JOURNAL OF FUNCTIONAL ANALYSIS 24, 336–352 (1977)

Measure Space Automorphisms, the Normalizers of their Full Groups, and Approximate Finiteness

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Communicated by the Editors

Received November 25, 1975; revised January 19, 1976

We deal with the normalizer $\mathcal{N}[T]$ of the full group $[T]$ of a nonsingular transformation T of a Lebesgue measure space in the group of all nonsingular transformations. We solve the conjugacy problem in $\mathcal{N}[T]/[T]$ for a measure preserving and ergodic T. Our results show that a locally finite extension of a solvable group is approximately finite.

1. INTRODUCTION

Let (X, μ) be a Lebesgue measure space, and let $\mathcal{A}$ be the group of automorphisms of (X, μ), where we mean by an automorphism a bi-measurable transformation that leaves μ quasi-invariant. The full group $[U]$ of a $U \in \mathcal{A}$ consists of those $Q \in \mathcal{A}$ with the property that f.a.a. $x \in X$ the Q-orbit of x is contained in the U-orbit of x. We consider an ergodic $T \in \mathcal{A}$ and we aim to elucidate the structure of the normalizer $\mathcal{N}[T] = \{R \in \mathcal{A} : R[T]R^{-1} = [T]\}$ of $[T]$ in $\mathcal{A}$. $\mathcal{N}[T]$ contains exactly those automorphisms $R \in \mathcal{A}$ that carry f.a.a. $x \in X$ the T-orbit of x onto the T-orbit of Rx. If, for an $R \in \mathcal{N}[T]$, there is an $n \in \mathbb{N}$ such that $R^n \in [T]$, then define the outer period $p(R)$ of R as the smallest such n. If, for all $n \in \mathbb{N}$, $R^n \notin [T]$, then say that R is outer aperiodic and set $p(R) = 0$. A countable group $\mathcal{G} \subset \mathcal{A}$ is called approximately finite if there exists a $U \in \mathcal{A}$ such that, f.a.a. $x \in X$, the $\mathcal{G}$-orbit of x coincides with the U-orbit of x. In Section 2 we shall prove that every $R \in \mathcal{N}[T]$ generates together with T an approximately finite group. In Section 3 we are concerned with the conjugacy problem in $\mathcal{N}[T]/[T]$ for an ergodic and measure preserving T. We briefly describe our conclusions. For this, we recall that every $R \in \mathcal{N}[T]$ has a module such that, with ν an invariant measure of T,

$$(dR^{-1}\nu/d\nu)(x) = \text{mod } R, \qquad \text{f.a.a. } x \in X.$$

ISSN 0022-1236

If $\nu(X) < \infty$, then mod $R = 1$ for all $R \in \mathcal{N}[T]$, but if $\nu(X) = \infty$, then every positive real appears as a module. If mod $R \neq 1$ then R must be outer aperiodic. Let us say that $R, R' \in \mathcal{N}[T]$ are outer conjugate if $R[T]$ and $R'[T]$ are conjugate in $\mathcal{N}[T]/[T]$; in other words, if there is a $P \in [T]$ and a $Q \in \mathcal{N}[T]$ such that $R' = QRPQ^{-1}$. The outer period, and outer aperiodicity, as well as the module, are invariants of outer conjugacy, and it turns out that these invariants are complete.

Our results have a number of applications. In particular, it follows that all locally finite extensions of solvable groups are approximately finite. We comment on these applications in the last section of the paper.

There is a close connection between the topics that we deal with here and certain developments in the theory of von Neumann algebras. At the origin of this is the group measure space construction, that produces out of an ergodic measure preserving automorphism T an approximately finite type II factor. Every element of $\mathcal{N}[T]$ induces an automorphism of this factor. For the automorphisms of a von Neumann algebra, one also has the notion of outer conjugacy where two automorphisms R and R' are called outer conjugate if R is conjugate to a product of R' by an inner automorphism. Outer conjugate elements of $\mathcal{N}[T]$ yield outer conjugate factor automorphisms. Hence the problem that we deal with here can be viewed as the measure theoretic analog of the outer conjugacy problem for the automorphisms of the approximately finite type II factors. Concerning the outer conjugacy for the automorphisms of the approximately finite type II factors, see [2, 3].

Also the methods that we employ here are similar to those encountered in the theory of von Neumann algebras. Thus we use unit systems (also called arrays) to prove outer conjugacy. These unit systems are particular instances of systems of matrix units. Moreover, we shall arrive at the approximate finiteness theorem by the use of an ultraproduct. This is similar to the use of the ultraproducts of von Neumann algebras that were described in [1]. For finite von Neumann algebras these ultraproducts were considered by McDuff [14]. See also the remark of Dixmier and Lance [6].

