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Classification of injective factors

Cases II_1, II_∞, III_λ, $\lambda \neq 1$

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Introduction

A von Neumann algebra M, acting in a Hilbert space $\mathcal{H}$, is called injective when, as a subspace of the Banach space $\mathcal{L}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$, M is the range of a projection of norm one. The main result of this paper is:

THEOREM 1. *All injective factors of type* II_1, *acting in a separable Hilbert space, are isomorphic.*

We now mention several applications, all of which solve problems which remained open for a long time.

COROLLARY 2. *All subfactors of the Murray and von Neumann hyperfinite factor* R ([34]) *are isomorphic to* R *or finite dimensional* (cf. [41], Pb. 4.4.27).

This shows that R can be characterized as the smallest infinite dimensional factor, it can be imbedded in all infinite dimensional factors and is, up to isomorphism, the only one. Also all von Neumann subalgebras of R are isomorphic to a product of von Neumann algebras $A_n \otimes M_n(\mathbf{C})$, $n = 1, 2, \cdots$ and a von Neumann algebra $A \otimes R$, where A and the A_n are abelian von Neumann algebras. Thus all von Neumann subalgebras of R are classified up to isomorphism by the number of atoms in the A_n and the A and the presence of a non-atomic projection in the A_n and A.

The factor R is the first one all of whose von Neumann subalgebras are classified. In [12] we showed that the group $\mathrm{Out}\, R = \mathrm{Aut}\, R/\mathrm{Int}\, R$ of classes of automorphisms of R modulo inner automorphisms is a simple group with countably many conjugacy classes indexed by a pair (p, γ), $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$.

Another remarkable property of the factor R is that if $S \subset R$ is any self-adjoint subset, the von Neumann subalgebra M of R generated by S can be characterized by a bicommutation property, analogue to the bicommutation in a type I factor:

$$x \in M \Longleftrightarrow [x, y_n] \xrightarrow[n \to \infty]{} 0 \text{ strongly for any bounded sequence}$$
$$(y_n)_{n \in \mathbf{N}} \text{ in } R \text{ such that } [s, y_n] \xrightarrow[n \to \infty]{} 0 \text{ strongly}, \qquad \text{for } s \in S.$$

COROLLARY 3. *Let $\mathcal{G}$ be a discrete countable group and λ the left regular representation of $\mathcal{G}$ in $l^2(\mathcal{G})$. Then if $\mathcal{G}$ is amenable the commutant $\lambda(\mathcal{G})'$ of λ is a von Neumann algebra of the form $\lambda(\mathcal{G})' = A \otimes R \oplus \sum_{n=1}^{\infty} A_n \otimes M_n(\mathrm{C})$ where the A_n and A are abelian von Neumann algebras* (cf. [41], Pb. 4.4.28).

It is known from the Schwartz property P that if $\mathcal{G}$ is not amenable the von Neumann algebra $\lambda(\mathcal{G})'$ is not of the above form. In particular if $\mathcal{G}$ has only infinite conjugacy classes the commutant of $\lambda(\mathcal{G})$ is isomorphic to R.

COROLLARY 4. *All injective factors of type II_∞, acting in a separable Hilbert space, are isomorphic* (cf. [41], Pb. 4.4.11).

Let $R_{0,1} = R \otimes M_\infty(\mathrm{C})$ be the corresponding factor. Corollary 4 implies that all factors of type II_∞ which are generated by an increasing sequence of finite dimensional * subalgebras are isomorphic to $R_{0,1}$. (The terminology used to qualify this approximation property of a von Neumann algebra by finite dimensional * algebras is usually "hyperfiniteness". However it is inadequate for nonfinite von Neumann algebras, in which case we shall follow [20] and use the adjective "approximately finite dimensional.")

COROLLARY 5. *Let G be a locally compact connected separable group and λ the left regular representation of G in $L^2(G)$. Then the commutant $\lambda(G)'$ is a von Neumann algebra of the form*:

$$\lambda(G)' = A \otimes R \otimes M_\infty(\mathrm{C}) \oplus \sum_{n=1}^{\infty} A_n \otimes M_n(\mathrm{C}) \oplus A_\infty \otimes M_\infty(\mathrm{C}) .$$

We now pass to more general representations of such groups. A theorem of J. Glimm and O. Marechal [30] asserts that for any infinite approximately finite dimensional factor M and for any non type I separable C^* algebra $\mathcal{A}$ there exists a representation π of $\mathcal{A}$ such that $\pi(\mathcal{A})'' = M$. Thus the class of approximately finite dimensional factors is the smallest class of factors suitable for representation theory of non type I C^* algebras (with $\mathcal{A}$ a uniformly hyperfinite C^* algebra, the $\pi(\mathcal{A})''$ are certainly approximately finite dimensional).

Our next result characterizes this class:

THEOREM 6. *For a factor M acting in a separable Hilbert space $\mathcal{H}$ the following properties are equivalent*:

(a) *M is approximately finite dimensional*:

(b) *M is injective*;

(c) *M has property P of Schwartz* [42];

(d) *M is semi discrete* [19].

The equivalence between those properties had been conjectured by several authors, in particular that (a) $\Leftrightarrow$ (c) was believed since [42], that (b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (a) was conjectured in [19]. The next corollary was conjectured by Kadison and Singer.

COROLLARY 7. *Let G be a solvable separable locally compact group or a connected locally compact separable group. Then any representation π of G in a Hilbert space generates an approximately finite dimensional von Neumann algebra.*

In fact, more generally we shall see that if G is locally compact separable and G_0, the connected component of the identity, is such that G/G_0 is amenable, then any $\pi(G)''$ is approximately finite dimensional. In [8], following the works of Powers, Araki, Woods and Krieger, we introduced invariants S and T for factors of type III, based on Tomita's theory. This allowed us to subdivide the class of type III factors in subclasses III_λ, $\lambda \in]0, 1[$, III_0 and III_1 corresponding to the subgroup of $\mathbf{R}_+^*$, $S(M)$. The important features of this theory are:

(1) Computability of the invariants S, T for all known constructions of factors; this is due to the characterization by the non-commutative Radon-Nikodym theorem [8] of the modular automorphism groups as the sections of an abstract kernel $\mathbf{R} \xrightarrow{\delta} \mathrm{Out}\, M$, where $\mathrm{Out}\, M = \mathrm{Aut}\, M/\mathrm{Int}\, M$, and to the equalities

$$S(M) = \text{Spectrum } \delta , \qquad T(M) = \text{Kernel } \delta .$$

(2) The existence and uniqueness of the discrete decomposition of a factor of type III_λ, $\lambda \in]0, 1[$ as cross product of a factor of type II_∞ by an automorphism θ with module λ. The discrete decomposition was also extended in [8] to the case III_0.

One criticism of [8] was that case III_1 was left untouched. However Lemma 1.2.5 [8] was really a crucial idea of Takesaki's solution of case III_1 by the introduction of continuous cross products.

A more serious criticism was the absence, at the time, of any reasonable classification:

(a) of factors of type II_∞,

(b) of their automorphisms with module λ.

In [12] we showed (Corollary 6) that all automorphisms θ of $R_{0,1} = R \otimes M_\infty(\mathrm{C})$ with the same module $\lambda \neq 1$ are conjugate. When M is an injective factor of type III_λ, the associated factor of type II_∞ in a discrete decomposition inherits the injectivity of M and we now apply Corollary 4 to conclude:

THEOREM 8. *For each* $\lambda \in]0, 1[$, *all injective factors of type* III_λ *are isomorphic to Powers' factor* R_λ.

In case III_0 the discrete decomposition of M allows one to show that M is obtained as the cross product of an abelian von Neumann algebra by a single automorphism. It then follows from a theorem of W. Krieger [29] that such factors are classified by ergodic (non-transitive) flows. In case III_1 the only known injective factor is the factor R_∞ of Araki-Woods. I am grateful to E. J. Woods for his kind invitation to Queens University where the present work was done.

CONTENTS

I. Preliminaries

I.1. *Joint distribution of pairs of positive square integrable operators affiliated with a semi-finite von Neumann algebra.* Let N be a semi-finite countably decomposable von Neumann algebra and τ a faithful semi-finite normal trace on N. We use the L^p spaces of Dixmier [14] and Segal [43], for the values $p = 1, 2$. An element x of $L^p(N, \tau)$ is a closed operator affiliated with N whose domain is strongly dense with respect to N ([43], Def. 2.1). A positive operator h affiliated with N is called integrable when its spectral measure: $\nu(B) = \tau(\chi_B(h))$, for any Borel subset B of $\mathbf{R}_+^* =]0, \infty[$, satisfies $\int \lambda d\nu(\lambda) < \infty$, and in this case one defines $\tau(h)$ as the value of $\int \lambda d\nu(\lambda)$. (See [43], Corollary 12.6.) Note that the spectral measure ν, is a Radon measure on $]0, +\infty[$, when h is integrable; $\nu(\{0\})$ can be $+\infty$ but this does not affect the equality $\tau(h) = \int_{\mathbf{R}_+^*} \lambda d\nu(\lambda)$ (see [43]). For $x \in L^p(N, \tau)$, $|x|^p$ is integrable and one has

$$(\tau(|x|^p))^{1/p} = \|x\|_p \qquad ([43], 12, 11; 3.8; 12.14)\,.$$

PROPOSITION. *Let N and τ be as above, $X = \mathbf{R}_+^2\backslash\{0\}$, H, K the continuous functions $H(x, y) = x$, $K(x, y) = y$, $(x, y) \in X$. Let h, k be positive elements of $L^2(N, \tau)$. Then there exists a positive Radon measure μ on X such that:*

(a) *For any positive Borel function f on $[0, +\infty[$ with $f(0) = 0$ and $f(h)$ integrable (resp. $f(k)$), the function $f(H)$ is integrable on X and $\tau(f(h)) = \mu(f(H))$ (resp. $\tau(f(k)) = \mu(f(K))$).*

(b) *For any pair f, g of complex Borel functions on $[0, +\infty[$, with $f(0) = g(0) = 0$, $f(h)$, $g(k) \in L^2(N, \tau)$ one has:*

$$\| f(h) - g(k) \|_2 = \| f(H) - g(K) \|_2\,.$$

Proof. For $\varepsilon > 0$, one has $\chi_{[\varepsilon,\infty[}(h)\varepsilon^2 \leqq h^2$ so that $\chi_A(h)$ is integrable for any Borel subset A of $[\varepsilon, \infty[$. Hence as in [33], Appendix, there exists a positive finite measure μ_ε on $[\varepsilon, \infty[\times[0, \infty[$ such that $\mu_\varepsilon(A\times B)=\tau(\chi_A(h)\chi_B(k))$, for any Borel sets $A\subset[\varepsilon, \infty[$, $B\subset[0, \infty[$. In the same way one gets a measure μ^ε on $[0, \infty[\times[\varepsilon, \infty[$ with, for $A\subset[0, \infty[$, $B\subset[\varepsilon, \infty[$, $\mu^\varepsilon(A \times B) = \tau(\chi_A(h)\chi_B(k))$. As the plane Borel sets form the least monotone class containing the disjoint unions of rectangles we see that all measures μ_ε, $\mu^{\varepsilon'}$ agree on the intersections of their domains and hence define a unique positive Radon measure μ on X such that $\mu(A \times B) = \tau(\chi_A(h)\chi_B(k))$ provided $0 \notin \bar{A}$ or $0 \notin \bar{B}$, $A, B\subset[0, \infty[$. So $\mu(\chi_A(H)) = \mu(A \times \mathbf{R}_+) = \tau(\chi_A(h))$, provided $0 \notin \bar{A}$, and the first assertion follows. Hence we have

$$\|f_1(H) - f_2(H)\|_2 = \|f_1(h) - f_2(h)\|_2\,,$$

for f_1, f_2 complex Borel functions on $[0, \infty[$ vanishing at 0 and making $f_j(h)$ square integrable. So to prove (b) we can assume that f (and g) is of the form: $f = \sum \lambda_j\chi_{A_j}$, $0 \notin \bar{A}_j$. Then $\tau(f(h)\bar{g}(k)) = \mu(f(H)\bar{g}(K))$ and hence (b) follows. Q.E.D.

Proposition I.1 summarizes the advantage of the L^2 norm over the other L^p norms, $p \neq 2$, for which it is no longer true. The next section contains the main technical lemma of this paper: Theorem 1.2.2.

I.2. *Stability of the polar decomposition of square integrable operators under square integrable perturbations.* Let $\mathcal{H} = L^2(N, \tau)$ be the Hilbert space of square integrable operators affiliated with the von Neumann algebra N with faithful normal semi-finite trace τ.

Each $x \in \mathcal{H}$ has a unique polar decomposition: $x = u(x)\,|x|$ where $u(x)$ is a partial isometry with support equal to the support of x and where $|x| = (x^*x)^{1/2}$.

The map $x \longrightarrow |x|$ was studied in [38] and it follows from [38] that for normal x and $y \in \mathcal{H}$ one has

$$\| |x| - |y| \|_2 \leqq \|x - y\|_2 .$$

In the non-abelian case this inequality is no longer true but the inequality of Powers-Stormer [36] replaces it:

PROPOSITION 1.2.1. *Let N and τ be as above.*
Let $x, y \in L^2(N, \tau)$, then

$$\| |x| - |y| \|_2^2 \leqq \| |x|^2 - |y|^2 \|_1 \leqq \|x - y\|_2 (\|x\|_2 + \|y\|_2) .$$

Proof. For the first inequality see [36] when N is a type I factor and [24], Lemma 2.10 for the general case. Also, $\| |x|^2 - |y|^2 \|_1$ is the distance in N_* between the vector states associated to x^* and y^* and hence is smaller than $\|x - y\|_2 (\|x\|_2 + \|y\|_2)$. Q.E.D.

Even when N is abelian the map $x \longrightarrow u(x)$ is quite discontinuous. For each $a > 0$ let u_a be the map from $\mathcal{H} = L^2(N, \tau)$ to the set of partial isometries of N defined by $u_a(x) = u(x)E_a(|x|)$ where E_a is the characteristic function of $]a, +\infty[\subset \mathbf{R}_+$. Still each u_a is discontinuous and even in the abelian case we cannot find for each $\delta > 0$ an $\varepsilon > 0$ such that for any $x \in L^2$ there exists an $a > 0$ satisfying $u_a(x) \neq 0$ and:

$$y \in L^2 , \quad \|y - x\|_2 \leqq \varepsilon \|x\|_2 \Longrightarrow \|u_a(x) - u_a(y)\|_2 \leqq \delta \|u_a(x)\|_2 .$$

However we shall prove the following continuity.

THEOREM 1.2.2. *Let N, τ be as above. Let $\delta \in]0, 1[$, $n \in \mathbf{N}$, put $\varepsilon = (\delta/6n)^8$. Then for any subset $(x_j)_{j=1,\cdots,n}$ of $L^2(N, \tau)$ of diameter less than $\varepsilon \|x_1\|_2$ there exists an $a > 0$ such that*

$$\|u_a(x_j) - u_a(x_1)\|_2 \leqq \delta \|u_a(x_1)\|_2 ,$$
$$\|x_1 - u_a(x_1)|x_1|\|_2 \leqq \delta \|x_1\|_2 .$$

As an easy corollary of Theorem 1.2.2 we have:

COROLLARY 1.2.3. *Let N, τ be as above, $\delta \in]0, 1[$, $n \in \mathbf{N}$ and $\varepsilon = (\delta/24n)^{16}$. Then for any unitary operators $(u_j)_{j=1,\cdots,n-1}$ in N and any equivalent projections $e_1, e_2 \in N$ such that $\|[u_k, e_s]\|_2 \leqq \varepsilon \|e_s\|_2$ for all k and s, one can find a projection $e \leqq e_1 \vee e_2$ such that*

$$\|ee_s - e_s\|_2 \leqq \delta \|e_s\|_2 , \qquad s = 1, 2 ,$$
$$\|[e, u_k]\|_2 \leqq \delta \|e\|_2 , \qquad k = 1, 2, \cdots, n-1 .$$

For the proof of 1.2.2 we begin by stating the elementary properties of the maps u_a: $u_a(x) = u(x)E_a(|x|)$.

LEMMA 1.2.4. *Let N, τ be as above and $a > 0$, $x \in L^2(N, \tau)$.*

(1) *$u_a(vx) = vu_a(x)$, $u_a(xv) = u_a(x)v$, v a unitary operator of N.*

(2) *$u_a(\theta(x)) = \theta(u_a(x))$, $\theta \in \operatorname{Aut} N$.*

(3) *Let $x^b = xE_b\,|x|$, $b > 0$, then $u_a(x^b) = u_c(x)$ and $u_a(x^b)\,|x^b| = u_c(x)\,|x|$, where $c = \sup(a, b)$.*

(4) *$u_a(h) = u_{f(a)}(f(h))$, $h \in L^2(N, \tau)^+$, f increasing bijection from $\mathbf{R}_+$ to $\mathbf{R}_+$.*

Proof. (1), (2) and (4) are clear. For (3), $|xE_b\,|x|\,| = |x|\,E_b\,|x|$ and $u(xE_b\,|x|) = u(x)E_b\,|x|$. For $t > 0$, $E_a(tE_b(t)) = 0$ if $t \leqq c$, and $=1$ if $t > c$, so $E_a(|x^b|) = E_c(|x|)$ and as $E_b(|x|)E_c(|x|) = E_c(|x|)$ we get (3). Q.E.D.

LEMMA 1.2.5. *Let N, τ be as above, $x \in L^2(N, \tau)$. Then*

$$\int_{\mathbf{R}_+^*} \|\, u_{a^{1/2}}(x) - u_{a^{1/2}}(|x|)\,\|_2^2\, da = \|\, x - |x|\,\|_2^2\ .$$

Proof. Let $u = u(x)$, $h = |x|$. For any Borel function f from $\mathbf{R}_+$ to $\mathbf{R}$ such that $f(h) \in L^2(N, \tau)$, $f(0) = 0$, we have

$$\|\, uf(h) - f(h)\,\|_2^2 = \tau(f(h)(u^* - 1)(u - 1)f(h)) = \tau(f(h)^2(2 - u - u^*))\ .$$

The expression that we want to compute is

$$\int_{\mathbf{R}_+^*} \|\, uE_{a^{1/2}}(h) - E_{a^{1/2}}(h)\,\|_2^2\, da = \int_{\mathbf{R}_+^*} \tau((E_{a^{1/2}}(h))^2(2 - u - u^*))da\ .$$

But $(E_{a^{1/2}}(h))^2 = E_{a^{1/2}}(h) = E_a(h^2)$ by 1.2.4 (4) and $\int_{\mathbf{R}_+^*} E_a(h^2)da$ is equal to h^2. Now $\tau(h^2(2 - u - u^*)) = \|\, uh - h\,\|_2^2$. Q.E.D.

LEMMA 1.2.6. *Let N and τ be as above, $h, k \in L^2(N, \tau)^+$. Then*

$$\int_{\mathbf{R}_+^*} \|\, u_{a^{1/2}}(h) - u_{a^{1/2}}(k)\,\|_2^2\, da \leqq \|\, h - k\,\|_2\, \|\, h + k\,\|_2\ .$$

Proof. As h and k are positive we have $u_a(h) = E_a(h)$ for each $a > 0$. So by Proposition 1.1 we can assume that $N = L^\infty(X, \mu)$, that $\tau = \mu$ and h and k are square integrable functions on X. We have $\int_{\mathbf{R}_+^*} |E_a(x) - E_a(y)|\, da = |x - y|$ for $x, y \in \mathbf{R}_+$, so by Fubini's theorem

$$\int_{\mathbf{R}_+^*} \|\, E_a(h^2) - E_a(k^2)\,\|_1\, da = \|\, h^2 - k^2\,\|_1\ .$$

As

$$|E_a(h^2) - E_a(k^2)| = |E_a(h^2) - E_a(k^2)|^2\ ,$$

we have

$$\|\, E_a(h^2) - E_a(k^2)\,\|_2^2 = \|\, E_a(h^2) - E_a(k^2)\,\|_1\ ,$$

so:

$$\int_{\mathbf{R}_+^*} \|\, E_a(h^2) - E_a(k^2)\,\|_2^2\, da = \|\, h^2 - k^2\,\|_1 \leqq \|\, h - k\,\|_2\, \|\, h + k\,\|_2\ .$$

Q.E.D.

