N_1 \cup N_2)''|N_1) \leqslant H(N_2).$$

3. Continuity of the relative entropy in the strong topology

Let R be a given finite von Neumann algebra with trace τ(*). If N, P are von Neumann subalgebras of R we shall write (cf. [12]) $N \overset{\delta}{\subset} P$ for positive δ, if and only if:

$$\forall x \in N,\ \|x\|_\infty \leqslant 1,\ \exists y \in P,\ \|y\|_\infty \leqslant 1,\ \|x-y\|_2 < \delta$$

where for each $p \in [1, \infty[$, and $a \in R$, $\|a\|_p = \tau(|a|^p)^{1/p}$ and $\|a\|_\infty$ is the C^* norm of a.

THEOREM 1. *Let R and τ be as above. For each integer $n < \infty$, and each $\varepsilon > 0$ there is a $\delta > 0$ such that for any pair of von Neumann subalgebras N, P of R:*

$$(\dim N = n,\quad N \overset{\delta}{\subset} P) \Rightarrow H(N|P) < \varepsilon.$$

The first part of the proof, Lemmas 2 to 9, does not involve the function H.

(*) τ is throughout a faithful normal trace such that $\tau(1) = 1$.

First part

LEMMA 2. *For each $m>0$ and $\eta>0$ there is a $\delta=\delta_m(\eta)$ such that for any von Neumann subalgebra P of R, any abelian von Neumann subalgebra $\mathcal{A}$ of R with minimal projections $(a_i)_{i=1,\ldots,m}$, $\sum_{i=1}^m a_i=1$, one has*

$$(\mathcal{A}\overset{\delta}{\subset}P)\Rightarrow(\|a_i-a_i'\|_2<\varepsilon,\quad i=1,\ldots,m,\ \sum_{i=1}^m a_i'=1$$

for some family of projections $a_i'\in P$).

Proof. For $m=1$ the lemma is obvious. We assume it has been proven for $m\leqslant m_0$ and we prove it for $m=m_0+1$. Choose η_1 such that:

$$9(3\eta_1)^{1/2}+\eta_1\leqslant\eta.$$

Then take δ smaller than $\delta_{m_0}(\eta_1)$ and than η_1. Let $\mathcal{A}$ and P be given, with $\mathcal{A}\overset{\delta}{\subset}P$ and call B the abelian von Neumann subalgebra of $\mathcal{A}$ whose list of minimal projections is $b_1=a_1,\ldots,$ $b_{m_0-1}=a_{m_0-1}$, $b_{m_0}=a_{m_0}+a_{m_0+1}$. There exist by hypothesis, projections $(b_j')_{j=1,\ldots,m_0}$, $b_j'\in P$ such that

$$\|b_j'-b_j\|_2\leqslant\eta_1,\quad j=1,\ldots,m_0.$$

As $\delta\leqslant\eta_1$, there is an $x\in P_+$, $\|x\|_\infty\leqslant1$ such that $\|x-a_{m_0}\|_2\leqslant\eta_1$.

Now $b_{m_0}'xb_{m_0}'$ is a positive element of R_+ such that

$$\|b_{m_0}'xb_{m_0}'-a_{m_0}\|_2=\|b_{m_0}'xb_{m_0}'-b_{m_0}a_{m_0}b_{m_0}\|_2\leqslant2\|b_{m_0}'-b_{m_0}\|_2+\|x-a_{m_0}\|_2\leqslant3\eta_1.$$

By [3] Lemma 4 p. 273 there is a spectral projection a_{m_0}' of $b_{m_0}'xb_{m_0}'$ such that:

$$\|a_{m_0}'-a_{m_0}\|_2\leqslant9(3\eta_1)^{1/2}.$$

We have $a_{m_0}'\in P$ and $a_{m_0}'\leqslant b_{m_0}'$, so that the family $(b_1',\ldots,b_{m_0-1}',a_{m_0}',b_{m_0}'-a_{m_0}')$ of projections of P satisfies the required conditions.

LEMMA 3. *Let e be a projection, $e\in R$, and let $y\in R$ be such that $y^*y\leqslant e$. Let $y=u\varrho$ be the polar decomposition of y. Then there exists a spectral projection f of ϱ such that*

$$\|f-e\|_2\leqslant9\|y^*y-e\|_2^{1/2},\ \|uf-y\|_2\leqslant8\|y^*y-e\|_2^{1/2}.$$

Proof. By [3] Lemma 4, p. 273, if we put $\varepsilon=\|\varrho-e\|_2^{1/2}$ and suppose $\varepsilon<\frac{1}{2}$, the spectral projection f of ϱ corresponding to $[1-\varepsilon,1]$ will satisfy the first of the required inequalities. Moreover by [3], p. 274, we have $\|\varrho-f\|_2\leqslant8\varepsilon$ so that $\|uf-u\varrho\|_2\leqslant8\varepsilon$.