Remark 1.2.7. One can show that there is no constant $K > 0$ such that the inequality

$$\int_{\mathbf{R}_+^*} \| E_a(h) - E_a(k) \|_1 \, da \leqq K \| h - k \|_1$$

holds for any $h, k \in L^1(N, \tau)$ (unless N is of bounded type I).

Proof of Theorem 1.2.2. We assume that $\| x_1 \|_2 = 1$. When $b \to 0+$, the $x_j^b = x_j E_b | x_j |$ converge in L^2 to x_j, so using 1.2.4 (3) we see that we can assume that all x_j and hence x_1 are of finite rank. Then replacing all x_j by vx_j where v is a unitary operator such that $vx_1 > 0$ we can assume, using 1.2.4 (1), that $x_1 > 0$. By 1.2.1:

$$\| x_j - | x_j | \|_2 \leqq \| x_j - x_1 \|_2 + \| | x_j | - | x_1 | \|_2 \leqq \varepsilon + 2\varepsilon^{1/2} \leqq 3\varepsilon^{1/2} .$$

So we get, by Lemma 1.2.5:

$$\int_{\mathbf{R}_+^*} \| u_{a^{1/2}}(x_j) - u_{a^{1/2}}(| x_j |) \|_2^2 \, da \leqq 9\varepsilon .$$

As $\| | x_j | - x_1 \|_2 \leqq 2\varepsilon^{1/2}$, Lemma 1.2.6 shows that:

$$\int_{\mathbf{R}_+^*} \| u_{a^{1/2}}(| x_j |) - u_{a^{1/2}}(x_1) \|_2^2 \, da \leqq (2\varepsilon^{1/2}) \times 3 = 6\varepsilon^{1/2} .$$

The inequality $\| x + y \|_2^2 \leqq 2 \| x \|_2^2 + 2 \| y \|_2^2$, $x, y \in L^2(N, \tau)$ shows that:

$$\int_{\mathbf{R}_+^*} \| u_{a^{1/2}}(x_j) - u_{a^{1/2}}(x_1) \|_2^2 \, da \leqq 18\varepsilon + 12\varepsilon^{1/2} \leqq 30\varepsilon^{1/2} .$$

Now let G be the decreasing function from $]0, +\infty[$ to $[0, +\infty[$ such that

$$G(a) = \tau(E_a(x_1^2)) = \tau(E_{a^{1/2}}(x_1)) = \| E_{a^{1/2}}(x_1) \|_2^2 = \| u_{a^{1/2}}(x_1) \|_2^2 .$$

As $\| x_1 \|_2^2 = 1$ we have

$$\int_{\mathbf{R}_+^*} G(a) da = \tau\Big(\int_{\mathbf{R}_+^*} E_a(x_1^2) da \Big) = \tau(x_1^2) = 1$$

so that $G(a)da$ is a probability measure on $]0, +\infty[$. Let

$$\mathcal{E}_j = \{ b > 0, \| u_{b^{1/2}}(x_j) - u_{b^{1/2}}(x_1) \|_2^2 > \varepsilon^{1/4} \| u_{b^{1/2}}(x_1) \|_2^2 \} .$$

For $a \in \mathcal{E}_j$ we have

$$G(a) < \varepsilon^{-1/4} \| u_{a^{1/2}}(x_j) - u_{a^{1/2}}(x_1) \|_2^2$$

so that

$$\int_{\mathcal{E}_j} G(a) da < \varepsilon^{-1/4} 30 \varepsilon^{1/2} = 30\varepsilon^{1/4} .$$

Let $\mathcal{V}$ be an open set in $\mathbf{R}_+$ containing 0, all $\mathcal{E}_j$, $j = 1, \cdots, n$ and with $\int_{\mathcal{V} \cap \mathbf{R}_+^*} G(a) da \leqq 30 n \varepsilon^{1/4}$. Then the smallest $b > 0$, $b \in \mathcal{V}^c$ satisfies

$$\int_{]0,b[} G(a)da \leqq 30n\varepsilon^{1/4} < 1 ;$$

hence $b < \infty$. Let the a required in 1.2.2 be $a = b^{1/2}$. As $b \notin \mathfrak{S}_j$ we have

$$\| u_a(x_j) - u_a(x_1) \|_2 \leqq \varepsilon^{1/8} \| u_a(x_1) \|_2 \qquad \text{for all } j = 1, \cdots, n .$$

Moreover for any $t \in \mathbf{R}_+$ one has

$$\int_0^b E_s(t^2)ds = t^2(1 - E_b(t^2)) + bE_b(t^2) .$$

Hence

$$\| x_1^2(1 - E_b(x_1^2)) \|_1 = \tau(x_1^2(1 - E_b(x_1^2))) \leqq \tau\Big(\int_0^b E_s(x_1^2)ds\Big)$$
$$= \int_0^b \tau(E_s(x_1^2))ds = \int_0^b G(s)ds \leqq 30n\varepsilon^{1/4} .$$

As $E_b(x_1^2) = E_a(x_1) = u_a(x_1)$, $\| x_1 \|_2 = 1$ we get:

$$\| x_1 - u_a(x_1)x_1 \|_2 \leqq 6n^{1/2}\varepsilon^{1/8} \| x_1 \|_2 . \qquad \text{Q.E.D.}$$

Proof of Corollary 1.2.3. Let $x_0 = (e_1 + e_2)^{1/2}$, $x_j = u_j x_0 u_j^*$, $j = 1, \cdots, n-1$. All the x_j are positive and we have:

$$\| u_j e_k u_j^* - e_k \|_1 \leqq 2\varepsilon \| e_k \|_1 \qquad \text{(Proposition 1.2.1)} ,$$
$$\| u_j x_0^2 u_j^* - x_0^2 \|_1 \leqq 2\varepsilon \| x_0^2 \|_1 ,$$
$$\| u_j x_0 u_j^* - x_0 \|_2 \leqq (2\varepsilon)^{1/2} \| x_0 \|_2 \qquad \text{(Proposition 1.2.1)} .$$

So the x_j form a set of diameter less than $4\varepsilon^{1/2} \| x_0 \|_2$. Let, by 1.2.2, $a > 0$ be such that

$$\| E_a(x_0) - E_a(x_j) \|_2 \leqq 6n(4\varepsilon^{1/2})^{1/8} \| E_a(x_0) \|_2 ,$$
$$\| x_0 E_a(x_0) - x_0 \|_2 \leqq 6n(4\varepsilon^{1/2})^{1/8} \| x_0 \|_2 .$$

Let $e = E_a(x_0)$. Then as $6n(4\varepsilon^{1/2})^{1/8} \leqq \delta/2$ we have

$$\| [e, u_j] \|_2 \leqq \delta \| e \|_2 , \qquad j = 1, \cdots, n-1 ;$$
$$\tau(x_0^2(1 - e)) \leqq \tfrac{1}{4}\delta^2\tau(x_0^2) ;$$
$$\tau(e_1(1 - e)) \leqq \tfrac{1}{4}\delta^2\tau(x_0^2) \leqq \delta^2\tau(e_1) ;$$
$$\| e_1(e - 1) \|_2 \leqq \delta \| e_1 \|_2 . \qquad \text{Q.E.D.}$$

I.3. *Representation of the ultraproduct of factors of type* II_1. Let $(N_k)_{k \in \mathbf{N}}$ be a sequence of factors with finite normalized traces τ_k. We define as in [17], p. 451 the ultraproduct $\prod_\omega N_k$, for an ultrafilter ω on $\mathbf{N}$, as the quotient of the product von Neumann algebra $\prod_1^\infty N_k$ by the 0-ideal of the trace τ_ω: $\tau_\omega((x_k)) = \lim_{k\to\omega} \tau_k(x_k)$. As in [17], p. 451 the ultraproduct is a factor with normalized trace τ_ω.

Let each N_k act canonically in $\mathcal{H}_k = L^2(N_k, \tau_k)$. Let $\mathcal{H}_\omega$ be the ultraproduct of the Hilbert spaces $\mathcal{H}_k$ for ω so that a $\xi \in \mathcal{H}_\omega$ is represented by a sequence: $(\xi_k)_{k \in \mathrm{N}}$, $\xi_k \in \mathcal{H}_k$, $\|\xi\| = \lim_\omega \|\xi_k\|$. Each bounded sequence of operators $A_k \in \mathcal{L}(\mathcal{H}_k)$ defines an operator $A \in \mathcal{L}(\mathcal{H}_\omega)$ by $A(\xi_k)_{k \in \mathrm{N}} = (A_k \xi_k)_{k \in \mathrm{N}}$. A sequence $(A_k)_{k \in \mathrm{N}}$ defines 0 if and only if $\lim_{k \to \omega} \|A_k\| = 0$. It follows that, in general, $\prod_\omega N_k$ *does not act* on $\mathcal{H}_\omega$. However $\prod_\omega N_k$ acts on a subspace of $\mathcal{H}_\omega$ identified by the following equi-integrability condition:

PROPOSITION 1.3.1. *Let N_k, $\mathcal{H}_k = L^2(N_k, \tau_k)$ be as above. Let ${}_\omega\mathcal{H}$ be the set of $\xi \in \mathcal{H}_\omega$ which satisfy, with $\xi = (\xi_k)_{k \in \mathrm{N}}$:*

(*) *For any $\varepsilon > 0$, there exists $a > 0$ such that*

$$\lim_{k \to \omega} \| E_a(|\xi_k|)\, |\xi_k| \|_2 < \varepsilon \,.$$

Then ${}_\omega\mathcal{H}$ is a closed subspace of $\mathcal{H}_\omega$ and $\prod_\omega N_k$ acts on ${}_\omega\mathcal{H}$ in a standard way with the vector $1 = (1)_{k \in \mathrm{N}}$ as cyclic and separating trace vector and the map $(\xi_k)_{k \in \mathrm{N}} \longrightarrow (J_k \xi_k)_{k \in \mathrm{N}}$ as canonical involution.

Proof. We just have to check that ${}_\omega\mathcal{H}$ is the closure in $\mathcal{H}_\omega$ of the set of vectors $(x_k)_{k \in \mathrm{N}}$, $\|x_k\|_\infty$ bounded. Assume that $\xi = (\xi_k)_{k \in \mathrm{N}}$ satisfies (*) and let $\varepsilon > 0$. Then for some $a > 0$ one has $\lim_{k \to \omega} \|\xi_k E_a\, |\xi_k|\,\|_2 < \varepsilon$ so that the vector η, $\eta_k = \xi_k(1 - E_a)\,|\xi_k|$ is at less than ε of ξ and satisfies $\|\eta_k\|_\infty \leqq a$ for all $k \in \mathrm{N}$. Conversely let $\varepsilon \in]0, 1[$ and $a > 0$ and assume that $\|\xi_k\|_2 \leqq 1$, $\|\xi_k - x_k\|_2 \leqq \varepsilon$ for all k, where $\|x_k\|_\infty \leqq a$ for all $k \in \mathrm{N}$. By 1.2.1

$$\|\, |\xi_k| - |x_k|\, \|_2 \leqq (3\varepsilon)^{1/2} \,.$$

Using Proposition 1.1 an easy computation gives $\|\, |\xi_k|\, E_{2a}(|\xi_k|)\,\|_2 \leqq 2(3\varepsilon)^{1/2}$.

Q.E.D.

In the special case $N_k = N$ for all $k \in \mathrm{N}$ we denote by N^ω the ultraproduct $\prod_\omega N$ and, as in [11], we denote by N_ω the relative commutant of N in N^ω where N is canonically imbedded in N.

I.4 *A technical lemma.* Most of the norms that one uses on a von Neumann algebra M satisfy the conditions:

$$\operatorname{norm}(a) = \operatorname{norm}(|a|)\,, \qquad \text{for } a \in M,\ a \text{ normal}\,;$$

$$a, b \in M^+,\ ab = ba \quad \text{and} \quad a \leqq b \text{ imply norm } a \leqq \operatorname{norm} b \,.$$

In particular all L^p norms $p \in [1, \infty]$ relative to a trace satisfy those conditions. This shows the interest of the following:

LEMMA 1.4. *Let M be a von Neumann algebra, $e, f \in M$ be two finite equivalent projections. Then there exists a unitary $W \in M$ such that:*

(a) $WeW^* = f$; (b) *W commutes with* $|e - f|$; (c) $|W - 1| \leqq 3\,|e - f|$.

Proof. Let c be the largest projection of the center of the von Neumann algebra N generated by e, f, such that N_c is of type I_2. As N_{1-c} is abelian, one can find projections e', e'', f', f'':

$$e'' = (1-c)e - e \wedge f\,, \quad e' + e'' = e\,, \quad f'' = (1-c)f - e \wedge f\,, \quad f' + f'' = f$$

such that $e' \wedge (1-f') = 0$, $(1-e') \wedge f' = 0$, that $e'', f'', e' \vee f'$ are pairwise orthogonal and that $|e-f| = e'' + f'' + |e'-f'|$.

Let u be the partial isometry of the polar decomposition of $f'e'$, then $|u - e'| \leqq \sqrt{2}\,|e'-f'|$. In fact one needs only to check it for one dimensional projections $e', f' \in M_2(\mathbf{C})$, in which case $|e'-f'|$ is the scalar $\sin\theta$ where $\theta \in [0, \pi/2[$ is the angle between e', f'. Moreover in this case $\|u-e'\| = 2\sin\theta/2$ so that the inequality is just $(\cos\theta/2)^{-1} \leqq \sqrt{2}$ for $\theta \in [0, \pi/2[$. One has $u^*u = e'$, $uu^* = f'$; let v satisfy $v^*v = e''$, $vv^* = f''$. Then $|v - e''| = \sqrt{2}\,e''$ since $e''f'' = 0$. Then $W_1 = u + v$ satisfies $W_1^* W_1 = e$, $W_1 W_1^* = f$, $[W_1, |e-f|] = 0$ and $|W_1 - e| \leqq \sqrt{2}\,|e-f|$.

As $(e \vee f - e) \sim (e \vee f - f)$ we get a partial isometry W_2 commuting with $|e-f|$ and satisfying

$$|W_2 - ((e \vee f) - e)| \leqq \sqrt{2}\,|e-f|,\ W_2^* W_2 = e \vee f - e,\ W_2 W_2^* = e \vee f - f\,.$$

It follows that, with $a = W_1 - e$, $b = W_2 - (e \vee f - e)$ one has:

$$|a + b| \leqq 2\sqrt{2}\,|e-f| \leqq 3\,|e-f|\,.$$

(If M acts in $\mathcal{H}$, one has $\|\,|a+b|\xi\| = \|a\xi + b\xi\| \leqq 2\sqrt{2}\,\|\,|e-f|\,\xi\|$ for any $\xi \in \mathcal{H}$ and as the square root is operator monotone one gets the above inequality.) Finally $W = (1 - e \vee f) + W_1 + W_2$ satisfies (a), (b), (c). Q.E.D.

II. Property Γ and the C^* algebra generated by a finite factor and its commutant

Let N be a finite factor, τ the canonical trace on N. We recall that N has property Γ of Murray and von Neumann if and only if for any $x_1, \cdots, x_m \in N$ and any $\varepsilon > 0$ there exists a unitary operator u, $\tau(u) = 0$ such that $\|[u, x_j]\|_2 \leqq \varepsilon$ for all j.

THEOREM 2.1. *Let N be a factor of type* II_1 *acting in a standard way in the Hilbert space $\mathcal{H} = L^2(N, \tau)$; let J be the canonical involution of the cyclic and separating vector $1 \in L^2(N, \tau)$.*

The following conditions on N are equivalent:

(a) *N has property* Γ;

(b) *For any finitely generated subgroup $\mathcal{G} \subset \mathrm{Int}\, N$ there is a non-normal $\mathcal{G}$-invariant state on N*;

(c) *For any unitary operators $u_1, \cdots, u_n \in N$ there is a sequence*

$$(\xi_k)_{k \in \mathbf{N}},\ \xi_k \in \mathcal{H},\ \|\xi_k\| = 1,\ (u_j - Ju_j^* J)\xi_k \xrightarrow[k \to \infty]{} 0$$

for all $j = 1, \cdots, n$ *but such that* $|\langle \xi_k, 1\rangle|$ *does not tend to* 1 *when* $k \to \infty$;

(d) *The* C^* *algebra* $C^*(N, N')$ *generated by* N *and* N' *in* $\mathcal{H}$ *contains no non-zero compact operator*: $C^*(N, N') \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

The possibility $C^*(N, N') \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$ was first shown in [2], by C. Akemann and P. Ostrand.

In [9] we defined a full factor as a factor M for which the inner automorphisms $\operatorname{Int} M$ form a closed subgroup of the group $\operatorname{Aut} M$ of automorphisms of M with the topology of norm pointwise convergence in M_*. We showed that for II_1 factors with separable predual, fullness is equivalent to the negation of property Γ. (See also [40].)

From 2.1 it follows that N is full if and only if $C^*(N, N') \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, and hence if and only if $\mathcal{K}(\mathcal{H}) \subset C^*(N, N')$ by irreducibility of $C^*(N, N')$.

COROLLARY 2.2. *Let* N *be a semi-discrete factor of type* II_1; *then* N *has property* Γ.

Proof. When N is semi-discrete the canonical map $\sum_{i=1}^n a_i \otimes b_i \to \sum_{i=1}^n a_i b_i$ from the algebraic tensor product $N \odot N'$ in $\mathcal{L}(\mathcal{H})$ extends to an isomorphism of the C^* tensor product of N by N' onto $C^*(N, N')$. By a theorem of M. Takesaki, as both N and N' are simple C^* algebras, so is their C^* tensor product ([44]), and hence so is $C^*(N, N')$. So

$$C^*(N, N') \cap \mathcal{K}(\mathcal{H}) \neq \{0\} \Longrightarrow C^*(N, N') = \mathcal{K}(\mathcal{H})$$

but this implies that $1 \in \mathcal{K}(\mathcal{H})$ so that $\mathcal{H}$ is finite dimensional. Q.E.D.

The next corollary was used in [18] but, however, the proof given there is false.

COROLLARY 2.3. *Let* N_1, N_2 *be factors of type* II_1, *then* $N_1 \otimes N_2$ *is full if and only if* N_1 *and* N_2 *are full.*

Proof. If N_1 or N_2 has property Γ so does $N_1 \otimes N_2$. Assume that both N_1 and N_2 are full and let N_j act in $\mathcal{H}_j$ as usual. By 3.1 (d) let $K_1 \in C^*(N_1, N_1') \cap \mathcal{K}(\mathcal{H}_1)$, $K_1 \neq 0$ and K_2 accordingly. As the C^* algebra generated by $N_1 \otimes N_2$, $N_1' \otimes N_2'$ contains $C^*(N_1, N_1') \otimes 1$ and $1 \otimes C^*(N_2, N_2')$, we see that it contains the non-zero compact operator $K_1 \otimes K_2$. Q.E.D.

We now prove some lemmas for the proof of Theorem 2.1.

LEMMA 2.4. *Let* N *be a factor of type* II_1 *satisfying* 2.1 (b), *then for any unitary operators* $u_1, \cdots, u_n \in N$ *and any* $\varepsilon > 0$ *there exists a non-zero projection* $e \in N$, $\tau(e) \leqq \varepsilon$ *such that*

$$\|[u_j, e]\|_2 \leqq \varepsilon \|e\|_2 , \qquad j = 1, \cdots, n .$$

Proof. By hypothesis there exists a singular state $\phi \in N^*$, invariant under the $\operatorname{Ad} u_j$, $j = 1, \cdots, n$ (use [41], 1.17.7). Let then, by [45], $f \in N$ be a projection with $\phi(f) = 1$, $\tau(f) \leqq \varepsilon$ ($\varepsilon < 1/2$). Let $\mathcal{V} = \{\psi$ state on N, $\psi(f) \geqq 1 - \varepsilon\} \subset N^*$. We identify the dual of N^n with $(N^*)^n$ by the duality

$$(\psi_1, \cdots, \psi_n)(x_1, \cdots, x_n) = \sum \psi_k(x_k) .$$

With this identification $(N_*)^n$ has N^n as dual Banach space and N^n has $(N^*)^n$ as dual Banach space.

Let

$$\mathcal{W} = \{(\psi - \psi \circ \operatorname{Ad} u_1, \cdots, \psi - \psi \circ \operatorname{Ad} u_n), \psi \in \mathcal{V} \cap N_*\} .$$

By construction $\mathcal{W}$ is a convex subset of $(N_*)^n$, and the closure of $\mathcal{W}$ in $(N^*)^n$, for the weak topology of $(N^*)^n$ corresponding to N^n, contains

$$(\phi - \phi \circ \operatorname{Ad} u_1, \cdots, \phi - \phi \circ \operatorname{Ad} u_n) = (0, \cdots, 0) ,$$

because ϕ belongs to the weak closure in N^* of $\mathcal{V} \cap N_*$.

So the weak closure of $\mathcal{W}$ in $(N_*)^n$ contains 0 and as $\mathcal{W}$ is convex the norm closure of $\mathcal{W}$ in $(N_*)^n$ contains 0.

Let $\psi \in N_* \cap \mathcal{V}$ satisfy $\| \psi \circ \operatorname{Ad} u_j - \psi \| \leqq \varepsilon$ for $j = 1, \cdots, n$. Let $\psi'(x) = (1/\psi(f))\psi(fxf)$. As $\psi(f) \geqq 1 - \varepsilon$, one has, for any $x \in N$,

$$\psi((1 - f)x) \leqq (\psi(1 - f))^{1/2} \| x \| \leqq \varepsilon^{1/2} \| x \| ,$$

so that $\| \psi' - \psi \| \leqq 3\varepsilon^{1/2}$. Let $h \in L^2(N, \tau)$, $h \geqq 0$, $\tau(h^2 \cdot) = \psi'$. We have support $h \leqq f$ and

$$(\| u_j h u_j^* - h \|_2)^2 \leqq \| \psi' - \psi' \circ \operatorname{Ad} u_j \| \leqq 7\varepsilon^{1/2} \tag{1.2.1}$$

Provided $3\varepsilon^{1/4} < (1/6n)^8$, there exists by Theorem 1.2.2 an $a > 0$ such that

$$u_a(h) \neq 0 \quad \text{and} \quad \| u_a(h_j) - u_a(h) \|_2 \leqq \delta \| u_a(h) \|_2 ,$$

where $\delta = 6n(3\varepsilon^{1/4})^{1/8} < 1$ and $h_j = u_j h u_j^*$. Let $e = u_a(h)$; then $u_a(h_j) = u_j e u_j^*$, so $\| [u_j, e] \|_2 \leqq \delta \| e \|_2$. Moreover $\tau(e) \leqq \tau(\text{support } h) \leqq \tau(f) \leqq \varepsilon$. But ε is arbitrary. Q.E.D.

LEMMA 2.5. *Let N be a II_1 factor satisfying* (b), *and f be a non-zero projection $f \in N$. Then N_f satisfies* (b).

Proof. Let $u_1, \cdots, u_n$ be unitary operators in N_f, $\bar{u}_j = u_j + 1 - f$. Let K be a type I_q subfactor of N with a minimal projection $e \leqq f$ and $\mathcal{G}$ the subgroup of $\operatorname{Int} N$ generated by the $\operatorname{Ad} \bar{u}_j$ and two inner automorphisms $\operatorname{Ad} v_j$, where the v_j generate K. Then as $e \leqq f$ the restriction of any singular $\mathcal{G}$-invariant state to N_f, is non-zero, singular, and invariant under the $\operatorname{Ad} u_j$. Q.E.D.

LEMMA 2.6. *Let N be a II_1 factor, ω a free ultrafilter on* $\mathbf{N}$, *N^ω as in* 1.3 *and $u_1, \cdots, u_n$ unitary operators in N. Then if the commutant of the*

u_j in N^ω is finite dimensional, there are unitary operators $u_{n+1}, \cdots, u_q \in N$ such that the commutant of the u_j, $j = 1, \cdots, q$ in N^ω is the scalars.

Proof. We just have to show that the commutant in N^ω of the u_j, $j = 1, \cdots, n$ is necessarily contained in N, if it is finite dimensional. Let us assume that some $x \in N^\omega$, $\|x\| \leqq 1$, $[x, u_j] = 0$ for all j, satisfies $\|x - y\|_2 \geqq \varepsilon > 0$ for any $y \in N$, $\|y\| \leqq 1$. Let $x = (x_k)_{k \in \mathbf{N}}$. We get, for any $y \in N$, $\|y\| \leqq 1$, that $\lim_{k\to\omega} \|x_k - y\|_2 \geqq \varepsilon$. So by induction on $p \in \mathbf{N}$ we construct sequences $(y_k^p)_{k \in \mathbf{N}}$ with

(1) $\|y_k^p - y_k^q\|_2 \geqq \varepsilon$ for all k and $q < p$;

(2) $\|y_k^p\| \leqq 1$ and $\|[y_k^p, u_j]\|_2 \leqq 1/k$ for all k and p.

Each time one takes, y_k^p among the x_m, $m \in \mathbf{N}$. Let $y^p \in N^\omega$ be represented by the sequence $(y_k^p)_{k \in \mathbf{N}}$. We see that all y^p belong to the unit ball of the commutant of the u_j in N^ω and as $\|y^p - y^q\|_2 \geqq \varepsilon$ for $p \neq q$ that this commutant is not finite dimensional. Q.E.D.

Proof of (a) $\Rightarrow$ (d) *in Theorem* 2.1. Let M be a factor acting in a Hilbert space $\mathcal{H}$. We shall prove that if there exists a central sequence $(v_k)_{k \in \mathbf{N}}$ of unitary operators of M which is not trivial ([41], Definitions 4.4.33 and 4.4.35, p. 213), one has:

$$C^*(M, M') \cap \mathcal{K}(\mathcal{H}) = \{0\} .$$

We can assume that for some $\xi_0 \in \mathcal{H}$, $\|\xi_0\| = 1$, and $\varepsilon > 0$, one has

$$\lim_{k\to\infty} |\langle v_k\xi_0, \xi_0\rangle| \leqq 1 - \varepsilon .$$

Let π, $\pi(\xi) = \langle \xi, \xi_0\rangle\xi_0$ for all $\xi \in \mathcal{H}$. Then if $C^*(M, M') \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$ we have $\pi \in C^*(M, M')$ by irreducibility of the identity representation of $C^*(M, M')$ in $\mathcal{H}$. Hence there exist $a_1, \cdots, a_n \in M$, $b_1, \cdots, b_n \in M'$ such that

$$\|\textstyle\sum_1^n a_j b_j - \pi\| \leqq \varepsilon/3 .$$

As $[a_j, v_k]b_j\xi_0 \xrightarrow[k\to\infty]{} 0$ for all j, we have

$$\|\textstyle\sum a_j b_j v_k \xi_0 - v_k \sum a_j b_j \xi_0\| \xrightarrow[k\to\infty]{} 0$$

so that

$$\|\textstyle\sum a_j b_j v_k \xi_0\| \xrightarrow[k\to\infty]{} \|\sum a_j b_j \xi_0\| .$$

However

$$\|\textstyle\sum a_j b_j \xi_0 - \pi\xi_0\| \leqq \varepsilon/3 \quad \text{and} \quad \pi\xi_0 = \xi_0 .$$

But

$$\|\textstyle\sum a_j b_j v_k \xi_0 - \pi(v_k\xi_0)\| \leqq \varepsilon/3 .$$

As $\|\pi(v_k\xi_0)\| = |\langle v_k\xi_0, \xi_0\rangle|$ we get a contradiction. Q.E.D.

Proof of (d) $\Rightarrow$ (c). We show that if (c) does not hold then $C^*(N, N')$ contains the one dimensional projection π of $L^2(N, \tau)$ on 1. As N does not satisfy (c) let $u_1, \cdots, u_n$ be the corresponding unitary operators. By adjoining the u_j^*, $j = 1, \cdots, n$ we see that $T = \sum_1^n u_j J u_j J$ can be assumed to be self-adjoint. Now $T \in C^*(N, N')$, $\| T \| = n$, $T(1) = n1$. We claim that $T - n$ is invertible on the orthogonal of the eigenvector 1. Otherwise, as T is self-adjoint, there would be a sequence $(\xi_k)_{k \in \mathbf{N}}$ in $\mathcal{H} = L^2(N, \tau)$, $\| \xi_k \| = 1$, $\langle \xi_k, 1 \rangle = 0$, such that $\| (T - n)\xi_k \| \to 0$. As $\| u_j J u_j J \xi_k \| = \| \xi_k \| = 1$ for all j, k, and as $\| T\xi_k \| \to n$, we see from the strict convexity of $\mathcal{H}$ that $\| u_i J u_i J \xi_k - u_j J u_j J \xi_k \| \to 0$ for all i and j, and hence $\| (u_j - J u_j^* J)\xi_k \| \to 0$ for all $j = 1, \cdots, n$, thus showing (c). We have shown that n is an isolated point and a simple point for the spectrum of T, so $C^*(N, N')$ contains a one dimensional projection. Q.E.D.

Proof of (c) $\Rightarrow$ (b). Let $u_1, \cdots, u_n$ be unitary operators in N. We want to find a non-normal state on N invariant under the $\operatorname{Ad} u_j$, $j = 1, \cdots, n$. Let ω be a free ultrafilter on $\mathbf{N}$ and N^ω the corresponding ultraproduct. By Lemma 2.6 we can distinguish two cases:

(1) The relative commutant of $u_1, \cdots, u_n$ in N^ω is $\mathbf{C}$. Let $(\xi_k)_{k \in \mathbf{N}}$ be a sequence of elements of norm 1 in $L^2(N, \tau)$ such that $\langle \xi_k, 1 \rangle = 0$ for all k and that $\| u_j J u_j J \xi_k - \xi_k \| \xrightarrow[k \to \infty]{} 0$ for all j. If $(\xi_k)_{k \in \mathbf{N}}$ satisfies condition (*) of Proposition 1.3.1, we get an element ξ of $L^2(N^\omega, \tau_\omega)$ with $\langle \xi, 1 \rangle = 0$, $\| \xi \| = 1$ and $u_j J u_j J \xi = \xi$ where $J = J_{N^\omega}$. But this contradicts the hypothesis on the relative commutant of the u_j in N^ω. So $(\xi_k)_{k \in \mathbf{N}}$ does not satisfy 1.3.1 (*). Hence for some $\varepsilon > 0$ we can find a subsequence $(\eta_k)_{k \in \mathbf{N}}$ of the sequence $(\xi_k)_{k \in \mathbf{N}}$, such that $\tau\big((E_k \mid \eta_k \mid^2) \mid \eta_k \mid^2\big) \geqq \varepsilon$ for all $k \in \mathbf{N}$ and E_a, $a > 0$, as in 1.2. For each k let $\phi_k \in N_*$ correspond to $\mid \eta_k \mid^2 \in L^1(N, \tau)$, and $e_k = E_k(\mid \eta_k \mid^2)$. Each ϕ_k is a state on N and by the choice of $(\xi_k)_{k \in \mathbf{N}}$ one has $\| [\phi_k, u_j] \| \xrightarrow[k \to \infty]{} 0$ for any $j = 1, \cdots, n$.

As $E_k(\mid \eta_k \mid^2) \mid \eta_k \mid^2 \geqq k E_k(\mid \eta_k \mid^2)$, we have $\tau(e_k) \leqq 1/k$ and hence $e_k \xrightarrow[k \to \infty]{} 0$ strongly. But $\phi_k(e_k) \geqq \varepsilon$ for all k, which shows that the set $(\phi_k)_{k \in \mathbf{N}}$ is not weakly relatively compact in $N_*\big([1], 2.3\big)$. Let ϕ be a non-normal element of the closure of $(\phi_k)_{k \in \mathbf{N}}$ in N^*, for the weak topology. Then ϕ is a state on N commuting with the u_j.

(2) The relative commutant of $u_1, \cdots, u_n$ in N^ω is infinite dimensional. Then this commutant must contain an infinite dimensional abelian von Neumann subalgebra, and hence contains non-zero projections e_k, $k \in \mathbf{N}$, with $\tau_\omega(e_k) \leqq 1/k$. As each projection in N is represented by a sequence of projec-

tions of N, we can find a sequence $(f_k)_{k\in N}$, of projections of N such that:

$$f_k \neq 0,\ \tau(f_k) \leqq 1/k,\ \|[u_j, f_k]\|_1 \leqq 1/k\, \|f_k\|_1 \qquad \text{for all } j, k\,.$$

Let then ϕ_k be the state on N corresponding to $\tau(f_k)^{-1}f_k \in L^1(N, \tau)$. Then exactly as above we find a weak limit ϕ of the ϕ_k in N^* showing that N satisfies (b).

Proof of (b) $\Rightarrow$ (a). Let $(u_j)_{j=1,\cdots,n}$ be a finite subset of the unitary group of N and let $\delta > 0$. We show that there exists a projection $e \in N$, $\tau(e) = 1/2$ such that $\|[u_j, e]\|_2 \leqq \delta$, for all j. Let $\mathcal{R}$ be the set of families $r = (E, U_1, \cdots, U_n)$ such that:

(α) E is a projection, $E \in N$, $\tau(E) \leqq 1/2$.

(β) Each U_j is a unitary operator in N commuting with E.

(γ) $\|U_j - u_j\|_1 \leqq \delta\tau(E)$, $j = 1, \cdots, n$.

Given two elements r, r' of $\mathcal{R}$ we write $r \leqq r'$ when

$$(1)\ E \leqq E', \quad (2)\ \|U'_j - U_j\|_1 \leqq \delta\tau(E' - E)\,, \quad j = 1, \cdots, n\,.$$

If $r \leqq r'$ and $E = E'$ then $r = r'$. If $r \leqq r' \leqq r''$ then $E \leqq E''$ and

$$\|U''_j - U_j\|_1 \leqq \delta\big(\tau(E'' - E') + \tau(E' - E)\big) = \delta\tau(E'' - E)$$

so $r \leqq r''$. We claim that $\leqq$ is inductive on $\mathcal{R}$. Any totally ordered subset of $\mathcal{R}$ has a cofinal sequence. We have to show that any increasing sequence $(r_n)_{n\in N}$ in $\mathcal{R}$ is majorized. We have $E_n \leqq E_{n+1}$ so that E_n converges strongly, when $n \to \infty$, to a projection $E \in N$, and $\tau(E) \leqq 1/2$. Now

$$\|U_{m,j} - U_{k,j}\|_1 \leqq \delta\tau(E_m - E_k)\,, \qquad k \leqq m\,,$$

so that $(U_{k,j})_{k\in N}$ is a Cauchy sequence in $L^1(N, \tau)$ which strongly converges to a unitary operator U_j. One has $[E, U_j] = \lim_{k\to\infty}[E_k, U_{k,j}]$ in the strong topology, so that the family $(E, U_1, \cdots, U_n) = r$ satisfies (α) and (β). We have

$$\|U_j - u_j\|_1 = \lim_{k\to\infty}\|U_{k,j} - u_j\|_1 \leqq \lim_{k\to\infty}\delta\tau(E_k) = \delta\tau(E)$$

so r satisfies (γ) and belongs to $\mathcal{R}$. For each k and j one has

$$\|U_{k,j} - U_j\|_1 = \lim_{m\to\infty}\|U_{k,j} - U_{m,j}\|_1 \leqq \lim_{m\to\infty}\delta\tau(E_m - E_k) = \delta\tau(E - E_k)$$

so $r_k \leqq r$ for all k, which proves the inductivity of $\mathcal{R}$.

By Zorn's lemma, let $r = (E, U_1, \cdots, U_n)$ be a maximal element in $\mathcal{R}$. We assume that $\tau(E) < 1/2$ and we contradict condition (b) of 2.1. Let $F = 1 - E$, $M = N_F$, $v_j = U_j(1 - E) = (1 - E)U_j \in M$ (using (β)). By Lemma 2.5, M has property 2.1 (b). Take $\varepsilon > 0$, $\tau(E) + \varepsilon < 1/2$, and $6\varepsilon \leqq \delta$. By Lemma 2.4 there exists a non-zero projection e, $e \in M$, $\tau'(e) < \varepsilon$, $\|[v_j, e]\|'_2 < \varepsilon\|e\|'_2$, where the $'$ corresponds to the normalization of τ in M. But, as

$\tau(F) \leqq 1$, we have $\tau(e) = \tau(F)\tau'(e) < \varepsilon$. As $\| v_j e v_j^* - e \|'_1 \leqq 2\varepsilon \| e \|'_1$, let w_j (Lemma 1.4) be a unitary operator in M such that

$$w_j v_j e v_j^* w_j^* = e \ , \quad \| w_j - F \|'_1 \leqq 6\varepsilon\tau'(e) \ .$$

Let $E' = E + e$; it is a projection strictly larger than E, such that $\tau(E') \leqq 1/2$. Let $U'_j = U_j E + w_j v_j$. As $w_j v_j$ is a unitary operator of $N_{(1-E)}$ and as $U_j E = E U_j$ we see that U'_j is a unitary operator in N. As $w_j v_j$ commutes with e, the U'_j commute with E'. For $x \in M$ we have $\| x \|'_1 = 1/\tau(F) \| x \|_1$, so we get $\| w_j - (1 - E) \|_1 \leqq \delta\tau(e)$ and:

$$\| w_j v_j - v_j \|_1 \leqq \delta\tau(e) \ , \qquad \| U'_j - U_j \|_1 \leqq \delta\tau(e) \ .$$

By hypothesis $\| U_j - u_j \|_1 \leqq \delta\tau(E)$ so that

$$\| U'_j - u_j \|_1 \leqq \delta\tau(E + e) = \delta\tau(E') \ .$$

We have shown that the family (E', U'_j) satisfies (α), (β), (γ) and hence defines an element r' of $\mathcal{R}$. The couple r, r' satisfies (1), (2) and as $e \neq 0$ we have contradicted the maximality of r. We have shown for each $\delta > 0$ the existence of a projection $E \in N$, $\tau(E) = 1/2$ such that E commutes with U_j, $\| U_j - u_j \|_1 \leqq \delta$, so that:

$$\| [u_j, E] \|_2 \leqq 2 \| U_j - u_j \|_2 + \| [U_j, E] \|_2 \leqq 2(2\delta)^{1/2}$$

because for $x \in N$, $\| x \|_\infty \leqq 2$ one has $\| x \|_2^2 \leqq 2 \| x \|_1$. As δ is arbitrary we have shown that N satisfies property Γ. Q.E.D.

III. A characterization of approximately inner automorphisms

Let N be a factor of type II_1 with normalized trace τ. We let $\operatorname{Aut} N$, the automorphism group of N, be gifted with the topology of strong pointwise convergence in N, which, as N is finite, is the same as the topology of norm pointwise convergence in N_*.