LEMMA 4. *Let $n>0$ and $\varepsilon>0$ be given. Then there is a $\delta>0$ such that for any pair of von Neumann subalgebras Q and P of R with $Q\overset{\delta}{\subset}P$, $\dim Q=n$, and any system of matrix* *units* $(e_{ij}^{(k)})_{i,j=1,\dots,n_k}$, $k=1,\dots,s$, *with* $\Sigma_{i,k}\, e_{ii}^{(k)}=1$, of Q, *there exists a system of matrix units*

$$(p_{ij}^{(k)})_{i,j=1,\dots,n_k},\ k=1,\dots,s,$$

of P *such that*

$$\|e_{j1}^{(k)}-p_{j1}^{(k)}\|_2<\varepsilon\ \forall\, j,k.$$

Proof. First choose $\eta>0$ such that

$$3\eta+9(7\eta)^{1/2}\leqslant\varepsilon/3\sqrt{n}.$$

Then choose $\delta=\delta_n(\eta)$ as in Lemma 2 with $\delta<\eta$. Let Q and P satisfy $Q\overset{\delta}{\subset}P$, and $(e_{ij}^{(k)})$ be a system of matrix units in Q. By Lemma 2 we can find a family $(b_j^{(k)})_{j=1,\dots,n_k,\,k=1,\dots,s}$ of projections of P with $\Sigma_{j,k}\, b_j^{(k)}=1$ and $\|e_{jj}^{(k)}-b_j^{(k)}\|_2\leqslant\eta$, $\forall\, j,k$. We now fix k and we construct an $n_k\times n_k$ system of matrix units $p_{ij}^{(k)}$ all belonging to $P_{\Sigma_j b_j^{(k)}}$. Let $j\in\{1,\dots,n_k\}$, and $x\in P$ such that $\|e_{j1}^{(k)}-x\|_2<\eta$. Then

$$\|b_j^{(k)}xb_1^{(k)}-e_{j1}^{(k)}\|_2\leqslant\|b_j^{(k)}-e_{jj}^{(k)}\|_2+\|b_1^{(k)}-e_{11}^{(k)}\|_2+\eta\leqslant3\eta$$

and

$$\|(b_j^{(k)}xb_1^{(k)})^*(b_j^{(k)}xb_1^{(k)})-b_1^{(k)}\|_2\leqslant7\eta.$$

Hence by Lemma 3 there exists a partial isometry $u_j^{(k)}\in P$ with initial support $f_j^{(k)}\leqslant b_1^{(k)}$ and such that:

$$u_j^{(k)}u_j^{(k)*}\leqslant b_j^{(k)},\quad \|u_j^{(k)}-e_{j1}^{(k)}\|_2\leqslant3\eta+8(7\eta)^{1/2}$$

and $\|b_1^{(k)}-f_j^{(k)}\|_2\leqslant9(7\eta)^{1/2}$.

Put $p_{11}^{(k)}=\wedge_j f_j^{(k)}$. Then $\tau(b_1^{(k)}-p_{11}^{(k)})\leqslant\sum_j\tau(b_1^{(k)}-f_j^{(k)})$ which follows from the relation $f-e\wedge f\sim e\vee f-e$ for e and f projections. So that $\|b_1^{(k)}-p_{11}^{(k)}\|_2\leqslant\varepsilon/3$, hence $\|e_{11}^{(k)}-p_{11}^{(k)}\|_2\leqslant2\varepsilon/3$. Also put $p_{j1}^{(k)}=u_j^{(k)}p_{11}^{(k)}$. Clearly the $p_{j1}^{(k)}p_{i1}^{(k)*}$ form a system of $n_k\times n_k$ matrix units, and we have:

$$\|p_{j1}^{(k)}-e_{j1}^{(k)}\|_2\leqslant\|u_j^{(k)}-e_{j1}^{(k)}\|_2+\|p_{11}^{(k)}-e_{11}^{(k)}\|_2\leqslant\varepsilon/3+2\varepsilon/3=\varepsilon.$$

Notation 5. Let N be a von Neumann subalgebra of R, then for any projection $f\in N'$ we put

$$N^f=\{xf+\lambda(1-f),\ x\in N,\ \lambda\in\mathbf{C}\}.$$

Clearly N^f is a von Neumann subalgebra of R.