We characterize the closure $\overline{\operatorname{Int}} N$ of the subgroup $\operatorname{Int} N$ of inner automorphisms by the following theorem:

THEOREM 3.1. *Let N be a factor of type* II_1 *with separable predual, acting in* $\mathcal{H} = L^2(N, \tau)$. *Then the following conditions are equivalent for* $\theta \in \operatorname{Aut} N$:

(a) $\theta \in \overline{\operatorname{Int}} N$;

(b) *There exists an automorphism of the C^* algebra generated by N and N' in $\mathcal{H}$ which is θ on N and identity on N';*

(c) *For any unitary operators $u_1, \cdots, u_n \in N$ and any $\varepsilon > 0$ there is a $\xi \in \mathcal{H}$, $\| \xi \| = 1$, $\| \theta(u_k) J u_k J \xi - \xi \| < \varepsilon$ for all $k = 1, \cdots, n$;*

(d) *There exists a bounded sequence $(x_n)_{n \in \mathbf{N}}$ in N, not converging* *strongly to* 0, *such that* $x_n a - \theta(a)x_n$ *converges to* 0 *strongly for any* $a \in N$.

The condition (b) means that for any $a_1, \cdots, a_k \in N$ and $b_1, \cdots, b_k \in N'$ one has $\|\sum_1^k \theta(a_j)b_j\| = \|\sum_1^k a_j b_j\|$ where the norms are operator norms in $\mathcal{H}$. The condition (c) means that for any unitaries $u_1, \cdots, u_n \in N$ and $\varepsilon > 0$ there is an $x \in N$, $\|x\|_2 = 1$ such that

$$\|\theta(u_k)x - xu_k\|_2 < \varepsilon .$$

COROLLARY 3.2. *Let* N *be a semi-discrete factor of type* II_1, *then* $\operatorname{Aut} N = \overline{\operatorname{Int}}\, N$.

Proof. Let $\theta \in \operatorname{Aut} N$, $a_1, \cdots, a_k \in N$, $b_1, \cdots, b_k \in N'$. By hypothesis, ([19]): $\|\sum_{i=1}^k a_i b_i\| = \|\sum_{i=1}^k a_i \otimes b_i\|$ where $a_i \otimes b_i$ acts on $\mathcal{H} \otimes \mathcal{H}$. But

$$\|\sum_{i=1}^k a_i \otimes b_i\| = \|\sum_{i=1}^k \theta(a_i) \otimes b_i\|$$

because $\theta \otimes 1$ is an automorphism of the C^* tensor product of N by N'. So θ satisfies 3.1 (b). Q.E.D.

At this point it is important to remark that conditions (a) and (b) make sense for arbitrary factors M acting standardly in a Hilbert space $\mathcal{H}$, but, by the above proof of 3.2, that they are not equivalent in this generality. In fact 3.2 shows that any automorphism of a semi-discrete, but not necessarily finite, factor satisfies (b). When M is the product factor of type II_∞ the automorphisms of M which do not preserve the trace do not satisfy (a).

COROLLARY 3.3. *Let* N_1, N_2 *be factors of type* II_1, $\theta_j \in \operatorname{Aut} N_j$. *Then* $\theta_1 \otimes \theta_2 \in \overline{\operatorname{Int}}\, N_1 \otimes N_2$ *if and only if* $\theta_j \in \overline{\operatorname{Int}}\, N_j$, $j = 1, 2$.

Proof. If $\theta_j \in \overline{\operatorname{Int}}\, N_j$ for all j, then easily $\theta_1 \otimes \theta_2 \in \overline{\operatorname{Int}}\, N_1 \otimes N_2$. Let N_j act standardly in $\mathcal{H}_j$ so that $N_1 \otimes N_2$ acts standardly in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $a_1, \cdots, a_k \in N_1$, $b_1, \cdots, b_k \in N_1'$. Then

$$a_1 \otimes 1, \cdots, a_k \otimes 1 \in N_1 \otimes N_2 , \quad b_1 \otimes 1, \cdots, b_k \otimes 1 \in (N_1 \otimes N_2)'$$

and also:

$$\|\sum a_i b_i\| = \|\sum (a_i \otimes 1)(b_i \otimes 1)\| .$$

As $\theta_1 \otimes \theta_2$ satisfies 3.1 (b) we have that:

$$\|\sum (\theta_1(a_i) \otimes 1)(b_i \otimes 1)\| = \|\sum (a_i \otimes 1)(b_i \otimes 1)\| .$$

Hence we have shown that θ_1 satisfies (b). Q.E.D.

The proof of Theorem 3.1 (of (c) $\Rightarrow$ (d)) relies essentially on the following lemma.

LEMMA 3.4. *Let* N, τ *be as above*, $\theta \in \operatorname{Aut} N$ *satisfying condition* (c), *and* $u_1, \cdots, u_n$ *be unitary operators in* N. *For any* $\varepsilon > 0$ *there exists a non-zero* *projection* $e \in N$ *and an* $x \in N$, $\|x\|_\infty \leqq 1$, $\|x\|_1 \geqq 1/4\tau(e)$, $xe = x$, $\theta(e)x = x$, *such that*:

$$\|[u_j, e]\|_1 \leqq \varepsilon \|e\|_1\,, \quad j = 1, \cdots, n\,,$$

and

$$\|xu_j - \theta(u_j)x\|_1 \leqq \varepsilon\tau(e)\,, \qquad j = 1, \cdots, n\,.$$

Proof. Let $\varepsilon' \in]0, 1/8[$, $20\varepsilon' < \varepsilon$, let $2\delta = (\varepsilon'/24(n+1))^{16}$, $\delta' = (\delta/6n)^8$. Let $\xi \in \mathcal{H} = L^2(N, \tau)$, $\|\xi\| = 1$ such that for all $j = 1, \cdots, n$ one has $\|\theta(u_j)Ju_jJ\xi - \xi\| \leqq \delta'$. By Theorem 1.2.2 there exists $a > 0$ such that the partial isometry $w = u_a(\xi)$ (if one uses 1.2.4) satisfies

$$\|\theta(u_j)wu_j^* - w\|_2 \leqq \delta \|w\|_2 \quad \text{and} \quad w \neq 0\,.$$

We get that $\|u_jw^*wu_j^* - w^*w\|_2 \leqq 2\delta\|w\|_2$ and that

$$\|\theta(u_j)ww^*\theta(u_j^*) - ww^*\|_2 \leqq 2\delta\|w\|_2\,.$$

Let $e_1 = w^*w$, $e_2 = \theta^{-1}(ww^*)$. We have $\|e_k\|_2^2 = \tau(e_k) = \tau(w^*w) = \|w\|_2^2$, and hence: $\|[u_j, e_k]\|_2 \leqq 2\delta\|e_k\|_2$ for all j and k.

So, by Corollary 1.2.3, let $e \leqq e_1 \vee e_2$ be a projection, $e \in N$, such that $\|[u_j, e]\|_2 \leqq \varepsilon'\|e\|_2$ for all j and that $\|ee_k - e_k\|_2 \leqq \varepsilon'\|e_k\|_2$. As $\varepsilon' < 1/2$ we have $e \neq 0$, $\|e\|_2 \geqq \|ee_k\|_2 \geqq 1/2\|e_k\|_2$. Let $x = \theta(e)we$. We have $\|x\|_\infty \leqq 1$,

$$\begin{aligned}\|x - w\|_2 &\leqq \|\theta(e)ww^* - ww^*\|_2 + \|w^*we - w^*w\|_2\\ &= \|\theta(e)\theta(e_2) - \theta(e_2)\|_2 + \|e_1e - e_1\|_2 \leqq 2\varepsilon'\|w\|_2.\end{aligned}$$

So

$$\|x - w\|_1 = \|(x - w)(e_1 \vee e_2)\|_1 \leqq 2\|x - w\|_2\|w\|_2 \leqq 4\varepsilon'\|w\|_1$$

and

$$\|x\|_1 \geqq (1 - 4\varepsilon')\tau(e_1) \geqq 1/2(1 - 4\varepsilon')\tau(e) \geqq 1/4\tau(e)\,.$$

Also $\|xu_j - \theta(u_j)x\|_2 \leqq (\delta + 4\varepsilon')\|w\|_2$ and, as above

$$\|xu_j - \theta(u_j)x\|_1 \leqq (\delta + 4\varepsilon')\|w\|_2 2\|e\|_2 \leqq \varepsilon\tau(e)\,. \qquad \text{Q.E.D.}$$

LEMMA 3.5. *Let* N, τ *be as above and* $\theta \in \operatorname{Aut} N$ *satisfying* 3.1(c). *Let* $f \in N$ *be a non-zero projection,* $v \in N$ *with* $vv^* = f$, $v^*v = \theta(f)$. *Then the automorphism* ${}_v\theta$, $x \in N_f \to v\theta(x)v^* \in N_f$ *also satisfies* 3.1(c).

Proof. First we assume that $f = 1$. For any unitary operator $u \in N$ and any $x \in N$ we have:

$$(vx)u - {}_v\theta(u)(vx) = v(xu - \theta(u)x)\,;$$

hence if θ satisfies 3.1(c) so does ${}_v\theta$ if $vv^* = v^*v = 1$.

Now, in general, let $m \in \mathbf{N}$, $1/m \leqq \tau(f)$ and $(e_{ij})_{i,j=1,\cdots,m}$ be a system of $m \times m$ matrix units in N with $f = e + e'$ where e, e' are projections, $e \geqq e_{11}$, belongs to the I_m factor of the e_{ij}, and $e' \leqq e_{22}$. By multiplying θ by an inner automorphism we can assume that $\theta(e_{ij}) = e_{ij}$ for all i, j and also that $\theta(e') = e'$. Then $\theta(f) = f$ and we can assume that $v = f$. Now let $u_1, \cdots, u_n$ be unitary operators in N_f, $\bar{u}_j = u_j + 1 - f$ the corresponding unitary operators of N and $\bar{u}_{n+1}$, $\bar{u}_{n+2}$ unitary operators generating the I_m factor of the e_{ij}. Let $(x_k)_{k \in \mathbf{N}}$ be a sequence of elements of N (the sequence is not necessarily bounded) such that $\| x_k \|_2 = 1$ and $\| x_k \bar{u}_s - \theta(\bar{u}_s) x_k \|_2 \xrightarrow[k \to \infty]{} 0$. It follows that $\| [e_{ij}, x_k] \|_2 \xrightarrow[k \to \infty]{} 0$, because $\theta(\bar{u}_{n+q}) = \bar{u}_{n+q}$, $q = 1, 2$.

So

$$\| e_{jj} x_k \|_2 - \| e_{11} x_k \|_2 \xrightarrow[k \to \infty]{} 0 \qquad \text{for all } j = 1, \cdots, m ,$$

because

$$\| e_{j1} x_k \|_2 = \| e_{11} x_k \|_2 \quad \text{and} \quad \| x_k e_{j1} \|_2 = \| x_k e_{jj} \|_2 .$$

Hence $\| e_{11} x_k \|_2^2 \longrightarrow 1/m$ when $k \longrightarrow \infty$. Let $y_k = f x_k f \in N_f$. As $e_{11} \leqq e \leqq f$ we get

$$\underline{\lim} \| y_k \|_2 \geqq \underline{\lim} \| e_{11} y_k e_{11} \|_2 = m^{-1/2}$$

because

$$\| e_{11} y_k e_{11} \|_2 = \| e_{11} x_k e_{11} \|_2 \longrightarrow m^{-1/2} \quad \text{when } k \longrightarrow \infty .$$

We have

$$y_k u_j - \theta(u_j) y_k = f\big(x_k \bar{u}_j - \theta(\bar{u}_j) x_k\big) f . \qquad \text{Q.E.D.}$$

Proof of (d) $\Rightarrow$ (a). Let ω be a free ultrafilter on $\mathbf{N}$, N^ω be the ultraproduct associated to N and ω as in I.2. Let F_2 be the algebra of 2×2 matrices over $\mathbf{C}$ and σ be the homomorphism of N in $N^\omega \otimes F_2$ such that $\sigma(x) = x \otimes e_{11} + \theta^\omega(x) \otimes e_{22}$, where as usual N is identified with its canonical image in N^ω.

Let $(y_n)_{n \in \mathbf{N}}$ be a bounded sequence in N representing $y \in N^\omega$; then the equality $\lim_{n \to \omega} y_n x - \theta(x) y_n = 0$, in the strong topology, for any $x \in N$, is equivalent to $y \otimes e_{21} \in P$, where P is the relative commutant of $\sigma(N)$ in $N^\omega \otimes F_2$.

From our hypothesis we want to deduce that $1 \otimes e_{11}$ is equivalent to $1 \otimes e_{22}$ in P. The relative commutant of N in N^ω is equal to N_ω with the notations of I.2, so the map $x \longrightarrow x \otimes e_{11}$ is an isomorphism of N_ω on the reduction of P by $1 \otimes e_{11}$. Let $a \in N_\omega$ be a non-zero projection, $\tau_\omega(a) = \alpha > 0$. Let $(a_n)_{n \in \mathbf{N}}$ be a representing sequence of projections, $\tau(a_n) \geqq \alpha$. Let $(y_k)_{k \in \mathbf{N}}$ be a bounded sequence of elements of N, and $\beta > 0$ such that: $\tau(y_k^* y_k) \geqq \beta$ for all k, $y_k x - \theta(x) y_k \xrightarrow[k \to \infty]{} 0$ strongly, for all $x \in N$. Now $(y_k^* y_k)_{k \in \mathbf{N}}$ is a central sequence and hence $\underline{\lim}_{k\to\infty}\tau(a_n^* y_k^* y_k a_n) \geqq \alpha\beta$, for $n \in \mathbf{N}$. Hence

$$\underline{\lim}_{k\to\infty} \| \theta(a_n) y_k a_n \|_2 \geqq \beta^{1/2}\alpha^{1/2} ,$$

because $\| \theta(a_n) y_k a_n - y_k a_n \|_2 \to 0$ as $k \to \infty$. For any $n \in \mathbf{N}$, take $k_n \geqq n$ with $\| \theta(a_n) y_{k_n} a_n \|_2 \geqq 1/2\beta^{1/2}\alpha^{1/2}$. The sequence $(y_{k_n})_{n\in\mathbf{N}}$ represents an element z of N^ω such that $z \otimes e_{21} \in P$ and that $(\theta^\omega(a) \otimes e_{22})(z \otimes e_{21})(a \otimes e_{11}) \neq 0$.

Then for each projection $a \in$ Center of N_ω, the central support of $a \otimes e_{11}$ in P is necessarily equal to the projection $a \otimes e_{11} + \theta^\omega(a) \otimes e_{22}$. It follows that all elements of the center of P are of the form

$$c = x \otimes e_{11} + \theta^\omega(x) \otimes e_{22},\ x \in \text{Center } N_\omega .$$

So for any such element c one checks that:

$$(\tau_\omega \times \mathrm{Tr})(c(1 \otimes e_{11})) = \tau_\omega(x) ,$$
$$(\tau_\omega \times \mathrm{Tr})(c(1 \otimes e_{22})) = \tau_\omega(\theta^\omega(x)) = \tau_\omega(x) .$$

We have shown that $1 \otimes e_{11}$ is equivalent to $1 \otimes e_{22}$ in P and so there is a unitary operator $u \in N^\omega$ with $u \otimes e_{21} \in P$.

Let $(u_n)_{n\in\mathbf{N}}$ be a representing sequence of unitary operators for u; then $\theta = \lim_{n\to\omega} \mathrm{Ad}\, u_n$ in Aut N. Q.E.D.

Proof of (a) $\Rightarrow$ (b). Let $\theta = \lim_{n\to\infty} \mathrm{Ad}\, u_n$ be an element of $\overline{\mathrm{Int}}\, N$. For each $n \in \mathbf{N}$, let α_n be the inner automorphism of the C^* algebra $C^*(N, N')$ implemented by u_n. For any element $x = \sum_1^k x_j y_j$ of the algebraic tensor product of N and N', the sequence $(\alpha_n(x))_{n\in\mathbf{N}}$ of elements of $\mathfrak{L}(\mathcal{H})$ converges strongly to $\sum_1^k \theta(x_j) y_j$.

So the automorphism $\theta \odot 1$ of the algebraic tensor product $N \odot N'$ is norm preserving for the norm of $\mathfrak{L}(\mathcal{H})$ and thus extends uniquely to the norm closure $C^*(N, N')$ as an automorphism of $C^*(N, N')$ satisfying the required conditions. Q.E.D.

Proof of (b) $\Rightarrow$ (c). Let $u_1, \cdots, u_n$ be unitary operators in N and consider the two operators

$$T = 1 + u_1 J u_1 J + \cdots + u_n J u_n J ,\quad S = 1 + \theta(u_1) J u_1 J + \cdots + \theta(u_n) J u_n J .$$

Our hypothesis on the automorphism θ and the norm preserving property of automorphisms of C^* algebras show that $\| S \| = \| T \|$. But $\| T \| = n + 1$ because the unit vector 1 in $\mathcal{H} = L^2(N, \tau)$ satisfies $u_j J u_j J 1 = u_j u_j^* = 1$ and because each term in the sum defining T is unitary. So $\| S \| = n + 1$. Hence for any $\varepsilon > 0$ we can find $\xi \in \mathcal{H}$, $\| \xi \| = 1$ such that

$$\| \xi + \theta(u_1) J u_1 J \xi + \cdots + \theta(u_n) J u_n J \xi \| \geqq (n + 1) - \varepsilon .$$

As $\| \theta(u_j) J u_j J \xi \| = 1$ for all j, we see, using the strict convexity of the unit ball of a Hilbert space, that for any $\eta > 0$, there exists a $\xi \in \mathcal{H}$, $\|\xi\| = 1$ such that $\|\theta(u_j)Ju_jJ\xi - \xi\| \leqq \eta$, $j = 1, \cdots, n$. Q.E.D.

Proof of (c) $\Rightarrow$ (d). Let $u_1, \cdots, u_n$ be unitary operators in N and $\delta > 0$. We show the existence of unitary operators $U_1, \cdots, U_n$ such that $\|U_j - u_j\|_1 \leqq \delta$, of a unitary operator $V \in N$ and of an element X of N, $\|X\|_\infty \leqq 1$, $\|X\|_1 \geqq 1/4$ such that:

$$\|XU_j - V\theta(U_j)V^*X\|_1 \leqq \delta\,, \quad i = 1, \cdots, n\,.$$

With $y = V^*X$ and replacing the U_j by u_j, we have

$$\|y\|_\infty \leqq 1\,, \quad \|y\|_1 \geqq 1/4\,, \quad \|yu_j - \theta(u_j)y\|_1 \leqq 3\delta\,,$$

so, then, assertion (d) follows from the separability of N_*.

We let $\mathcal{R}$ be the set of all families $r = (E, U_1, \cdots, U_n, V, X)$ of elements of N which satisfy the following conditions:

(1) E is a projection, the U_j are unitary operators commuting with E.

(2) $\|U_j - u_j\|_1 \leqq \delta\tau(E)$, $j = 1, \cdots, n$.

(3) $VV^* = E$, $V^*V = \theta(E)$, so that ${}_V\theta$ is an automorphism of N_E.

(4) $X \in N_E$, $\|X\|_\infty \leqq 1$, $\|X\|_1 \geqq 1/4\tau(E)$.

(5) $\|XU_j - {}_V\theta(U_jE)X\|_1 \leqq \delta\tau(E)$, $j = 1, \cdots, n$.

We define for $r, r' \in \mathcal{R}$ the relation $r \leqq r'$ by the conditions:

(a) $E \leqq E'$;

(b) $\|U'_j - U_j\|_1 \leqq \delta\tau(E' - E)$ for all j;

(c) $EV' = V'\theta(E) = V$, so that ${}_{V'}\theta(E) = E$;

(d) $X'E = EX' = X$.

We check that $r \leqq r'$, $E = E' \Rightarrow r = r'$ and that $r \leqq r' \leqq r''$ implies $r \leqq r''$. We want to show that $\mathcal{R}$ is inductive.

Any totally ordered subset of $\mathcal{R}$ contains a cofinal sequence. We have to show that any sequence $r^k = (E^k, U_1^k, \cdots, U_n^k, V^k, X^k)$, $r^k \leqq r^{k+1}$, of elements of $\mathcal{R}$ is majorized. We let $E = \lim_{k\to\infty} E^k$ strongly, $U_j = \lim_{k\to\infty} U_j^k$ in L^1 (and hence strongly) and we check (1), (2), (a), (b). The sequence V^k converges strongly because $(E^qV^k)_{k\in\mathbf{N}}$ and $\big(V^k\theta(E^q)\big)_{k\in\mathbf{N}}$ are stationary sequences for each $q \in \mathbf{N}$. The limit V is a partial isometry, with $VV^* = E$, $V^*V = \theta(E)$ and $E^qV = V\theta(E^q) = V^q$ for all $q \in \mathbf{N}$, so that we get (3) and (c). For the same reason the sequence X^k converges to an $X \in N_E$ such that $\|X\|_\infty \leqq 1$, $XE^q = E^qX = X^q$ for all $q \in \mathbf{N}$. So $\|X\|_1 \geqq \|X^q\|_1 \geqq 1/4\tau(E^q)$ for all q and hence $\|X\|_1 \geqq 1/4\tau(E)$.

For each $q \in \mathbf{N}$ we have

$$\|X^qU_j^q - V^q\theta(U_j^qE^q)V^{q*}X^q\|_1 \leqq \delta\tau(E^q)$$

so that by continuity we have $\| XU_j - V\theta(U_jE)V^*X\|_1 \leqq \delta\tau(E)$ which ends the proof of inductivity of $\mathcal{R}$.

Let $r = (E, U_1, \cdots, U_n, V, X)$ be a maximal element of $\mathcal{R}$ such that $E \neq 1$. We shall contradict condition (c) for θ, thus proving (c) $\Rightarrow$ (d). Let $F = 1 - E \neq 0$ and let Y be a partial isometry in N such that $YY^* = F$, $Y^*Y = \theta(F)$. By Lemma 3.5 the automorphism ${}_Y\theta$ of N_F also satisfies condition (c). Let $v_j = U_jF = FU_j$ be for each j the restriction of U_j to F $\big(U_j$ and E commute, by 1)$\big)$, so that the v_j are unitary operators in N_F. Let $\varepsilon > 0$ such that $3 \times 3\varepsilon \leqq \delta$.

By Lemma 3.4, there exists a non-zero projection $e \in N_F$ and an $x \in N_F$, $\| x \|_\infty \leqq 1$, $\| x \|_1 \geqq 1/4\tau(e)$, $xe = x$, ${}_Y\theta(e)x = x$ such that $\| v_jev_j^* - e \|_1 \leqq \varepsilon \| e \|_1$ for all j, and $\| xv_j - {}_Y\theta(v_j)x \|_1 \leqq \varepsilon\tau(e)$ for all j. By 1.4 there are unitary operators v'_j of N_F commuting with e and such that $\| v'_j - v_j \|_1 \leqq (1/3)\delta\tau(e)$.

Let Y' be a partial isometry of N_F with initial support ${}_Y\theta(e)$ and final support e. As the final support F of Y is larger than ${}_Y\theta(e)$, the initial support of $Y'Y$ is $Y^*{}_Y\theta(e)Y = \theta(e)$ and its final support is e. Let $E' = E + e$, $U'_j = U_jE + v'_j$, $V' = V + Y'Y$ and $X' = X + Y'x$. As $e \leqq 1 - E$, and as the v'_j are unitary operators of $N_{(1-E)}$ which commute with e, we have (1). Also

$$\| U_j - U'_j \|_1 = \tau\big(| U_j(1 - E) - v'_j |\big) = \tau(| v_j - v'_j |) \leqq \delta\tau(e) .$$

So the couple (r, r') in the obvious notation satisfies (a), (b) and r satisfies (2).

The initial support of V' is

$$V^*V + (Y'Y)^*(Y'Y) = \theta(E) + \theta(e) = \theta(E') ;$$

its final support is $E + e = E'$, and one gets (3) and (c). The final support of $Y'x$ is smaller than e so, as $xe = e$, one has $Y'x \in N_e$, $X + Y'x \in N_{E'}$, and $EX' = X'E = X$.

$$\| X + Y'x \|_\infty = \mathrm{Sup}\,(\| X \|_\infty, \| Y'x \|_\infty) \leqq 1 .$$
$$\| X + Y'x \|_1 = \| X \|_1 + \| Y'x \|_1 \geqq 1/4\tau(E) + 1/4\tau(e) ,$$

which gives (d) for r, r' and (4) for r'. We want to check (5), we have:

$$X'U'_j = (X + Y'x)(U_jE + v'_j) = XU_j + Y'xv'_j$$

for all j. Also

$$U'_jE' = (U_jE + v'_j)(E + e) = U_jE + v'_je$$

and

$$V'\theta(U_jE)V'^* = \big(V'\theta(E)\big)\theta(U_jE)\big(V'\theta(E)\big)^* = V\theta(U_jE)V^* .$$

As $V^*X' = V^*EX' = V^*X$, by (3) for r and (d), we get $V'\theta(U_jE)V'^*X' = V\theta(U_jE)V^*X$. We have

$$V'\theta(v_j'e)V'^* = \big(V'\theta(e)\big)\theta(v_j'e)\big(V'\theta(e)\big)^* = Y'Y\theta(v_j'e)(Y'Y)^* .$$

Also $V'\theta(x_j'e)Y'^*X' = Y'Y\theta(v_j'e)Y^*Y'^*Y'x$ because the initial support of Y'^* is e and $eX' = Y'x$. As $Y'^*Y'x = {}_Y\theta(e)x = x$ we get

$$V'\theta(v_j'e)V'^*X' = Y'Y\theta(v_j'e)Y^*x = Y'{}_Y\theta(v_j')x .$$

Hence

$$\begin{aligned}{}_{Y'}\theta(U_j'E')X' &= V'\theta(U_jE)V'^*X' + V'\theta(v_j'e)V'^*X' \\ &= V\theta(U_jE)V^*X + Y'Y\theta(v_j')Y^*x ,\end{aligned}$$

so that

$$X'U_j' - {}_{Y'}\theta(U_j'E')X' = \big(XU_j - {}_Y\theta(U_jE)X\big) + \big(Y'xv_j' - Y'{}_Y\theta(v_j')x\big) .$$

By (5) for r we have

$$\| XU_j - {}_Y\theta(U_jE)X\|_1 \leqq \delta\tau(E) .$$

As $\| v_j - v_j' \|_1 \leqq (1/3)\delta\tau(e)$ we have

$$\begin{aligned}\| xv_j' - {}_Y\theta(v_j')x \|_1 &\leqq 2\,\| x \|_\infty \| v_j - v_j' \|_1 + \| xv_j - {}_Y\theta(v_j)x \|_1 \\ &\leqq 2/3\,\delta\tau(e) + \varepsilon\tau(e) \leqq \delta\tau(e)\end{aligned}$$

since $\varepsilon < \delta/3$.

We have shown that $r' \in \mathcal{R}$ and that $r < r'$, so this contradicts the maximality of r. Q.E.D.

IV. Tensor product of centrally trivial automorphisms of finite factors

Let N be a factor of type II_1 with normalized trace τ.

DEFINITION 4.1. *Let θ be an automorphism of N. Then let $c(\theta)$ be the supremum of the set of positive real c such that for any elements $x_1, \cdots, x_n$ of N and any $\varepsilon > 0$, there exists $\xi \in \mathcal{H} = L^2(N, \tau)$, $\| \xi \| = 1$ such that $\| \theta(\xi) - \xi \| \geqq c$ while*

$$\| (x_j - Jx_j^*J)\xi \| \leqq \varepsilon \qquad \textit{for } j = 1, \cdots, n .$$

We have $0 \leqq c(\theta) \leqq 2$. In this definition one could restrict the x_j to belong to any subset of N which generates N as a C^* algebra, and in particular to the unitary operators of N.

PROPOSITION 4.2. *Let N and θ be as above.*

(a) *For any $\theta' \in \mathrm{Aut}\, N$, outer conjugate to θ, one has $c(\theta') = c(\theta)$.*

(b) *For any other finite factor M one has $c(\theta \otimes 1_M) \leqq c(\theta)$.*

(c) *For any θ-invariant non-zero projection $e \in N$, one has $c(\theta^e) = c(\theta)$.*

Proof. (a) Take $\theta' = \mathrm{Ad}\, u.\theta$; then, given $x_1, \cdots, x_n$ and $\varepsilon > 0$ take $\xi \in \mathcal{H}$, $\| \xi \| = 1$ with

$$\|\theta(\xi) - \xi\| \geqq c(\theta) - \varepsilon\,, \qquad \|(x_j - Jx_j^*J)\xi\| \leqq \varepsilon$$

for all j, and $\|(u - Ju^*J)\xi\| \leqq \varepsilon$. Then

$$\|\theta'(\xi) - \xi\| = \|\theta(\xi) - u^*Ju^*J\xi\| \geqq c(\theta) - 2\varepsilon$$

so that $c(\theta') \geqq c(\theta)$.

(b) We identify $\mathfrak{L} = L^2(N \otimes M, \tau_N \otimes \tau_M)$, with $\mathcal{H} \otimes \mathcal{K}$, where $\mathcal{H} = L^2(N, \tau_N)$, $\mathcal{K} = L^2(M, \tau_M)$. So $J_{\mathfrak{L}} = J_{\mathcal{H}} \otimes J_{\mathcal{K}}$ and for $x \in N$ one has

$$J_{\mathcal{H}}x^*J_{\mathcal{H}} \otimes 1 = J_{\mathfrak{L}}(x^* \otimes 1)J_{\mathfrak{L}}\,.$$

Also the unitary transformation of $\mathfrak{L}$ corresponding to $\theta \otimes 1_M$ is $\theta \otimes 1_{\mathcal{K}}$. We assume that $c(\theta \otimes 1) > 0$, and let $c > 0$ such that $c < c(\theta \otimes 1)$ and $\eta \in]0, 1[$. Next we let $x_1, \cdots, x_n \in N$ and $\varepsilon > 0$. Let $\delta > 0$ such that $n\delta < 1/4\,c^2(1 - \eta^2)$ and $\delta < \varepsilon^2$.

By hypothesis, let $\xi \in \mathfrak{L} = \mathcal{H} \otimes \mathcal{K}$ such that

$$\|\xi\| = 1\,, \quad \|(\theta \otimes 1)\xi - \xi\| \geqq c \quad \text{and} \quad \|(T_j \otimes 1)\xi\| \leqq \delta\,,$$

where $T_j = x_j - J_{\mathcal{H}}x_j^*J_{\mathcal{H}}$ for all j. Let $\mathcal{B}$ be an orthonormal basis of $\mathcal{K}$ and $(\xi_b)_{b \in \mathcal{B}}$ the components of ξ on $\mathcal{B}$. Let μ be the discrete measure on $\mathcal{B}$ such that $\mu(b) = \|\xi_b\|^2$ and:

$$E_j = \{b \in \mathcal{B},\ \|T_j\xi_b\|^2 \geqq \delta\,\|\xi_b\|^2\}\,,$$
$$G = \{b \in \mathcal{B},\ \|\theta(\xi_b) - \xi_b\|^2 \geqq (\eta c)^2\,\|\xi_b\|^2\}\,.$$

We have, for each j, that

$$\mu(E_j) = \textstyle\sum_{b \in E_j} \|\xi_b\|^2 \leqq \delta^{-1} \sum \|T_j\xi_b\|^2 \leqq \delta\,.$$

Also

$$\textstyle\sum_G \|\theta(\xi_b) - \xi_b\|^2 \leqq 4 \sum_G \|\xi_b\|^2 = 4\mu(G) \quad \text{and} \quad \sum_{G^c} \|\theta(\xi_b) - \xi_b\|^2 \leqq (\eta c)^2\,.$$

So we have

$$\eta^2c^2 + 4\mu(G) \geqq c^2,\ \mu(G) \geqq 1/4\,c^2(1 - \eta^2) > \textstyle\sum \mu(E_j)$$

so that $G \backslash \bigcup_{j=1}^n E_j$ is non-empty, thus showing that $c(\theta) \geqq \eta c$ and proving (b).

When M is finite dimensional the $x \otimes y$, $x \in N$, $y \in M$ generate $N \otimes M$ as a C^* algebra so that the equality $c(\theta \otimes 1) = c(\theta)$ follows easily.

(c) Let $\alpha \in]0, 1]$ and e be a projection in N with $\tau(e) = \alpha$. Let u be a unitary operator in N such that ${}_u\theta(e) = e$. Then by (a) the number $C(\alpha) = c({}_u\theta^e)$ just depends on θ and α, not u and e. The equality $c(\theta \otimes 1_K) = c(\theta)$ for finite dimensional K shows that $C(\alpha) = c(\theta)$ for any rational number α. Let $\delta < 1$, to show that $C(\alpha) \geqq \delta^3 c(\theta)$ we can assume that for some integer q one has $\delta^2 \leqq 1 - 1/q \leqq \alpha < 1$. Let K be a type I_q subfactor of N with matrix units e_{ij} and e a projection in N, $\tau(e) = \alpha$, $1 - e \leqq e_{11}$. We can assume that

$\theta(x) = x$ for all x in K, and that $\theta(e) = e$. Let $\varepsilon > 0$; $x_1, \cdots, x_n \in N_e$. Let $\xi \in N$, $\|\xi\|_2 = 1$, $\|[x_j, \xi]\|_2 \leqq \varepsilon$ for all j, such that $\|\theta(\xi) - \xi\|_2 \geqq \delta c(\theta)$.

As the $e_{ij} \in N$ we can also assume that $[e_{ij}, \xi] = 0$ for all $i, j = 1, \cdots, q$. So ξ and $\theta(\xi)$ belong to K'. As for $a \in K' \cap N$ we have

$$\|(1 - e_{11})a(1 - e_{11})\|_2 = \|(1 - e_{11})a\|_2 = \|1 - e_{11}\|_2 \|a\|_2 \geqq \delta \|a\|_2 .$$

We get:

$$e\xi e \in N_e , \quad \|e\xi e\|_2 \geqq \|(1 - e_{11})\xi(1 - e_{11})\|_2 \geqq \delta ,$$
$$\|[x_j, e\xi e]\|_2 = \|e[x_j, \xi]e\|_2 \leqq \varepsilon \qquad \text{for all } j = 1, \cdots, n ,$$
$$\|e\xi e - \theta(e\xi e)\|_2 = \|e(\xi - \theta(\xi))e\|_2 \geqq \|(1 - e_{11})(\xi - \theta(\xi))(1 - e_{11})\|_2 \geqq \delta^2 c(\theta) ,$$

because $\xi - \theta(\xi)$ commutes with K. If we replace $\|\ \|_2$ by the L^2 norm $\|\ \|_2'$ of N_e, we have $\|y\|_2 \leqq \|y\|_2' \leqq \delta^{-1}\|y\|_2$ for all $y \in N_e$ which shows that $c(\theta^e) \geqq \delta^3 c(\theta)$. Q.E.D.

We now state the main result of this section:

THEOREM 4.3. *Let θ be an automorphism of a factor of type* II_1, *N, and let $p = p_a(\theta)$ be the asymptotic period of θ* (i.e., *its period in* $\operatorname{Aut} N/CtN$) (cf. [12]). *Then*

$$c(\theta) = \operatorname{Sup}_{z \in \mathbf{C}, z^p = 1} |z - 1| .$$

For $p = 0$ the notation means $\operatorname{Sup}_{z \in \mathbf{C}, z^0 = 1} |z - 1| = 2$. From 4.3 it follows that $c(\theta) < \sqrt{3} \Leftrightarrow c(\theta) = 0 \Leftrightarrow \theta \in CtN$.

COROLLARY 4.4. *Let N_1, N_2 be factors of type* II_1, *θ_j be automorphisms of N_j, $j = 1, 2$. Then $\theta_1 \otimes \theta_2 \in Ct(N_1 \otimes N_2) \Leftrightarrow \theta_j \in CtN_j$, $j = 1, 2$.*

Proof. If $\theta_1 \notin CtN_1$ then easily $\theta_1 \otimes \theta_2 \notin Ct(N_1 \otimes N_2)$. If $\theta_1 \in CtN_1$ then by 4.2 and 4.3, $c(\theta_1 \otimes 1) = 0$ so, by 4.3, $\theta_1 \otimes 1 \in CtN_1 \otimes N_2$. But $Ct(N_1 \otimes N_2)$ is a subgroup of $\operatorname{Aut}(N_1 \otimes N_2)$. Q.E.D.

LEMMA 4.5. *Let N be a factor of type* II_1, *$\theta \in \operatorname{Aut} N$, $0 \leqq c < c(\theta)$ and $u_1, \cdots, u_n$ be unitary operators in N. For any $\varepsilon > 0$, there exists a non-zero projection $e \in N$ and an $x \in N_e$ such that:*

(1) $\|[e, u_j]\|_1 \leqq \varepsilon\|e\|_1$, $j = 1, \cdots, n$ *and* $\|\theta(e) - e\|_1 \leqq \varepsilon\|e\|_1$;

(2) $\|x\|_\infty \leqq 1$, $\|x\|_1 \geqq 2^{-4}\tau(e)$, $\|x\|_2 \geqq 1/4\|e\|_2$;

(3) $\|\theta(x) - x\|_2 \geqq c\|x\|_2$;

(4) $\|[x, u_j]\|_1 \leqq \varepsilon\tau(e)$.

Proof. Let $(v_k)_{k \in \mathbf{N}}$ be a countable family of unitary operators of N, invariant globally under θ and θ^{-1}, and containing the u_j, $j = 1, \cdots, n$. As $c < c(\theta)$, let for each $k \in \mathbf{N}$, $\xi_k \in \mathcal{H} = L^2(N, \tau)$, $\|\xi_k\| = 1$,

$$\|(v_q - Jv_q^* J)\xi_k\| \leqq 1/k\ , \quad q = 1, \cdots, k\ ,$$
$$\|\theta(\xi_k) - \xi_k\| \geqq c\ , \qquad \text{for all } k\ .$$

Let $\mathcal{K}$ be the subspace of the ultraproduct $\mathcal{H}_\omega$, corresponding to the free ultrafilter ω, of all $\xi = (\xi_k)_{k\in\mathbf{N}}$ such that $(v_q - Jv_q^*J)\xi_k \xrightarrow[k\to\omega]{} 0$ for all $q \in \mathbf{N}$. $\mathcal{K}$ is closed. By construction $\mathcal{K}$ reduces the unitary operator $(\xi_k)_{k\in\mathbf{N}} \to (\theta(\xi_k))_{k\in\mathbf{N}}$, and we let U be the reduced unitary operator. We have $\|U - 1\| \geqq c$.

As U is unitary on $\mathcal{K}$, we let $\lambda \in$ Spectrum U, $|\lambda - 1| \geqq c$. For any $\delta > 0$ there exists $\xi \in \mathcal{K}$, $\|\xi\| = 1$, $\|U\xi - \lambda\xi\| \leqq \delta$. Let $\delta_1 \in]0, 1/4[$, $20\delta_1 < \varepsilon$, $\delta_2 = 1/2(\delta_1/24(n+2))^{16}$, $2\delta = (\delta_2/6(n+2))^8$. Let $\xi \in L^2(N, \tau)$, $\|\xi\| = 1$, such that $\|(u_j - Ju_j^*J)\xi\| \leqq \delta$, for all j, and $\|\theta(\xi) - \lambda\xi\| \leqq \delta$.