LEMMA 6. *Let* $n>0$ *and* $\alpha>0$ *be given. Then there is a* $\delta>0$ *such that for any pair of von Neumann subalgebras* N *and* P *of* R *with* $N\overset{\delta}{\subset}P$, $\dim N=n$, *there exist a von Neumann subalgebra* $\tilde N$ *of* P, $\dim\tilde N\leqslant n+1$, *a pair of projections* $f\in N'$, $\tilde f\in\tilde N'$ *and a unitary* $u\in R$ *such that*

(1) $ufu^*=\tilde f$, $uN^fu^*=\tilde N^{\tilde f}$, (2) $\|f-1\|_2\leqslant\alpha$, $\|\tilde f-1\|_2\leqslant\alpha$, (3) $\|u-1\|_2\leqslant\alpha$.

Proof. First choose $\varepsilon>0$ such that

$$\varepsilon<\alpha/8n,\ 14\varepsilon^{1/4}\leqslant\alpha/8n.$$

Then apply Lemma 4 for this n and this ε to get δ. Let N and P be given, $N \overset{\delta}{\subset} P$, and let $(e_{ij}^{(k)})$ be a system of matrix units in N. By Lemma 4 there exists a system of matrix units $(p_{ij}^{(k)})$ of P such that

$$\| e_{11}^{(k)} - p_{11}^{(k)} \|_2 < \varepsilon, \ \forall\, i, k.$$

Let $\tilde{N}$ be the von Neumann subalgebra of P generated by the $p_{ij}^{(k)}$. Clearly $\dim \tilde{N} \leqslant n+1$. (It is equal to $n+1$ if the sum of the $p_{jj}^{(k)}$ is different from 1.)

The proof of Lemma 5 p. 274 of [3] does not use the hypothesis "factor". Hence we get for each $k \in \{1, \ldots, s\}$ a partial isometry w_k such that: $w_k^* w_k \leqslant e_{11}^{(k)}$, $w_k w_k^* \leqslant p_{11}^{(k)}$, $\|w_k - e_{11}^{(k)}\|_2 \leqslant \alpha/8n$. We take $w = \sum_k \sum_j p_{j1}^{(k)} w_k e_{1j}^{(k)}$. The partial isometries $p_{j1}^{(k)} w_k e_{1j}^{(k)}$ have pairwise orthogonal initial projections and pairwise orthogonal final projections. So w is a partial isometry,

$$w^* w = \sum_{j,k} e_{j1}^{(k)} w_k^* w_k e_{1j}^{(k)}, \quad ww^* = \sum_{j,k} p_{j1}^{(k)} w_k w_k^* p_{1j}^{(k)}.$$

Hence $f = w^* w$ belongs to N' and $\tilde{f} = ww^*$ belongs to $(\tilde{N})'$. For each i, j, k we have $we_{ij}^{(k)} w^* = p_{i1}^{(k)} w_k e_{1i}^{(k)} e_{ij}^{(k)} e_{j1}^{(k)} w_k^* p_{1j}^{(k)} = p_{i1}^{(k)} w_k w_k^* p_{1j}^{(k)} = \tilde{f} p_{ij}^{(k)}$.

Let u be a unitary in R such that $uf = w$. Then

$$ufu^* = ww^* = \tilde{f}, \quad ufe_{ij}^{(k)} fu^* = \tilde{f} p_{ij}^{(k)} \tilde{f} \ \forall\, i, j, k.$$

Hence $uN^f u^* = \tilde{N}^{\tilde{f}}$.

Also

$$\| w - \sum_k \sum_j e_{j1}^{(k)} e_{11}^{(k)} e_{1j}^{(k)} \|_2 \leqslant n\varepsilon + n \frac{\alpha}{8n} \leqslant \frac{\alpha}{4},$$

so that

$$\| w - 1 \|_2 \leqslant \frac{\alpha}{4}, \quad \| f - 1 \|_2 \leqslant \frac{\alpha}{2}, \quad \| \tilde{f} - 1 \|_2 \leqslant \frac{\alpha}{2},$$

and

$$\| u - 1 \|_2 \leqslant \| uf - 1 \|_2 + \| u(1-f) \|_2 \leqslant \frac{\alpha}{4} + \frac{\alpha}{2} \leqslant \alpha.$$

LEMMA 7. *Let $\alpha > 0$, u be a unitary in R such that $\|u - 1\|_2 \leqslant \alpha$, then there exist unitaries v and v' such that*

$$\|v - 1\|_\infty \leqslant \alpha^{1/2}, \quad \tau(\text{Support } (v' - 1)) \leqslant \alpha^{1/2} \quad \text{and} \quad vv' = u.$$

Proof. Let e' be the spectral projection of u such that $|u-1|^2 e' \geqslant \alpha e'$, $|u-1|^2 (1-e') \leqslant \alpha(1-e')$. Then take $v = u(1-e') + e'$, $v' = (1-e') + ue'$ and use the inequality $\tau(|u-1|^2 e') \leqslant \|u-1\|_2^2 \leqslant \alpha^2$ to get $\tau(e') \leqslant \alpha$.