By Theorem 1.2.2, let $a > 0$ such that, with $v = u_a(\xi)$, one has: $v \neq 0$ (because $\delta_2 < 1$), $\|u_j v u_j^* - v\|_2 \leqq \delta_2 \|v\|_2$, for all j, and $\|\bar{\lambda}\theta(v) - v\|_2 \leqq \delta_2 \|v\|_2$. Let $e_1 = v^*v$, $e_2 = vv^*$. We have

$$\|u_j v^* u_j^* u_j v u_j^* - v^* v\|_2 \leqq 2\delta_2 \|v\|_2$$

so that we get, for all k, j,

$$\|[e_k, u_j]\|_2 \leqq 2\delta_2 \|e_k\|_2 \quad \text{and} \quad \|\theta(e_k) - e_k\|_2 \leqq 2\delta_2 \|e_k\|_2\ .$$

Let (by 1.2.3), e be a projection, $e \leqq e_1 \vee e_2$ such that 4.5 (1) holds and that $\|ee_k - e_k\|_2 \leqq \delta_1 \|e_k\|_2$ for all k. Put $x = eve$ so $x \in N_e$,

$$\|x\|_\infty \leqq 1\ , \quad \|x - v\|_2 \leqq 2\delta_1 \|v\|_2\ , \quad \|x\|_2 \geqq 1/2\,\|v\|_2 \geqq 1/4\,\|e\|_2$$

and hence

$$\|x\|_1 \geqq \|x\|_2^2 \geqq 2^{-4}\tau(e)$$

which gives (2).

Also

$$\|[x, u_j]\|_2 \leqq (4\delta_1 + \delta_2)\|v\|_2 \leqq 5\delta_1 \|v\|_2 \leqq \varepsilon/2\,\|e\|_2\ ,$$

because $\|v\|_2 \leqq 2\|x\|_2 \leqq 2\|e\|_2$. So we get

$$\|[x, u_j]\|_1 \leqq \|[x, u_j]\|_2 \|e \vee u_j e u_j^*\|_2 \leqq \varepsilon \|e\|_2^2 = \varepsilon\tau(e)\ .$$

So x satisfies 4.5 (4). Finally

$$\|\theta(x) - \lambda x\|_2 \leqq (\delta_2 + 4\delta_1)\|v\|_2 \leqq 5\delta_1 \times 2\,\|x\|_2$$

so that 4.5 (3) holds with $c - \varepsilon/2$ instead of c. Q.E.D.

Proof of the inequality $c(\theta) \leqq \mathrm{Sup}_{z\in\mathbf{C}, z^p=1} |z - 1|$. Let $u_1, \cdots, u_n$ be unitary operators in N, $c < c(\theta)$, and $\delta > 0$. Let us show the existence of unitary operators $U_1, \cdots, U_n \in N$ such that $\|U_j - u_j\|_1 \leqq \delta$ for all j, of an $X \in N$, $\|X\|_\infty < 1$, $\|X\|_1 \geqq 2^{-4}$, $\|[X, U_j]\|_1 \leqq \delta$, and of a unitary operator $P \in N$, $\|P - 1\|_1 \leqq \delta$ such that $\|{}_P\theta(X) - X\|_2 \geqq c\,\|X\|_2$. We then get a non-trivial central sequence $(X_k)_{k\in\mathbf{N}}$, such that $\| \theta(X_k) - X_k \|_2 \geqq c \| X_k \|_2$ for all $k \in \mathbf{N}$. It follows that the unitary transformation V of $L^2(N_\omega, \tau_\omega)$ attached to the action θ_ω of θ on the asymptotic centralizer N_ω, satisfies $\| V - 1 \| \geqq c$. But by [12], Proposition 2.1.2 and Theorem 2.1.3 the spectrum of V is $\{z, z \in \mathbf{C}, z^p = 1\}$ where $p = p_a(\theta)$.

Now let $\mathcal{R}$ be the set of all families $(E, U_1, \cdots, U_n, X, P)$ where:

(1) E is a projection in N commuting with the unitary operators U_j.

(2) $\| U_j - u_j \|_1 \leqq \delta\tau(E)$ for all j.

(3) $X \in N_E$, $\| X \|_\infty \leqq 1$, $\| X \|_1 \geqq 2^{-4}\tau(E)$, $\| [X, U_j] \|_1 \leqq \delta\tau(E)$.

(4) P is unitary, $\| P - 1 \|_1 \leqq \delta\tau(E)$, $P\theta(E)P^* = E$.

(5) $\| P\theta(X)P^* - X \|_2 \geqq c \| X \|_2$.

Moreover an ordering on $\mathcal{R}$ is defined as follows: $r \leqq r'$ when

(a) $E \leqq E'$ and $\| U_j - U_j' \|_1 \leqq \delta\tau(E' - E)$ for all j;

(b) $X'E = EX' = X$;

(c) $EP' = P'\theta(E) = EP$ and $\| P' - P \|_1 \leqq \delta\tau(E' - E)$.

As above one checks that $\mathcal{R}$, $\leqq$ is an inductive ordered set. Let r be a maximal element, and assume that $F = 1 - E \neq 0$. By Proposition 4.2 (c), let c', $c < c' < c({}_P\theta^F)$ where ${}_P\theta^F$ is the restriction of ${}_P\theta$ to N_F (we use (4)). Let $\varepsilon > 0$, $2^4 \times 6 \times 3\varepsilon \leqq c'^2 - c^2$, $7\varepsilon \leqq \delta$; then, by Lemma 4.5 there exists a non-zero projection $e \in N_F$ such that:

$$\| [e, U_j] \|_1 \leqq \varepsilon \| e \|_1 \quad \text{for all } j \,; \quad \| {}_P\theta(e) - e \|_1 \leqq \varepsilon \| e \|_1$$

and an $x \in N_e$ satisfying the other conditions of 4.5 for ${}_P\theta^F$, and $c' > c$. By Lemma 1.4, let U_j' be a unitary operator of N such that

$$U_j'E = EU_j' = U_jE,\ U_j'e = eU_j' \quad \text{and} \quad \| U_j' - U_j \|_1 \leqq 3\varepsilon \| e \|_1 \,.$$

With $E' = E + e$ we have (1), (2) and (a). With $X' = X + x$ we have

$$\| [X', U_j'] \|_1 \leqq \| [X, U_j] \|_1 + \| [x, U_j] \|_1 + 6\varepsilon \| e \|_1 \leqq \delta\tau(E) + 7\varepsilon\tau(e) \leqq \delta\tau(E')$$

and we get (3) and (b). Now e and ${}_P\theta(e)$ both belong to N_F and there exists a unitary operator Q of N such that $QE = EQ = E$ and that $Q\,{}_P\theta(e)Q^* = e$, $\| Q - 1 \|_1 \leqq 3\varepsilon\tau(e)$. Let $P' = QP$. Then

$$EP' = EP,\ P'\theta(E) = QP\theta(E) = EP \quad \text{and} \quad \| P' - P \|_1 \leqq \delta\tau(E' - E)$$

which gives (c). We have

$$\| P' - 1 \|_1 \leqq \delta\tau(E'),\ P'\theta(E + e)P'^* = E + e \,,$$

hence (4).

$$\begin{aligned} \| P'\theta(X')P'^* - X' \|_2^2 &= \tau\big((P'\theta(X'^*)P'^* - X'^*)(P'\theta(X')P'^* - X')\big) \\ &= \tau\big(P'\theta(X'^*X')P'^* - X'^*P'\theta(X')P'^* \\ &\qquad - P'\theta(X'^*)P'^*X' + X'^*X'\big) \end{aligned}$$

so that

$$\| P\theta(X')P^* - X' \|_2^2 \leqq \| P'\theta(X')P'^* - X' \|_2^2 + 6 \times 3\varepsilon\tau(e) .$$

Now $X' = X + x$, $X \in N_E$, $x \in N_F$ and ${}_P\theta(E) = E$ so, as x satisfies 4.5(3) with respect to ${}_P\theta$ and c', we get:

$$\begin{aligned} \| {}_P\theta(X') - X' \|_2^2 &= \| {}_P\theta(X) - X \|_2^2 + \| {}_P\theta(x) - x \|_2^2 \\ &\geqq c^2 \| X \|_2^2 + c'^2 \| x \|_2^2 \geqq c^2 \| X' \|_2^2 + (c'^2 - c^2)2^{-4}\tau(e) . \end{aligned}$$

As $(c'^2 - c^2)2^{-4}\tau(e) \geqq 6 \times 3\varepsilon\tau(e)$ by the choice of ε, we have shown that $\| {}_{P'}\theta(X') - X' \|_2^2 \geqq c^2 \| X' \|_2^2$. We thus have contradicted the maximality of r. Q.E.D.

Proof of the inequality $c(\theta) \geqq \mathrm{Sup}_{z \in \mathbf{C}, z^p=1} | z - 1 |$. Let $z \in \mathbf{C}$, $z^p = 1$. Then as already seen above z is in the point spectrum of the unitary transformation of $L^2(N_\omega, \tau_\omega)$ associated to the action θ_ω of θ on the asymptotic centralizer of N at $\omega \in \beta\mathbf{N}\backslash\mathbf{N}$. Thus there exists a central sequence $(y_k)_{k \in \mathbf{N}}$ in N such that $\| \theta(y_k) - z y_k \|_2 \xrightarrow[k \to \infty]{} 0$ and $\| y_k \|_2 = 1$ for all k. Q.E.D.

V. All injective factors of type II_1 are isomorphic

Let $\mathcal{H}$ be a separable Hilbert space. By definition the centralizer of a non-necessarily normal state ϕ on $\mathcal{L}(\mathcal{H})$ is

$$\mathcal{L}(\mathcal{H})_\phi = \{x \in \mathcal{L}(\mathcal{H}),\ \phi(xy) = \phi(yx) \quad \text{for all } y \in \mathcal{L}(\mathcal{H})\} .$$

We let $\mathcal{H}^c$ be the Hilbert space whose underlying real vector space is the same as for $\mathcal{H}$ and such that the identity map $\xi \to \xi^c$ is an antilinear isometry. For $x \in \mathcal{L}(\mathcal{H})$ we let $x^c \in \mathcal{L}(\mathcal{H}^c)$ be such that $x^c\xi^c = (x\xi)^c$ for all $\xi \in \mathcal{H}$. Let Tr be the usual trace on $\mathcal{L}(\mathcal{H})$, so that for a projection e, $\mathrm{Tr}\, e$ is the dimension of the range of e. We denote by $\| x \|_{\mathrm{HS}}$, and $\| x \|_{\mathrm{Tr}}$ the Hilbert-Schmidt and trace norm of an $x \in \mathcal{L}(\mathcal{H})$: $\| x \|_{\mathrm{HS}} = \mathrm{Tr}(x^*x)^{1/2}$, $\| x \|_{\mathrm{Tr}} = \mathrm{Tr}| x |$. We endow the space $\mathcal{L}(\mathcal{H})_{\mathrm{HS}}$ of Hilbert-Schmidt operators of the scalar product $\langle x, y \rangle_{\mathrm{HS}} = \mathrm{Tr}(y^*x)$ and we note that $\mathcal{L}(\mathcal{H})_{\mathrm{HS}}$ thus becomes a Hilbert algebra. In particular:

$$\| ax \|_{\mathrm{HS}} \leqq \| a \| \, \| x \|_{\mathrm{HS}},\ \| xa \|_{\mathrm{HS}} \leqq \| a \| \, \| x \|_{\mathrm{HS}} , \qquad a \in \mathcal{L}(\mathcal{H}),\ x \in \mathcal{L}(\mathcal{H})_{\mathrm{HS}} .$$

Also to each normal state ϕ on $\mathcal{L}(\mathcal{H})$ corresponds a unique density matrix $\rho > 0$, $\| \rho \|_{\mathrm{Tr}} = \mathrm{Tr}(\rho) = 1$ such that

$$\mathrm{Tr}(\rho x) = \phi(x) \qquad \text{for } x \in \mathcal{L}(\mathcal{H}) .$$

In this setting the Powers-Stormer inequality [36], Lemma 4.1 shows that $\| \rho_\phi^{1/2} - \rho_\psi^{1/2} \|_{\mathrm{HS}}^2 \leqq \| \phi - \psi \|$ where ϕ and ψ are normal states on $\mathcal{L}(\mathcal{H})$.

The main result of this section is the next theorem, whose applications will be discussed in the next sections.

THEOREM 5.1. *Let R be the hyperfinite factor of type* II_1 *and N a factor of type* II_1 *acting standardly in $\mathcal{H}$ and with normalized trace τ. The following are equivalent*:

1. *N is isomorphic to R.*
2. *N is isomorphic to $N \otimes R$ and given $x_1, \cdots, x_n \in N$, $\varepsilon > 0$ there are $z_1, \cdots, z_n \in R$ and a unitary operator $X \in N \otimes R$ with*:

$$\| x_j \otimes 1 - X(1 \otimes z_j)X^* \|_2 \leqq \varepsilon\,, \qquad j = 1, \cdots, n\,.$$

3. *The symmetry σ_N, $x \otimes y \to y \otimes x$ is in $\overline{\mathrm{Int}}\,(N \otimes N)$.*
4. $\| \sum x_j y_j \|_{\mathcal{H}} = \| \sum x_j \otimes y_j \|_{\mathcal{H}\otimes\mathcal{H}}$, $x_1, \cdots, x_n \in N$, $y_1, \cdots, y_n \in N'$.
5. $| \tau(\sum a_i b_i^*) | \leqq \| \sum a_i \otimes b_i^c \|_{\mathcal{H}\otimes\mathcal{H}^c}$, $a_1, \cdots, a_n$, $b_1, \cdots, b_n \in N$.
6. *Given $x_1, \cdots, x_n \in N$ and $\varepsilon > 0$, there exists a non-zero finite dimensional projection $e \in \mathcal{L}(\mathcal{H})$ such that, for all j*:

$$\| [x_j, e] \|_{\mathrm{HS}} \leqq \varepsilon \| e \|_{\mathrm{HS}}\,, \quad | \tau(x_j) - \langle x_j e, e\rangle_{\mathrm{HS}}/\langle e, e\rangle_{\mathrm{HS}} | \leqq \varepsilon\,.$$

7. *N is contained in the centralizer of a state ϕ on $\mathcal{L}(\mathcal{H})$.*

The hard part is to go from 7. to 1.:

Proof of 1. $\Rightarrow$ 7. As the commutant of R has property P [42] there exists a projection P of norm one of $\mathcal{L}(\mathcal{H})$ onto $N = R$ such that $P(axb) = aP(x)b$, $a, b \in N$, $x \in \mathcal{L}(\mathcal{H})$ so $\tau \cdot P = \phi$ is a state on $\mathcal{L}(\mathcal{H})$ and $N \subset \mathcal{L}(\mathcal{H})_\phi$. Q.E.D.

Proof of 7. $\Rightarrow$ 6. We can assume that $x_1, \cdots, x_n$ are unitary operators. Assume that for any finite subset F of the unitary group of N and any $\delta > 0$, there exists a state $\psi_0 \in N_*$ such that $\| [\psi_0, u] \| \leqq \delta$, for all $u \in F$ while $| \psi_0(x_j) - \tau(x_j) | \geqq \varepsilon$ for some $j \in \{1, \cdots, n\}$. Then by the weak compactness of the state space of N (non-normal states) we could get a tracial state on N different from τ which is not possible.

So let $F = (u_j)_{j=1,\dots,p}$ and $\delta < \varepsilon$ be such that each x_k belongs to F and that, for any (normal) state ψ_0 on N:

$$\bigl(\| [\psi_0, u_j] \| \leqq 2\delta \text{ for all } j\bigr) \Longrightarrow | \psi_0(x_k) - \tau(x_k) | \leqq \varepsilon\,. \tag{5.2}$$

Let $\mathcal{L}(\mathcal{H})_*$ be the predual of $\mathcal{L}(\mathcal{H})$, let $\mathcal{L}(\mathcal{H})_*^p$ be the Banach space $\mathcal{L}(\mathcal{H})_*^p$ with norm $\| (\phi_1, \cdots, \phi_p) \| = \sum \| \phi_j \|$. Then

$$\sum \phi_j(x_j) = \langle (\phi_1, \cdots, \phi_p), (x_1, \cdots, x_p)\rangle$$

identifies the product von Neumann algebra $(\mathcal{L}(\mathcal{H}))^p$ with the dual of $\mathcal{L}(\mathcal{H})_*^p$.

Let W be the set of all $(\psi - \psi \cdot \mathrm{Ad}\, u_1, \cdots, \psi - \psi \cdot \mathrm{Ad}\, u_p)$, for ψ a normal state on $\mathcal{L}(\mathcal{H})$. Then W is a convex subset of $\mathcal{L}(\mathcal{H})_*^p$ and hence its weak and norm closures coincide. As by the bipolar theorem the set of normal states is weakly dense in the state space of $\mathcal{L}(\mathcal{H})$, there is a net $(\psi_\alpha)_{\alpha \in I}$ of normal states converging weakly to the state ϕ invariant under the $\mathrm{Ad}\, u_j$. So we have shown that the weak, and hence norm, closure of W contains $(0, \cdots, 0)$. Let then $\eta = (\delta/6p)^8$ and ψ be the normal state on $\mathcal{L}(\mathcal{H})$ with:

$$(5.3) \qquad \| \psi - \psi \cdot \mathrm{Ad}\, u_k \| \leqq \eta^2 , \quad k = 1, \cdots, p .$$

Let ρ be the unique positive Hilbert-Schmidt operator $\| \rho \|_{\mathrm{HS}} = 1$ such that $\psi(x) = \langle x\rho, \rho \rangle_{\mathrm{HS}}$, for $x \in \mathcal{L}(\mathcal{H})$. Then by (5.3) and the Powers-Stormer inequality:

$$(5.4) \qquad \| u_k \rho u_k^* - \rho \|_{\mathrm{HS}} \leqq \eta \| \rho \|_{\mathrm{HS}} = \eta .$$

Now let $\rho_k = u_k \rho u_k^*$ and apply Theorem 1.2.2 to get an $a \geqq 0$ such that, as $\delta < 1$,

$$(5.5) \qquad E_a(\rho) \neq 0 , \quad \| E_a(\rho_k) - E_a(\rho) \|_{\mathrm{HS}} \leqq \delta \| E_a(\rho) \|_{\mathrm{HS}} .$$

We $e = E_a(\rho)$ is a finite dimensional projection because $\mathrm{Tr}(e) < \infty$ and one has:

$$(5.6) \qquad \| [u_k, e] \|_{\mathrm{HS}} \leqq \delta \| e \|_{\mathrm{HS}} , \quad k = 1, \cdots, p .$$

Let ψ_0 be the normal state on N such that

$$(5.7) \qquad \psi_0(y) = \langle ye, e \rangle_{\mathrm{HS}} / \langle e, e \rangle_{\mathrm{HS}} = \mathrm{Tr}(eye)/\mathrm{Tr}(e),\ y \in N .$$

Then for each k, with $e_k = u_k^* e u_k$ and $y \in N$ one has

$$\psi_0(\mathrm{Ad}\, u_k(y)) = \langle y u_k^* e, u_k^* e \rangle_{\mathrm{HS}} / \langle e, e \rangle_{\mathrm{HS}} = \langle y e_k, e_k \rangle_{\mathrm{HS}} / \langle e, e \rangle_{\mathrm{HS}} ,$$

so, as $\| e_k - e \|_{\mathrm{HS}} \leqq \delta \| e \|_{\mathrm{HS}}$ we get: $\| \psi_0 - \psi_0 \cdot \mathrm{Ad}\, u_k \| \leqq 2\delta$, and by (5.2) we get the last inequality of 6. Q.E.D.

Proof of 6. $\Rightarrow$ 5. Let $a_1, \cdots, a_n \in N$, $\| a_j \| \leqq 1$ and $b_1, \cdots, b_n \in N$. Let $\varepsilon > 0$ be arbitrary and by 6., let $e \neq 0$, be a finite dimensional projection, such that:

$$(5.8) \qquad \| [b_j^*, e] \|_{\mathrm{HS}} \leqq \varepsilon \| e \|_{\mathrm{HS}} , \qquad j = 1, \cdots, n ,$$

$$(5.9) \qquad | \tau(\textstyle\sum a_j b_j^*) - \langle \sum a_j b_j^* e, e \rangle_{\mathrm{HS}} / \langle e, e \rangle_{\mathrm{HS}} | \leqq \varepsilon .$$

By (5.8) and $\| a_j \| \leqq 1$ we have $\| a_j(e b_j^* - b_j^* e) e \|_{\mathrm{HS}} \leqq \varepsilon \| e \|_{\mathrm{HS}}$; hence

$$| \langle (e a_j e)(e b_j e)^* e, e \rangle_{\mathrm{HS}} - \langle a_j b_j^* e, e \rangle_{\mathrm{HS}} | \leqq \varepsilon \| e \|_{\mathrm{HS}}^2 ,$$

and hence

$$| \langle \textstyle\sum a_j' b_j'^* e, e \rangle_{\mathrm{HS}} - \langle \sum a_j b_j^* e, e \rangle_{\mathrm{HS}} | \leqq n\varepsilon \| e \|_{\mathrm{HS}}^2 \qquad a_j' = e a_j e,\ b_j' = e b_j e .$$

Let $\mathcal{K} = e\mathcal{H}$ be the range of e and Q the reduced von Neumann algebra of $\mathcal{L}(\mathcal{H})$ by e. Then $\mathcal{K}$ is finite dimensional as Q, and $a_j', b_j' \in Q$. The normalized trace of any $x \in Q$ is

$$(5.10) \qquad \tau_Q(x) = \mathrm{Tr}(x)/\mathrm{Dim}\, e = \langle xe, e \rangle_{\mathrm{HS}} / \langle e, e \rangle_{\mathrm{HS}} .$$

So the above inequality, together with (5.9) gives:

(5.11) $$|\tau_q(\sum a'_j b'^*_j) - \tau(\sum a_j b^*_j)| \leqq (n+1)\varepsilon .$$

Let J be the isometric involution of $\mathcal{K} \otimes \mathcal{K}^c$ such that $J(\xi \otimes \eta^c) = \eta \otimes \xi^c$, $\xi, \eta \in \mathcal{K}$. Then for $x \in Q$ one has $J(x \otimes 1)J = 1 \otimes x^c$. Let m be the dimension of $\mathcal{K}$ and $\xi_1, \cdots, \xi_m$ an orthonormal basis for $\mathcal{K}$,

$$\xi = \frac{1}{\sqrt{m}} \sum \xi_j \otimes \xi^c_j \in \mathcal{K} \otimes \mathcal{K}_c .$$

Then $\|\xi\| = 1$ and $\langle x \otimes 1\xi, \xi\rangle = \tau_q(x)$ for all $x \in Q$.

Moreover $J = J_{\xi, Q\otimes 1}$, so that $J\xi = \xi$ and $J(x \otimes 1)\xi = (x^* \otimes 1)\xi$ for all $x \in Q$, as can be checked directly for the canonical matrix units e_{ij}: $\xi_j \longrightarrow \xi_i$, $i, j = 1, \cdots, m$ of Q.

Hence, we have

$$\langle \sum ((a'_j \otimes 1)J(b'_j \otimes 1)J)\xi, \xi\rangle = \langle((\sum a'_j b'^*_j) \otimes 1)\xi, \xi\rangle = \tau_q(\sum a'_j b'^*_j)$$

while it is also equal to $\langle(\sum a'_j \otimes b'^c_j)\xi, \xi\rangle$. Now, $\mathcal{K} \otimes \mathcal{K}^c$ is included in $\mathcal{H} \otimes \mathcal{H}^c$ and $\xi \in \mathcal{K} \otimes \mathcal{K}^c$ satisfies $(e \otimes e^c)\xi = \xi$, so we have, as $(e \otimes e^c)(\sum a_j \otimes b^c_j)(e \otimes e^c) = \sum a'_j \otimes b'^c_j$:

(5.12) $$\tau_q(\sum a'_j b'^*_j) = \langle \sum a_j \otimes b^c_j \xi, \xi\rangle .$$

As $\|\xi\| = 1$, this, with (5.11) completes the proof. Q.E.D.

Proof of 5. $\Rightarrow$ 4. (implicitly contained in [19], proof of Proposition 4.5).

Let $N \odot N'$ be the algebraic tensor product of N by N' and $N \otimes_{\min} N'$ the C^* algebra generated by $N \odot N'$ in $\mathcal{H} \otimes \mathcal{H}$. Let η be the homomorphism from $N \odot N'$ to $\mathfrak{L}(\mathcal{H})$ which satisfies

$$\eta(\sum_{i=1}^n x_i \otimes y_i) = \sum x_i y_i , \qquad x_i \in N,\ y_i \in N' .$$

Let ξ be a cyclic and separating trace vector of N, and J the corresponding involution. Then for $a_1, \cdots, a_n, b_1, \cdots, b_n \in N$:

$$\tau(\sum a_j b^*_j) = \langle \sum a_j b^*_j \xi, \xi\rangle = \langle \sum a_j J b_j J\xi, \xi\rangle$$

because as ξ is a trace vector $Jb_j\xi = b^*_j\xi$. So using 5. we have:

$$|\langle \sum_1^n a_j J b_j J\xi, \xi\rangle| \leqq \|\sum_1^n a_j \otimes J b_j J\|_{\mathcal{H}\otimes\mathcal{H}} .$$

As $JNJ = N'$ we get:

(5.13) $$|\langle \sum_1^n x_j y_j \xi, \xi\rangle| \leqq \|\sum_1^n x_j \otimes y_j\| , \qquad x_j \in N,\ y_j \in N' .$$

In other words the map from $N \odot N'$ to C which satisfies $w(A) = \langle \eta(A)\xi, \xi\rangle$ is bounded and hence extends uniquely to a state of $N \otimes_{\min} N'$.

So we have for any $A, B \in N \odot N'$, that:

(5.14) $$w(B^*A^*AB) \leqq \|A\|^2_{\min} w(B^*B) .$$

As η is a * homomorphism we have $w(A^*A) = \|\eta(A)\xi\|^2$ and by (5.14)

$$\|\eta(A)\eta(B)\xi\| \leqq \|A\|_{\min} \|\eta(B)\xi\| \qquad \text{for all } A, B \in N \odot N' .$$

As the $\eta(B)\xi$ are dense in $\mathcal{H}$ we get that $\|\eta(A)\|_{\mathcal{H}} \leqq \|A\|_{\min}$ and, as min is the minimal C^* norm on $N \odot N'$ we get the desired equality by [44], Theorem 2. Q.E.D.

Proof of 3. $\Leftrightarrow$ 4. By Corollary 3.2 above, by [19], Corollaries 4, 6 and Proposition 3.9 we have 4. $\Rightarrow$ 3.

Conversely, let N act in $\mathcal{H}$, $N \otimes N$ in $\mathcal{H} \otimes \mathcal{H}$, and σ be the symmetry of $N \otimes N$. By hypothesis $\sigma \in \overline{\mathrm{Int}}\, N \otimes N$ so, Theorem 3.1 shows that, for $a_1, \cdots, a_n \in N \otimes N$, $b_1, \cdots, b_n \in N' \otimes N'$ one has

$$\|\textstyle\sum a_i b_i\|_{\mathcal{H}\otimes\mathcal{H}} = \|\textstyle\sum \sigma(a_i) b_i\|_{\mathcal{H}\otimes\mathcal{H}} . \tag{5.15}$$

Let $x_1, \cdots, x_n \in N$, $y_1, \cdots, y_n \in N'$, $a_i = x_i \otimes 1$, $b_i = y_i \otimes 1$; then (5.15) means, as $\sigma(x_i \otimes 1) = 1 \otimes x_i$,

$$\|(\textstyle\sum x_i y_i) \otimes 1\|_{\mathcal{H}\otimes\mathcal{H}} = \|\textstyle\sum y_i \otimes x_i\|_{\mathcal{H}\otimes\mathcal{H}} . \qquad \text{Q.E.D.}$$

Proof of 3. $\Rightarrow$ 2. Let N be a factor of type II_1 satisfying 3. Then it satisfies 4. and by Corollary 2.2 it has property Γ. So $\mathrm{Int}\, N \neq \overline{\mathrm{Int}}\, N$ by [9], Theorem 3.1, p. 429. Let σ be the symmetry of $N \otimes N$. Let $\theta \in CtN$, then by Corollary 4.4 we have $\theta \otimes 1 \in CtN \otimes N$ and by [12], Lemma 2.2.2 we have $(\theta \otimes 1)\sigma(\theta \otimes 1)^{-1}\sigma^{-1} \in \mathrm{Int}\, N \otimes N$. So $\theta \otimes \theta^{-1}$ is inner and so is θ ([28]). Therefore we have $CtN = \mathrm{Int}\, N \neq \overline{\mathrm{Int}}\, N$ and hence, with ω a free ultrafilter on $\mathbf{N}$ the asymptotic centralizer N_ω is not abelian by [12], Theorem 2.2.1, (c)$\Rightarrow$ (d). So by [17], N is isomorphic to $N \otimes R$.

We now construct an approximate imbedding of N in R. Apparently such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups. However the construction below relies on condition 6. First observe that 4. implies 7. by [19], Corollary 4.6 and (5.10,) and the proof of 1. $\Rightarrow$ 7. We need some notation and lemmas.

Notation 5.16. For each $n \in \mathbf{N}$ let $\mathbf{Z}^{*n}$ be the free group on n generators $g_1, \cdots, g_n$. For $m \in \mathbf{Z}^{*n}$ let the length of m be the sum of the absolute values of the exponents of the g_i in the reduced form of m. Finally for unitary operators $u_1, \cdots, u_n$ in $\mathcal{H}$, let, for $m \in \mathbf{Z}^{*m}$, $u(m)$ be the unitary operator obtained in replacing each g_i by the corresponding u_i and finding the product in $\mathcal{L}(\mathcal{H})$.

So $m \longrightarrow u(m)$ is the homomorphism of $\mathbf{Z}^{*m}$ in $\mathcal{L}(\mathcal{H})$ such that $u^{g_i} = u_i$, $u = (u_1, \cdots, u_n)$.

LEMMA 5.17. *Let N be a finite factor satisfying 6., $u_1, \cdots, u_n$ be unitary operators of N, $k \in \mathbf{N}$ and $\mathcal{K}$ be the set of all words $m \in \mathbf{Z}^{*n}$ whose length is less than k. Then for any $\varepsilon > 0$ there exists a finite dimensional factor Q and unitary operators $v_1, \cdots, v_n \in Q$ such that:*

$$|\tau(u(m)) - \tau_q(v(m))| \leqq \varepsilon \qquad \text{for all } m \in \mathcal{K} ,$$

where $u = (u_1, \cdots, u_n)$, $v = (v_1, \cdots, v_n)$ and τ_q is the normalized trace of Q.

Proof. Let $\delta = \varepsilon/3k$ and using 6., let $e \in \mathcal{L}(\mathcal{H})$ be a non-zero, finite dimensional projection such that:

$$\| [e, u_j] \|_{\mathrm{HS}} \leqq \delta \| e \|_{\mathrm{HS}} , \quad j = 1, \cdots, n . \tag{5.18}$$

$$|\tau(u(m)) - \langle u(m)e, e\rangle_{HS}/\langle e, e\rangle_{\mathrm{HS}}| \leqq \varepsilon , \quad m \in \mathcal{K} .$$

Let $e_j = u_j e u_j^*$; then $\| e_j - e \|_{\mathrm{HS}} \leqq \delta \| e \|_{\mathrm{HS}}$ and, by Lemma 1.4 there are unitary operators w_j in $\mathcal{L}(\mathcal{H})$ with $w_j e_j w_j^* = e$ and $\| w_j - 1 \|_{\mathrm{HS}} \leqq 3\delta \| e \|_{\mathrm{HS}}$. Let $v'_j = w_j u_j$; then we have:

$$v'_j e = e v'_j , \quad \| v'_j - u_j \|_{\mathrm{HS}} \leqq \varepsilon/k \| e \|_{\mathrm{HS}} , \qquad \text{for all } j . \tag{5.19}$$

Let $v_j = v'_j e = e v'_j$. Then each v_j is a unitary operator of the finite dimensional factor Q reduced from $\mathcal{L}(\mathcal{H})$ by $e \in \mathcal{L}(\mathcal{H})$. The normalized trace of Q is $\tau_q(x) = \langle xe, e\rangle_{\mathrm{HS}}/\langle e, e\rangle_{\mathrm{HS}}$, $x \in Q$. By induction on the length of the word m one gets:

$$\| u(m) - v'(m) \|_{\mathrm{HS}} \leqq \text{length } (m)\frac{\varepsilon}{k} \| e \|_{\mathrm{HS}}$$

as a consequence of (5.19), because for unitary operators $a, b \in \mathcal{L}(\mathcal{H})$ one has:

$$\| a u_j b - a v'_j b \|_{\mathrm{HS}} \leqq \varepsilon/k \| e \|_{\mathrm{HS}} , \qquad \text{for all } j . \tag{5.20}$$

Now by (5.19), e commutes with all v'_j, so that $(v'(m))e = v(m)$ for all words m. So we get:

$$|\langle u(m)e, e\rangle_{\mathrm{HS}} - \langle v(m)e, e\rangle_{\mathrm{HS}}| \leqq \varepsilon \| e \|^2_{\mathrm{HS}} , \qquad m \in \mathcal{K} . \tag{5.21}$$

By 5.18 and the definition of τ_q we get the conclusion. Q.E.D.

LEMMA 5.22. *Let N satisfy condition 6., let ω be a free ultrafilter on $\mathbf{N}$ and R^ω the ultraproduct associated to the hyperfinite factor R. Then there exists a normal homomorphism of N into R^ω.*

Proof. Let $\mathcal{F} = \mathbf{Z}^{*\infty}$ be the free group with countably many generators $g_1, \cdots, g_n, \cdots$. Let τ be the normalized trace on N, τ_0 on R. Let $u_1, \cdots, u_n, \cdots$ be a sequence of unitary operators of N generating N as a von Neumann algebra. For each $k \in \mathbf{N}$ let $\mathcal{K}_k \subset \mathcal{F}$ be the set of all words $m \in \mathcal{F}$ involving only $g_1, \cdots, g_k$ and with length less than k.

By Lemma 5.17, for each k, let $v_1^k, v_2^k, \cdots, v_k^k$ be unitary operators of R

such that:

$$(5.23)\qquad |\tau(u(m)) - \tau_0(v^k(m))| \leqq 1/k\ , \qquad \text{for all } m \in \mathcal{K}_k\ , \quad \text{where } v^k = (v_1^k, v_2^k, \cdots, v_k^k, 1, 1, \cdots)\ .$$

For each $j \in \mathbf{N}$, let v_j be the unitary operator of R^ω represented by the sequence $(v_j^k)_{k \in \mathbf{N}}$. Then as each word $m \in \mathcal{F}$ belongs to all $\mathcal{K}_k$, $k \geqq k(m)$, we get, with $v = (v_1, \cdots, v_j, \cdots)$:

$$(5.24)\qquad \tau_{0,\omega}(v(m)) = \lim_{k\to\omega} \tau_0(v^k(m)) = \tau(u(m))\ , \qquad m \in \mathcal{F}\ .$$

For $x \in \mathfrak{A} =$ * algebra generated by the u_j, $j \in \mathbf{N}$, let $\pi(x) = \sum \lambda_m v(m)$, when $x = \sum \lambda_m u(m)$. If $\sum \lambda_m u(m) = 0$ then $\sum \bar{\lambda}_m \lambda_{m'} \tau(u(m)^* u(m')) = 0$ and hence, by (5.24) $\sum \bar{\lambda}_m \lambda_{m'} \tau_{0,\omega}(v(m)^* v(m')) = 0$, so that the definition of π is unambiguous. By construction π is a trace-preserving * homomorphism of $\mathfrak{A}$ into R^ω. So by [35], we can extend π to a trace-preserving * homomorphism of N into R^ω; π is then normal, injective. Q.E.D.

LEMMA 5.25. *Let N satisfy condition 6., then there exists, for each free ultrafilter ω on $\mathbf{N}$, a normal isomorphism θ of $N \otimes N$ in the ultraproduct $(N \otimes R)^\omega$ such that:*

(a) *For each $x \in N$, $\theta(x \otimes 1)$ is represented by the sequence $(x \otimes 1_R)_{\nu \in \mathbf{N}}$.*

(b) *For each $y \in N$, $\theta(1 \otimes y)$ is represented by a sequence of the form $(1_N \otimes z_\nu)_{\nu \in \mathbf{N}}$, $z_\nu \in R$.*

Proof. Let π_1 be the unique isomorphism of N in $(N \otimes R)^\omega$ such that $\pi_1(x) = (x \otimes 1_R)_{\nu \in \mathbf{N}}$, $x \in N$. Let π be as in Lemma 5.22 and π_2 the isomorphism of N into $(N \otimes R)^\omega$ such that $\pi_2(y) = (1_N \otimes (\pi(y))_\nu)_{\nu \in \mathbf{N}}$ where $(\pi(y))_{\nu \in \mathbf{N}}$ is a representing sequence for $\pi(y) \in R^\omega$, $y \in N$. By construction $\pi_1(N)$ and $\pi_2(N)$ are commuting subfactors of $(N \otimes R)^\omega$, and as we are in a finite factor, we can identify $\pi_1(N) \otimes \pi_2(N)$ with the subfactor of $(N \otimes R)^\omega$ generated by $\pi_1(N)$, $\pi_2(N)$. Then $\pi_1 \otimes \pi_2$ is an isomorphism of $N \otimes N$ into $(N \otimes R)^\omega$ satisfying (a) and (b). Q.E.D.

End of the proof of 3. $\Rightarrow$ 2. Let $\theta\colon N \otimes N \to (N \otimes R)^\omega$ be as in (5.25). Let $x_1, \cdots, x_n \in N$, $\varepsilon > 0$, and by the hypothesis on the symmetry $\sigma\colon N \otimes N \to N \otimes N$, let v be a unitary operator of $N \otimes N$ such that:

$$(5.26)\qquad \| x_j \otimes 1 - v(1 \otimes x_j)v^* \|_2 \leqq \varepsilon/2\ , \qquad j = 1, \cdots, n\ .$$

As θ preserves the L^2 norm and is a * homomorphism:

$$(5.27)\qquad \| \theta(x_j \otimes 1) - \theta(v)\theta(1 \otimes x_j)\theta(v)^* \|_2 \leqq \varepsilon/2\ , \qquad j = 1, \cdots, n\ .$$

Let $(X_\nu)_{\nu \in \mathbf{N}}$ be a representing sequence of unitary operators of $N \otimes R$ for $X = \theta(v) \in (N \otimes R)^\omega$. Let, for each j, $(z_j^\nu)_{\nu \in \mathbf{N}}$ be a sequence of elements of R such that $(1 \otimes z_j^\nu)_{\nu \in \mathbf{N}}$ represents $\theta(1 \otimes x_j)$. Then we have, by (5.27):

$$\lim_{\nu\to\omega} \| x_j \otimes 1 - X_\nu(1 \otimes z_j^\nu)X_\nu^* \|_2 \leqq \varepsilon/2 , \qquad \text{for all } j . \tag{5.28}$$

So for a suitable $\nu \in \mathbf{N}$ we have conclusion 2. Q.E.D.