LEMMA 8. *For each $n \in \mathbf{N}$ there is $k(n) \in \mathbf{N}$ such that if N is a von Neumann subalgebra of R with $n = \dim N$, then for any projection $e \in R$ there is a projection $E \in N' \cap R$ such that $E \leqslant e$ and $\tau(1-E) \leqslant k(n)\tau(1-e)$.*

Proof. Consider N as imbedded as blocks along the main diagonal in the $n \times n$ matrices M_n. M_n is generated by the group H of unitaries with coefficients $0, \pm 1$. Let $k(n) = \text{card } H$. If $G = H \cap N$, then $(G)'' = N$. Let $E = \wedge_{u \in G} ueu^*$.

LEMMA 9. *Let n and $\varepsilon' > 0$ be given. Then there exists a $\delta > 0$ such that, for any pair of von Neumann subalgebras of R, with $N \overset{\delta}{\subset} P$ and $\dim N = n$, there exists a unitary v, $\|v - 1\|_\infty < \varepsilon'$, a projection E, $\tau(E) > 1 - \varepsilon'$ and a von Neumann subalgebra $\tilde{N}$ of P, $\dim \tilde{N} \leq n + 1$ such that:*

$$E \in N', \; vEv^* \in \tilde{N}' \quad \textit{and} \quad vN^E v^* = (\tilde{N})^{vEv^*}.$$

Proof. Choose $\alpha > 0$ such that if $k(n)$ is as in Lemma 8 then

$$\alpha^{1/2} \leq \varepsilon', \; \alpha^{1/2} + \alpha^2 \leq \varepsilon'/k(n).$$

Then choose δ corresponding to n and α by Lemma 6. Let $N \overset{\delta}{\subset} P$ and $\dim N = n$. Then let $\tilde{N} \subset P$, u, f, $\tilde{f}$ satisfy the conditions of Lemma 6. By Lemma 7 there are unitaries $v, v' \in R$, with $v'e = ev' = e$ and $e \leq f$. (Take $e = f \wedge (1 - \text{support } (1 - v'))$, so that $\tau(1-e) \leq \tau(1-f) + \tau$ (support $(1 - v')) \leq \|1 - f\|_2^2 + \alpha^{1/2} \leq \alpha^2 + \alpha^{1/2} \leq \varepsilon'/k(n)$.

Let $E \in N'$ be a projection such that $E \leq e$, $\tau(1 - E) \leq k(n)\tau(1 - e) \leq \varepsilon'$ (Lemma 8). As $E \leq f$ we have $(N^f)^E = N^E$, $((\tilde{N})^{\tilde{f}})^{uEu^*} = (uN^f u^*)^{uEu^*} = uN^E u^*$ and as $uEu^* \leq \tilde{f}$, $\tilde{N}^{uEu^*} = uN^E u^*$. Also $E \leq e$, so that $uEu^* = vEv^*$, and $uN^E u^* = vN^E v^*$, hence $vN^E v^* = \tilde{N}^{(vEv^*)}$.

Second part

LEMMA 10. *Let $n > 0$, $\varepsilon > 0$ be given, then there exists an $\varepsilon' > 0$ such that for any von Neumann subalgebra Q of R of dimension n one has:*

$$H(Q \mid vQv^*) < \varepsilon, \quad \forall\, v \textit{ unitary}, \quad \|v - 1\|_\infty < \varepsilon'.$$

Proof. Choose ε' such that for any family of positive reals α_i, $i = 1, \ldots, n$, $\Sigma\, \alpha_i = 1$, any families $\lambda_i, \lambda_i' > 0$, $i = 1, \ldots, n$, $\Sigma \lambda_i = \Sigma \lambda_i' = 1$,

$$\sum |\lambda_i - \lambda_i'| \alpha_i < 2\varepsilon' \Rightarrow \left| \sum (\eta(\lambda_i) - \eta(\lambda_i'))\, \alpha_i \right| < \varepsilon.$$

Then let Q and v be given. Let $x \in R_+$, $\tau(x) = 1$. We have $E_{vQv^*}(x) = vE_Q(v^*xv)v^*$, $\tau\eta E_{vQv^*}(x) = \tau\eta E_Q(v^*xv)$. Now $X = E_Q(x)$ and $X' = E_Q(v^*xv)$ both belong to Q^+ and satisfy $\tau(X) = \tau(X') = 1$, $\|X' - X\|_1 \leq \|v^*xv - x\|_1 \leq 2\varepsilon'$, since E_Q is a contraction in the $\| \; \|_1$-norm.

Let $q_1, \ldots, q_s$ be the list of atoms of the center of Q, where Q_{q_j} is of type n_j and $\Sigma n_j^2 = n$ so that $\Sigma n_j \leq n$. Let $X = \Sigma_k \Sigma_{j=1}^{n_k} \lambda_j^{(k)} e_j^{(k)}$, (resp. $X' = \Sigma_{k,j} \lambda_j'^{(k)} e_j'^{(k)}$) be the spectral decomposition of X (resp. X'), where for each k, j, $e_j^{(k)}$ and $e_j'^{(k)}$ are minimal projections of Q_{q_k}. We have, for a suitable choice of the decompositions [10], Theorem 5.6, that

$$\Sigma |\lambda_j^{(k)} - \lambda_j'^{(k)}| \tau(e_j^{(k)}) \leq 2\varepsilon',$$

hence that

$$|\tau\eta E_{vQv^*}(x)-\tau\eta E_Q(x)|=|\sum(\eta(\lambda_j^{(k)})-\eta(\lambda_j'^{(k)}))\,\tau(e_j^{(k)})|\leqslant\varepsilon.$$

We have shown that $\forall x\in R^+$, $|\tau\eta E_{vQv^*}(x)-\tau\eta E_Q(x)|\leqslant\varepsilon\tau(x)$; it follows easily that $H(Q\,|\,vQv^*)<\varepsilon$.

LEMMA 11. *Let N be a von Neumann subalgebra of R of dimension $\leqslant n$, and $E\in N'$ be a projection, then*

$$H(N^E\,|\,N)\leqslant\eta(\tau(E))+\eta(1-\tau(E)),$$

$$H(N\,|\,N^E)\leqslant\eta(\tau(E))+\eta(1-\tau(E))+\tau(1-E)\log n.$$

Proof. Let Q be the von Neumann algebra generated by N and the abelian von Neumann subalgebra $\mathcal{A}$ of N' generated by E. We have (property I) $H(Q\,|\,N)\leqslant H(\mathcal{A})=\eta\tau(E)+\eta\tau(1-E)$. As N^E and N^{1-E} are commuting and generate Q we have (property I)

$$H(Q\,|\,N^E)\leqslant H(N^{1-E})\leqslant\eta\tau(1-E)+\eta(\tau(E))+\tau(1-E)\log n.$$

Then

$$H(N\,|\,N^E)\leqslant H(Q\,|\,N^E)\leqslant\eta(\tau(1-E))+\eta\tau(E)+\tau(1-E)\log n$$

and

$$H(N^E\,|\,N)\leqslant H(Q\,|\,N)\leqslant\eta(\tau(E))+\eta(\tau(1-E)).$$

End of the proof of Theorem 1. First choose ε' such that Lemma 10 is satisfied with $n+1$ and $\varepsilon/2$, and such that, $\varepsilon'\leqslant 1/2$,

$$2(\eta(\varepsilon')+\eta(1-\varepsilon'))+\varepsilon'\log(n+1)\leqslant\varepsilon/2.$$

Then choose δ such that Lemma 9 is satisfied for n and ε'. Now let N and P be given, with $N\overset{\delta}{\subset}P$ and $\dim N=n$. By Lemma 9 we find $E\in N'$, $\tau(E)>1-\varepsilon'$, and $\tilde{N}$, $\tilde{N}\subset P$, $\dim\tilde{N}\leqslant n+1$, $v\in R$, $\|v-1\|_\infty<\varepsilon'$ with $vN^Ev^*=\tilde{N}^{(vEv^*)}$. By Lemma 11 we have:

$$H(N\,|\,N^E)\leqslant\eta(1-\varepsilon')+\eta(\varepsilon')+\varepsilon'\log(n+1).$$

By Lemma 10 we have

$$H(N^E\,|\,vN^Ev^*)\leqslant\varepsilon/2.$$

By Lemma 11 we have:

$$H(\tilde{N}^{vEv^*}\,|\,N)\leqslant\eta(1-\varepsilon')+\eta(\varepsilon').$$

So wet get, using property G and $H(\tilde{N}\,|\,P)=0$, that

$$H(N\,|\,P)\leqslant 2(\eta(1-\varepsilon')+\eta(\varepsilon'))+\varepsilon'\log(n+1)+\varepsilon/2\leqslant\varepsilon.$$

4. The non abelian Kolmogoroff-Sinai's theorem

Definition 1. Let θ be an automorphism of the finite von Neumann algebra R, preserving the faithful normal trace τ, $\tau(1)=1$, then

(a) For any finite dimensional von Neumann subalgebra N of R put

$$H(N,\theta)=\lim_{k\to\infty}\frac{1}{k}H(N,\theta(N),\ldots,\theta^k(N)).$$

(b) Put $$H(\theta)=\operatorname{Sup}_N H(N,\theta).$$

Note that in (a) the limit exists because of property B.