Proof of 2. $\Rightarrow$ 1. Let N satisfy 2. and hence let $(N_k)_{k\in\mathbf{N}}$ be an increasing generating sequence of subfactors of N, all isomorphic to N, with relative commutant isomorphic to R and giving factorizations $N = N_k \otimes N_k'$. Let $x_1, \cdots, x_n \in N$, $\varepsilon > 0$ and let $k \in \mathbf{N}$, $x_1', \cdots, x_n' \in N_k$ such that $\| x_j - x_j' \|_2 \leqq \varepsilon/2$. We want to prove the condition of approximation called condition C ([34]), for $x_1, \cdots, x_n$ and ε. By the above argument we can assume that all x_j belong to $M \otimes \mathbf{C}$ in some factorization $N = M \otimes R$ of N, with M isomorphic to N. Let then $z_1, \cdots, z_n \in R$ and X unitary, $X \in M \otimes R$ with

$$\| x_j - X(1_M \otimes z_j)X^* \|_2 \leqq \varepsilon/2 , \qquad \text{for all } j = 1, \cdots, n .$$

By the choice of R, let Q be a finite dimensional subfactor of R, and $q_1, \cdots, q_n \in Q$ with $\| z_j - q_j \|_2 \leqq \varepsilon/2$ for all j. Then $X(\mathbf{C} \otimes Q)X^*$ is a finite dimensional subfactor of $M \otimes R = N$ and $\| x_j - X(1_M \otimes q_j)X^* \|_2 \leqq \varepsilon$, $j = 1, \cdots, n$. Q.E.D.

Remark 5.29. Let N and $\mathcal{H}$ be as in the hypothesis of (5.1), let $\mathcal{U}$ be the unitary group of N with discrete topology and π the identity representation of $\mathcal{U}$ in $\mathcal{H}$. Then the representation of $\mathcal{U}$ in $\mathcal{L}(\mathcal{H})_{\mathrm{HS}}$ defined by $\rho(u)(y) = uyu^*$ is unitarily equivalent to $\pi \otimes \pi^c$, the tensor product of π by its conjugate. So a reformulation of 7. $\Leftrightarrow$ 1. of Theorem 5.1 is:

(5.30) *N hyperfinite $\Leftrightarrow \pi \otimes \pi^c$ weakly contains the trivial representation* .

With the notations of (5.1) we also have, by 7. $\Leftrightarrow$ 1.:

$$\text{(5.31)} \quad N \textit{ hyperfinite} \Leftrightarrow \textit{for any unitary operators } u_1, \cdots, u_n \in N , \quad \left\| \sum_1^n u_j \otimes u_j^c \right\|_{\mathcal{H}\otimes\mathcal{H}^c} = n .$$

Note also that the last condition of (5.31) was the only one used in the proof of Corollary 3.2.

Remark 5.32. The proof of 5. $\Rightarrow$ 4. given above extends to infinite factors and shows that a factor M is semi-discrete if and only if some (and then all) faithful normal state φ on M admits a purification (in the sense of Powers-Størmer [36] and Woronowitcz [48]) $\tilde{\varphi}$ on the C^* tensor product of M by M^0.

Remark 5.33. A Banach algebra B is called amenable [4], [26] when $H^1(B, Y) = \{0\}$ for any dual B-bimodule Y. A von Neumann algebra M is called amenable when $H^1(M, Y) = \{0\}$ for any dual *normal* B-bimodule [27]. In [27], Johnson, Kadison, and Ringrose showed that all approximately finite dimensional von Neumann algebras are amenable.

We show below that if N is an amenable factor of type II_1 it satisfies condition 7. of Theorem 5.1.

Let us take N acting in $\mathcal{H} = L^2(N, \tau)$ (τ the canonical trace) with 1 as unit trace vector. Let X be the Banach space of linear functionals ψ on $\mathcal{L}(\mathcal{H})$ such that:

(a) For some $K < \infty$, $|\psi(xAy)| \leqq K \| x \|_2 \| A \| \| y \|_2$ for all $x, y \in N$, $A \in \mathcal{L}(\mathcal{H})$. (The smallest such K is denoted by $\| \psi \|_X$.)

(b) $\psi(x) = 0$ for all $x \in N$.

It is easy to see that the unit ball of X is weakly compact (for the weak topology $\sigma(X, \mathcal{L}(\mathcal{H}))$) and that X is naturally a dual normal N bimodule, with operations:

$$(x\psi)(A) = \psi(Ax) , \quad (\psi x)(A) = \psi(xA) ,$$

for $x \in N$, $A \in \mathcal{L}(\mathcal{H})$. Now let $D \in Z^1(N, X)$ be the derivation of N in X such that:

$$D(x)(A) = \langle (xA - Ax)1, 1 \rangle , \quad A \in \mathcal{L}(\mathcal{H}) .$$

Then to say that $D \in B^1(N, X)$ is trivially equivalent to the existence of a state ψ on $\mathcal{L}(\mathcal{H})$ satisfying condition 5.1.7.

Remark 5.34. Let us call a state φ on $\mathcal{L}(\mathcal{H})$, such that $N \subset (\mathcal{L}(\mathcal{H}))_\varphi$, a hypertrace for the factor N. Then the existence of a hypertrace on N implies that N is finite and by Theorem 5.1, that N is hyperfinite, which agrees with the terminology of Dixmier [15].

Remark 5.35. One can consider condition 6. of Theorem 5.1 as an analogue of Følner's condition for amenable discrete groups [21] and a hypertrace as an analogue of an invariant mean on such a group. Then the proof, given above, of 7. $\Rightarrow$ 6. is exactly analogous to the proof, given by Namioka, of Følner's theorem [23].

VI. Stability properties of the class of injective von Neumann algebras

This section is an exposition of definitions and results due to J. T. Schwartz [42], Choda and Echigo [6], [7], Hakeda and Tomiyama [25], Arveson [3], Effros and Lance [19] and Choi and Effros [5]. If some of the statements are new they are simple to deduce from the quoted papers, except perhaps for Corollary 6.9 (c). One considers the category of C^* algebras with units, the morphisms being the completely positive, unit preserving, linear maps.

A C^* algebra A is called injective when the following analogue of the Hahn-Banach theorem is true (cf. [19]):

For any C^* algebras $B \subset C$ and any morphism θ of B in A, there is an extension $\bar{\theta}$ of θ to C: $\bar{\theta}: C \to A$.

Arveson's theorem [3] shows that $\mathcal{L}(\mathcal{H})$ is injective for any $\mathcal{H}$.

Definition 6.1. A von Neumann algebra M is called injective when it is injective as a C^* algebra.

This definition ties up with the extension property of [25] by:

PROPOSITION 6.2. *A von Neumann algebra M in $\mathcal{H}$ is injective if and only if it is the range of a projection E of norm one from $\mathcal{L}(\mathcal{H})$.*

As in many categories, the injectivity of von Neumann algebras is related to the existence of solutions to certain equations, more precisely Choi and Effros prove in [5] that:

PROPOSITION 6.3. *For each n let $F_n = M_n(\mathbf{C})$. Let M be a von Neumann algebra in $\mathcal{H}$. Then M is injective if and only if for each $s = s^* \in M \otimes F_n$, each $\sigma = \sigma^* \in F_n$ such that $b \otimes \sigma \leqq s$ for some $b = b^* \in \mathcal{L}(\mathcal{H})$, there exists an $x \in M$ such that $x \otimes \sigma \leqq s$, $\|x\| \leqq \|b\|$, $x = x^*$.*

The next proposition shows that the injective von Neumann algebras form a monotone class.

PROPOSITION 6.4. *Let $\mathcal{H}$ be a Hilbert space.*

(a) *If M in $\mathcal{H}$, is an injective von Neumann algebra, then so is its commutant M'* (cf. [25]).

(b) *The weak closure of an ascending union of injective von Neumann algebras is injective* (cf. [19]).

(c) *Same as* (b) *with intersections of decreasing families.*

In particular, as each finite dimensional or each type I von Neumann algebra is injective so are the approximately finite dimensional ones (see [20] for equivalent definitions of this class).

PROPOSITION 6.5. *Let $\mathcal{H}$ be a separable Hilbert space, X, a standard Borel space with probability measure μ and $\alpha \to M(\alpha)$ a Borel map from X to von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$. Then $M = \int M(\alpha)d\mu(\alpha)$ (elements of M are classes of essentially bounded Borel functions $(x_\alpha)_{\alpha \in X}$, $x_\alpha \in M(\alpha)$ for all $\alpha \in X$) is injective if and only if almost all $M(\alpha)$, $\alpha \in X$ are injective.*

Proof. By the Hahn-Banach separation theorem, and by 6.3, if a von Neumann algebra N in $\mathcal{H}$ is not injective, there is $n \in \mathbf{N}$, $s = s^* \in N \otimes F_n$, $b \in \mathcal{L}(\mathcal{H})$, $b = b^*$, $\|b\| \leqq 1$, $\sigma = \sigma^* \in F_n$ and $\phi \in (\mathcal{L}(\mathcal{H}) \otimes F_n)_*^+$, $\varepsilon > 0$, such that:

$$(6.6) \quad b \otimes \sigma \leqq s\,, \quad \phi(x \otimes \sigma) \geqq \phi(s) + \varepsilon\,, \qquad \text{for any } x = x^* \in N,\ \|x\| \leqq 1\,.$$

Replacing s by $s + \varepsilon/2$ one shows that (6.6), with $\varepsilon/3$ instead of ε, is still satisfied for σ in a (norm) open subset $\mathcal{V}$ of the self adjoint part of F_n.

For each $n \in \mathbf{N}$ let $(\sigma_{n,j})_{j\in\mathbf{N}}$ be a norm dense sequence in the self adjoint part of F_n. Then N is not injective if and only if for some integers n, j, q, k, there exist $s = s^* \in N \otimes F_n$, $b = b^*$, $b \in \mathcal{L}(\mathcal{H})$, $\|b\| \leqq 1$, and $\phi \in (\mathcal{L}(\mathcal{H}) \otimes F_n)_*^+$ with

$$\|s\| \leqq k\,, \quad \|\phi\| \leqq k\,, \quad b \otimes \sigma_{n,j} \leqq s$$

and

$$\phi(x \otimes \sigma_{n,j}) \geqq \phi(s) + 1/q\,, \qquad \text{for any } x = x^* \in N,\ \|x\| \leqq 1\,.$$

Now, if $M(\alpha)$ fails to be injective for a non-negligible set of α's there are integers n, j, q, k such that the corresponding set of α's is non-negligible, and, say, is equal to X. Then for each $\alpha \in X$ we choose (using [10], Appendix V) a $b_\alpha \in \mathcal{L}(\mathcal{H})$, $\|b_\alpha\| \leqq 1$, $b_\alpha = b_\alpha^*$, an $s_\alpha = s_\alpha^* \in M(\alpha)$, $\|s_\alpha\| \leqq k$ and a $\phi_\alpha \in (\mathcal{L}(\mathcal{H}) \otimes F_n)_*^+$, with $\|\phi_\alpha\| \leqq k$ and such that the corresponding families are Borel, while

$$b_\alpha \otimes \sigma_{n,j} \leqq s_\alpha\,, \quad \phi_\alpha(x \otimes \sigma_{n,j}) \geqq \phi_\alpha(s_\alpha) + 1/q \tag{6.7}$$
$$\text{for all } x \in M(\alpha),\ x = x^*,\ \|x\| \leqq 1\,.$$

The above selection is possible because one needs to check (6.7) only for a strongly dense set of $x \in M(\alpha)$. Then let

$$b = (b_\alpha)_{\alpha\in X} \in \int_X \mathcal{L}(\mathcal{H})d\mu = P\,, \quad s = (s_\alpha)_{\alpha \in X} \in M \otimes F_n\,,$$

and $\phi \in (M \otimes F_n)_*^+$ be such that for any $y = (y_\alpha)_{\alpha\in X}$, $y_\alpha \in M(\alpha) \otimes F_n$, one has $\phi(y) = \int \phi_\alpha(y_\alpha)d\mu(\alpha)$. Then

$$b \otimes \sigma_{n,j} \leqq s\,, \quad \phi(x \otimes \sigma_{n,j}) \geqq \phi(s) + 1/q \qquad \text{for all } x \in M,\ x = x^*,\ \|x\| \leqq 1\,.$$

This shows that M is not injective, by 6.3. The inverse implication follows by a similar argument. Q.E.D.

PROPOSITION 6.8. *Let M be a von Neumann algebra generated by a von Neumann subalgebra N and a subgroup $\mathcal{G}$ of the normalizer of N. Then if N is injective and $\mathcal{G}$ is amenable as a discrete group M is injective.*

Proof. By 6.4 (a) we just have to show that M' is injective but M' is equal to $(N')^{\mathcal{G}}$, for the natural action of $\mathcal{G}$ on N'. As $\mathcal{G}$ is amenable, there exists a projection of norm one from N' to $(N')^{\mathcal{G}}$. One then applies 6.4 (a) and 6.2. Q.E.D.

COROLLARY 6.9. (a) *Any continuous representation π of an amenable locally compact group $\mathcal{G}$ is such that $(\pi(\mathcal{G}))''$ and $\pi(\mathcal{G})'$ are injective.*

(b) *The group measure space construction from a triple X, μ, $\mathcal{G}$ with $\mathcal{G}$ amenable discrete, gives an injective von Neumann algebra.*

(c) *Any representation π of a separable connected locally compact group G is such that $\pi(G)'$ and $\pi(G)''$ are injective.*

Proof. (a) and (b) follow trivially by (6.6). To get (c) one can restrict to factor representations by 6.5, then to real connected Lie groups by [31], Proposition 2.2, and finally apply (6.6) using [16], Proposition 1.7 and Proposition 2.3. Q.E.D.

We are grateful to J. Dixmier for pointing out to us the proof of 6.7 (c).

VII. The classification of injective factors

We restrict our attention to factors acting in a separable Hilbert space. The types I_n, $n < \infty$ and I_∞ are well known; there is up to isomorphism only one factor in each of those types.

THEOREM 7.1. *Up to isomorphism, the hyperfinite factor is the only injective factor of type* II_1; *all its subfactors are hyperfinite and hence classified by their type* (I_n, $n < \infty$, *or* II_1).

Proof. Let N be an injective factor of type II_1 acting standardly in a separable Hilbert space $\mathcal{H}$. Let E be a projection of norm 1 of $\mathcal{L}(\mathcal{H})$ onto N; then by classical results of Tomiyama [47], it is a conditional expectation:

$$E(axb) = aE(x)b,\ x \in \mathcal{L}(\mathcal{H}),\ a,\ b \in N\ .$$

So $\tau \circ E$ (τ the normalized trace of N) satisfies condition 7. of Theorem 5.1:

$$\phi(uxu^*) = \phi(x),\ x \in \mathcal{L}(\mathcal{H})\ ,\quad u \text{ unitary in } N\ . \qquad \text{Q.E.D.}$$

By Theorem 7.1, R is the only infinite dimensional factor which is contained in all others. The terminology hyperfinite is also justified by condition 7. of Theorem 5.1: an infinite dimensional factor is the hyperfinite factor R if and only if there exists on $\mathcal{L}(\mathcal{H})$ a state invariant under inner automorphisms of R, a strengthening of the condition of existence of a trace on R.

COROLLARY 7.2. *Let $\mathcal{G}$ be a discrete countable group with infinite conjugacy classes; then the left regular representation of $\mathcal{G}$ generates the hyperfinite factor R if and only if $\mathcal{G}$ is amenable.*

Proof. If $\mathcal{G}$ is amenable, apply 6.7 (a) and 7.1. Otherwise see [42].

Q.E.D.

Combining Theorem 7.1 and Proposition 6.5, we see that any von Neumann subalgebra M of R is a direct integral of hyperfinite factors or of type I_n, $n < \infty$, factors. This observation has the following important consequence:

COROLLARY 7.3. *Let $\mathcal{S}$ be a self-adjoint subset of R. Then the von Neumann subalgebra of R generated by $\mathcal{S}$ is*: $M = \{x \in R,\ [x, y_n] \xrightarrow[n\to\infty]{} 0$ *strongly, for any bounded sequence of elements of R such that* $[y_n, s] \xrightarrow[n\to\infty]{} 0$ *strongly for* $s \in \mathcal{S}\}$.

Proof. Let N be the von Neumann algebra generated by $\mathcal{S}$. We have $N \subset M$. Let $(N_k)_{k \in \mathbf{N}}$ be an increasing sequence of finite dimensional subalgebras of N, generating N. Assume that for some $\varepsilon > 0$ and $y \in M$ one has $\| y - x \|_2 \geqq \varepsilon$ for all $x \in N$. Then as the relative commutant of $N_k' \cap R$ in R is equal to N_k for each $k \in \mathbf{N}$ (because N_k is finite dimensional), there exists a unitary $u_k \in N_k' \cap R$ such that $\| u_k y u_k^* - y \|_2 \geqq \varepsilon$ (because the strongly closed convex hull of the uyu^*, u unitary in $N_k' \cap R$, contains an element of $(N_k' \cap R)'$ by standard arguments). Then the sequence $(u_k)_{k \in \mathbf{N}}$ satisfies $[u_k, s] \xrightarrow[k\to\infty]{} 0$ strongly, for $s \in \mathcal{S}$ but $[u_k, y] \not\to 0$ strongly. Q.E.D.

By Proposition 6.4 the next theorem shows that up to isomorphism there is only one factor of type II_∞ which is the weak closure of an ascending union of finite dimensional von Neumann algebras. The terminology "hyperfinite" to designate the last property is inadequate; we shall follow [20] and call it "approximately finite dimensional".

THEOREM 7.4. *Let $R_{0,1} = R \otimes \mathcal{L}(\mathcal{H})$. Then up to isomorphism $R_{0,1}$ is the only injective factor of type* II_∞, *hence the only approximately finite dimensional one.*

Proof. Let $M = N \otimes \mathcal{L}(\mathcal{H})$ be injective, with N of type II_1. Then using a conditional expectation of M on N one gets that N is injective (by 6.2) and hence 7.1 applies. Q.E.D.

Combining Corollary 6.7 (c) and the theorem of Dixmier-Pukanzky ([37]) one gets that $R_{0,1}$ is, with $\mathcal{L}(\mathcal{H})$, the only factor which appears in the direct integral decomposition of the von Neumann algebra generated by the left regular representation of a connected separable locally compact group.

THEOREM 7.5. *Let M be an injective factor of type* III_0; *then M is a Krieger's factor*, i.e., *it is the cross product of an abelian von Neumann algebra by an ergodic automorphism.*

Proof. Let $M = W^*(N, \theta)$ be a discrete decomposition of M. By construction N is the centralizer of a weight on M and hence is injective as M. So, combining 6.5 with 7.4, we find that $R_{0,1}$ is the only factor occurring in the direct integral decomposition of N. Hence Theorem II.1 [10] applies.

Q.E.D.

COROLLARY 7.6. *Two injective factors of type* III_0 *are isomorphic if and only if their flows of weights are isomorphic.*

This is a known result of W. Krieger [29] combined with 7.5 and [13].

THEOREM 7.7. *For each* $\lambda \in]0, 1[$, *the Powers factor* R_λ *is, up to isomorphism, the only injective factor of type* III_λ *(see* [8]*), and in particular the only approximately finite dimensional one.*

Proof. Let $M = W^*(N, \theta)$ be a discrete decomposition of M (cf. [8]). Then, as above, N is injective and Theorem 7.4 shows that N is isomorphic to $R_{0,1}$. Then by [12] one gets the conclusion. Q.E.D.

THEOREM 7.8. *Any injective factor of type* III_1 *is semi-discrete and approximately finite dimensional.*

Proof. By [8], p. 167 and Theorem 7.7 the cross product of M by σ_{T_0}, the modular automorphism for a $T_0 \neq 0$, is isomorphic to R_λ, $T_0 \log \lambda = 2\pi$. The conclusions follow easily. Q.E.D.

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