THEOREM 2. *Let R be a hyperfinite von Neumann algebra of type II_1, and τ, θ as above. Let $(P_q)_{q\in N}$ be an increasing sequence of finite dimensional subalgebras of R with $(\bigcup_{q=1}^{\infty}P_q)^- = R$ (weak closure) then:*

$$H(\theta)=\lim_{q\to\infty}H(P_q,\theta).$$

Proof. Let N be a finite dimensional von Neumann subalgebra of R, and $\varepsilon>0$. Apply Theorem 1 of section 3 to get a q such that $H(N|P_q)<\varepsilon$. Then

$$\begin{aligned}H(N,\theta)&=\lim_{k\to\infty}\frac{1}{k}H(N,\theta(N),\ldots,\theta^k(N))\\&\leqslant\lim_{k\to\infty}\frac{1}{k}H(P_q,\theta(P_q),\ldots,\theta^k(P_q))+\overline{\lim_{k}}\frac{1}{k}\sum_{j=0}^{k}H(\theta^j(N)|\theta^j(P_q))\\&\leqslant H(P_q,\theta)+\varepsilon.\end{aligned}$$

THEOREM 3. *Let N_0 be a finite dimensional von Neumann algebra with faithful normalised trace τ_0. Put $(R,\tau)=\otimes_{\nu\in\mathbf{Z}}(N_0,\tau_0)_\nu$ and let S be the automorphism of R corresponding to the translation of 1 in $\mathbf{Z}$. Then S preserves τ and:*

$$H(S)=H(N_0).$$

Proof. Let π_j be the homomorphism of N_0 in R such that

$$\pi_j(x)=1\otimes\ldots\otimes 1\otimes \underset{j\text{th term}}{x}\otimes 1\otimes\ldots\otimes 1\otimes\ldots$$

We have by construction $S\pi_j=\pi_{j+1}$.

Let $k\in\mathbf{N}$ and $P_k=(\bigcup_{|j|\leqslant k}\pi_j(N))''$. Clearly the P_k form an increasing sequence of finite dimensional von Neumann subalgebras of R and $\bigcup_{k=1}^{\infty}P_k$ is weakly dense in R.

Property E shows that $H(P_k, SP_k, \ldots, S^nP_k)=(2k+n+1)H(N_0)$ hence Theorem 3 follows from Theorem 2.

We next extend Theorem 3 to a larger class of ergodic automorphisms of the hyperfinite factor R which are the analogues of Bernoulli shifts. Recall that if M_n is the I_n-factor with trace Tr_n such that $Tr_n(1)=n$, and ω is a state of M_n then $\omega(x)=Tr_n(hx)$, where h is a positive operator of trace 1, then the eigenvalue list of ω is the spectrum

$\{\lambda_1, \ldots, \lambda_n\}$ of h counted with multiplicity. If R_m is a von Neumann algebra and φ_m a normal state of R_m, $m \in \mathbf{Z}$, we denote as before by $\otimes_{m \in \mathbf{Z}}(R_m, \varphi_m)$ the von Neumann algebra obtained from the GNS representation of the state $\otimes\varphi_m$ on the C^*-algebra tensor product $\otimes R_m$. If φ is a state on a von Neumann algebra M its centralizer is the algebra $\{x \in M: \varphi(xy - yx) = 0, \forall\, y \in M\}$. The following theorem was also noted by W. Krieger.

THEOREM 4. *Let M_0 be the $n \times n$ matrices and φ_0 a faithful state on M_0 with eigenvalue list $\{\lambda_1, \ldots, \lambda_n\}$. For $m \in \mathbf{Z}$ let $M_m = M_0$ and $\varphi_m = \varphi_0$. Let $M = \otimes_{m \in \mathbf{Z}}(M_m, \varphi_m)$. Then:*

(1) *If R is the centralizer of the state $\otimes\varphi_m$ in M then R is the hyperfinite II_1-factor.*

(2) *The restriction of the shift on M to R is an ergodic automorphism with entropy $-\sum_{j=1}^{n} \lambda_j \log \lambda_j$.*

Definition 2. The automorphism of R constructed above is the Bernoulli shift defined by $\{\lambda_1, \ldots, \lambda_n\}$.

Proof. Let φ be the normal state $\otimes\varphi_n$ on M. Then it is well known that φ is faithful and M is a factor. If $p \leqslant q$ are integers let M_p^q be the image of $\otimes_p^q M_m$ in M under its natural imbedding. Let F_p^q be the centralizer of $\varphi \mid M_p^q$ in M_p^q. If $p \leqslant r \leqslant s \leqslant q$ then clearly $F_r^s \subset F_p^q$. We assert that $\bigcup_1^\infty F_{-p}^p$ is strongly dense in R.

Let $\sigma_t = \sigma_t^\varphi$ be the modular automorphism of M defined by φ, see [15]. Since φ is a product state, for each $p \leqslant q$ we have $\sigma_t = \sigma_t \mid M_p^q \otimes \sigma_t \mid (M_p^q)^c$, where $(M_p^q)^c = (M_p^q)' \cap M$, and M is identified with $M_p^q \otimes (M_p^q)^c$. Let Φ be the faithful strongly continuous σ-invariant conditional expection of M onto R [4]. Then $\Phi(M_p^q) = F_p^q$. Let $x \in R$. Since $\bigcup_1^\infty M_{-p}^p$ is strongly dense in M there is a net $\{x_\alpha\}$ in $\bigcup_1^\infty M_{-p}^p$ converging strongly to x. Thus $x = \Phi(x) =$ strong limit $\Phi(x_\alpha)$, with $\Phi(x_\alpha) \in \bigcup_1^\infty F_{-p}^p$. The assertion follows.

We show R is a factor, hence by the above paragraph R is the hyperfine II_1-factor. Let P be the group of finite permutations on $\mathbf{Z}$ and G the group of *-automorphisms of M defined by the action of P on the factors in the infinite tensor product. Since φ is G-invariant the automorphisms are well defined, and R is globally invariant for each $g \in G$. If $g \in G$ there are integers $p \leqslant q$ such that g is an automorphism of M_p^q and the identity on $(M_p^q)^c$. Since M_p^q is a type I-factor g is an inner automorphism of M, hence implemented by a unitary operator $u_g \in M$. Since φ is G-invariant $u_g \in R$. The arguments in [13] show that the C^*-algebra $\otimes M_m$ is asymptotically abelian with respect to G and that $\otimes\varphi_m$ is an extremal invariant state. Since φ is faithful on M, G acts ergodically on M, see [14]. Since $u_g \in R$ for all $g \in G$, R is a factor as asserted.

To show the second part of the theorem let α be the automorphism of M which shifts the factors in the infinite tensor product one factor to the right, so $\alpha(M_p^q) = M_{p+1}^{q+1}$, and let

H be the cyclic group generated by α. As above φ is H-invariant, the automorphism well defined, and $\otimes M_m$ asymptotically abelian with respect to H. Since M is a factor, $\otimes\varphi_m$ is ergodic [14], and H acts ergodically on M [14]. Let θ denote the restriction of α to R. Then θ is an ergodic automorphism of R.

By Theorem 2 $H(\theta) = \lim H(F^p_{-p}, \theta)$. Fix p; then

$$H(F^p_{-p}, \theta(F^p_{-p}), \ldots, \theta^k(F^p_{-p})) = H(F^p_{-p}, F^{p+1}_{-p+1}, \ldots, F^{p+k}_{-p+k}).$$

By property C

$$H(F^p_{-p}, \ldots, \theta^k(F^p_{-p})) \leqslant H(F^{p+k}_{-p}).$$

Let A^p_{-p} be a maximal abelian subalgebra of F^p_{-p} such that A^p_{-p} commutes with $\theta^j(A^p_{-p})$ for all $j \in \mathbf{Z}$, and such that $A = (A^p_{-p} \cup \theta(A^p_{-p}) \cup \ldots \cup \theta^k(A^p_{-p}))''$ is a maximal abelian subalgebra of F^{p+k}_{-p}. This choice is possible since φ is a product state. Thus by properties C, D, E together with the above inequality we have

$$H(F^p_{-p}, \ldots, \theta^k(F^p_{-p})) = H(A) = (2p+k)\left(-\sum_1^n \lambda_j \log \lambda_j\right).$$

The proof is complete.

Remark 4. Recently G. Emch has defined an entropy for automorphisms of von Neumann algebras. His definition is different from ours, and as far as his entropy cannot be computed in the above examples because of the lack of a Kolmogoroff-Sinai theorem we do not know whether the two definitions coincide in the case of shifts. In fact his definition of $H(N, \theta)$ is not even increasing in N, so that we do not believe that the analogue of our Theorem 2 can be proved in his context.

Remark 5. Another possible candidate for the entropy of an automorphism is the *abelian entropy*

$$H_a(\theta) = \sup H(\theta \mid \mathcal{A}),$$

where the sup is taken over all abelian von Neumann subalgebras $\mathcal{A}$ of R with $\theta(\mathcal{A}) = \mathcal{A}$, and $H(\theta \mid \mathcal{A})$ is the entropy of $\theta \mid \mathcal{A}$ defined in the abelian case. If θ is the Bernoulli shift defined by $\{\lambda_1, \ldots, \lambda_n\}$ we again get $H_a(\theta) = -\sum_{j=1}^n \lambda_j \log \lambda_j$. However, the definition is unsatisfactory in that it is not clear whether there exist large invariant abelian von Neumann subalgebras for a given θ. In the next remark (Remark 6) we show how to compute $H(\theta^k)$ from $H(\theta)$; it is highly improbable that the same formula holds for the abelian entropy H_a, because an abelian von Neumann algebra $\mathcal{A} \subset R$ globally invariant under θ^k is not necessarily globally invariant under θ.

Remark 6. Let R be a von Neumann algebra of type II_1, and θ an automorphism preserving a faithful normal trace τ satisfying $\tau(1)=1$. We show that if p is an integer then $|p|H(\theta)\geqslant H(\theta^p)$, and if R is hyperfinite then $|p|H(\theta)=H(\theta^p)$.

The case $p=0$ is trivial as is the identity $H(\theta^{-1})=H(\theta)$. We thus assume $p>0$. Let $\varepsilon>0$ and N be a finite dimensional von Neumann subalgebra of R and $n_0\in\mathbf{N}$ such that

$$H(\theta^p)<\frac{1}{r}H(N,\theta^pN,\ldots,\theta^{rp}N)+\varepsilon \quad \text{for} \quad r\geqslant n_0.$$

It is immediate from properties A and C that

$$H(N_1,\ldots,N_k)\leqslant H(N_1,\ldots,N_k,N_{k+1})$$

for finite dimensional von Neumann subalgebras $N_1,\ldots,N_{k+1}$ of R. Therefore for n_0 sufficiently large, and $r\geqslant n_0$,

$$H(\theta)+\varepsilon>\frac{1}{rp}H(N,\theta N,\ldots,\theta^{rp}N)\geqslant\frac{1}{rp}H(N,\theta^pN,\ldots,\theta^{rp}N>\frac{1}{p}H(\theta^p)-\varepsilon/p.$$

Thus $pH(\theta)\geqslant H(\theta^p)$.

For the converse inequality assume R is hyperfinite, so there is an increasing sequence $\{F_j\}$ of finite dimensional von Neumann subalgebras with union weakly dense in R. Let $\varepsilon>0$. By Theorem 2 there are k, $n_0\in\mathbf{N}$ such that

$$\left|H(\theta)-\frac{1}{n}H(F_k,\theta F_k,\ldots,\theta^nF_k)\right|<\varepsilon/p \quad \text{for} \quad n\geqslant n_0. \tag{1}$$

Let $m=\dim F_k$. Choose by Theorem 1, $\delta>0$ so small that for N and P finite dimensional von Neumann subalgebras of R we have

$$\dim N=m, N\overset{\delta}{\subset}P\Rightarrow H(N|P)<\varepsilon/p. \tag{2}$$

Since the sequence $\{F_j\}$ is increasing with union dense in R there exists $q\in\mathbf{N}$, $q\geqslant k$, such that $F_k,\theta F_k,\ldots,\theta^{p-1}F_k\overset{\delta}{\subset}F_q$. Choose $r\in\mathbf{N}$ so large that

$$rp\geqslant n_0, \tag{3}$$

$$\left|H(\theta^p,F_q)-\frac{1}{r}H(F_q,\theta^pF_q,\ldots,\theta^{rp}F_q)\right|<\varepsilon. \tag{4}$$

By Property C

$$H(F_q,\theta^pF_q,\ldots,\theta^{rp}F_q)\geqslant H(F_q,\ldots,F_q,\theta^pF_q,\ldots,\theta^pF_q,\ldots,\theta^{(r-1)p}F_q,\ldots,\theta^{(r-1)p}F_q,\theta^{rp}F_q),$$

where all repetitions occur p times. Using this together with Property F we obtain from (1)–(4),

$$
\begin{aligned}
H(\theta^p, F_q)+\varepsilon &> \frac{1}{r} H(F_q, \theta F_q, \ldots, \theta^{rp} F_q) \\
&\geqslant \frac{1}{r} H(F_k, \theta F_k, \ldots, \theta^{rp} F_k) - \frac{1}{r} \sum_{s=0}^{r-1} \sum_{t=0}^{p-1} H(\theta^{sp+t} F_k \,|\, \theta^{sp} F_q) - \frac{1}{r} H(\theta^{rp} F_k \,|\, \theta^{rp} F_q) \\
&= \frac{p}{rp} H(F_k, \theta F_k, \ldots, \theta^{rp} F_k) - \frac{1}{r} r \sum_{t=0}^{p-1} H(\theta^t F_k \,|\, F_q) - \frac{1}{r} H(F_k \,|\, F_q) \\
&> p(H(\theta) - \varepsilon/p) - \left(p + \frac{1}{r}\right)\varepsilon/p \geqslant pH(\theta) - 3\varepsilon.
\end{aligned}
$$

Thus $H(\theta^p) \geqslant pH(\theta)$, as asserted.

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