2. Approximate Finiteness

Let (X, μ) be a Lebesgue measure space, $\mu(X) = 1$. With a free ultrafilter ω on $\mathbb{N}$, we form the ultraproduct $(\mathscr{B}^\omega, \mu^\omega)$ of the measure algebra of (X, μ). Thus we consider sequences $A_n \subset X$, $n \in \mathbb{N}$, identifying two such sequences $(A_n)_{n\in\mathbb{N}}$ and $(A_n')_{n\in\mathbb{N}}$ if

$$\lim_{n\to\omega} \mu(A_n \triangle A_n') = 0.$$

The resulting set of equivalence classes is $\mathscr{B}^\omega$. This $\mathscr{B}^\omega$ carries Boolean operations. Here $(X)_{n\in\mathbb{N}}$ represents $1 \in \mathscr{B}^\omega$ and $(\varnothing)_{n\in\mathbb{N}}$ represents $0 \in \mathscr{B}^\omega$. If $(A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}^\omega$, then $(X - A_n)_{n\in\mathbb{N}} \in 1 - \hat{A} \in \mathscr{B}^\omega$, and if $(B_n)_{n\in\mathbb{N}} \in \hat{B} \in \mathscr{B}^\omega$, $(B_n')_{n\in\mathbb{N}} \in \hat{B}' \in \mathscr{B}^\omega$, then $(B_n \cap B_n') \in \hat{B} \cap \hat{B}'$. μ^ω is given by

$$\mu^\omega(\hat{A}) = \lim_{n\to\omega} \mu(A_n), \qquad (A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}^\omega.$$

$(\mathscr{B}^\omega, \mu^\omega)$ is a measure algebra. Every automorphism U of (X, μ) induces an automorphism U^ω of $\mathscr{B}^\omega$ by

$$(UA_n)_{n\in\mathbb{N}} \in U^\omega\hat{A}, \qquad (A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}^\omega.$$

Consider now an ergodic automorphism T of (X, μ). We denote by $\mathscr{B}_\omega$ the fixed point algebra of $\{S^\omega : S \in [T]\}$, and by μ_ω the restriction of μ^ω to $\mathscr{B}_\omega$. The elements of $\mathscr{B}_\omega$ are represented by sequences $A_n \subset X$, $n \in \mathbb{N}$, with the property

$$\lim_{n\to\omega} \mu(A_n \triangle SA_n) = 0, \qquad S \in [T].$$

(2.1) Lemma. *Let $A \in \mathscr{B}^\omega$, and let, for a uniformly dense countable subgroup $\mathscr{H}$ of $[T]$,*

$$H^\omega\hat{A} = \hat{A}, \qquad H \in \mathscr{H}. \tag{1}$$

Then $\hat{A} \in \mathscr{B}_\omega$.

Proof. Let $S \in [T]$, and $\epsilon > 0$. Choose a decomposition

$$X = \bigcup_{H\in\mathscr{H}} E_H,$$

such that

$$Sx = Hx, \qquad \text{f.a.a. } x \in E_H,\ H \in \mathscr{H},$$

and let then $\mathscr{H}_0$ be a finite subset of $\mathscr{H}$ such that

$$\mu\left(\bigcup_{H\in\mathscr{H}_0} E_H\right) > 1 - \epsilon, \qquad \mu\left(\bigcup_{H\in\mathscr{H}_0} HE_H\right) > 1 - \epsilon.$$

Setting $(E_H)_{n\in\mathbb{N}} \in \hat{E}_H \in \mathscr{B}^\omega$, $H \in \mathscr{H}_0$, and setting

$$\hat{A}' = \hat{A} \cap \left(\bigcup_{H\in\mathscr{H}_0} \hat{B}_H\right), \qquad \hat{A}'' = \hat{A} \cap \left(\bigcup_{H\in\mathscr{H}_0} H^\omega\hat{E}_H\right),$$

one has then

$$\mu^\omega(\hat{A} \triangle \hat{A}') < \epsilon, \qquad \mu^\omega(\hat{A} \triangle \hat{A}'') < \epsilon.$$

From (1),

$$\begin{aligned} S^\omega\hat{A}' &= \bigcup_{H\in\mathscr{H}_0} S^\omega(\hat{A} \cap \hat{E}_H) \\ &= \bigcup_{H\in\mathscr{H}_0} H^\omega(\hat{A} \cap \hat{E}_H) \\ &= \bigcup_{H\in\mathscr{H}_0} (\hat{A} \cap H^\omega\hat{E}_H) = \hat{A}''. \end{aligned}$$

Q.E.D.

We remark that for all $\hat{A} \in \mathscr{B}_\omega$, the $f_{\hat{A}} \in L_\infty(X, \mu)$ that is given by

$$f_{\hat{A}} = \text{weak*-}\lim_{n\to\omega} \chi_{A_n}, \qquad (A_n)_{n\in\mathbb{N}} \in \hat{A},$$

is T-invariant, and therefore a constant. It follows that

$$\mu_\omega(\hat{A}) = \text{weak*-}\lim_{n\to\omega} \chi_{A_n}, \qquad (A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}_\omega, \tag{2}$$

and also that

$$\mu_\omega(\hat{A}) = \lim_{n\to\omega} \nu(A_n), \qquad (A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}_\omega, \quad \nu \sim \mu, \quad \nu(X) = 1. \tag{3}$$

For $R \in \mathscr{N}[T]$, R^ω leaves $\mathscr{B}_\omega$ invariant and we denote the restriction of R^ω to $\mathscr{B}_\omega$ by R_ω. By (3), R_ω is a μ_ω-preserving automorphism of $\mathscr{B}_\omega$.

(2.2) Lemma. *Let $(A_n)_{n\in\mathbb{N}} \in \hat{A} \in \mathscr{B}_\omega$, and let $B \subset X$. Then*

$$\mu_\omega(\hat{A})\, \mu(B) = \lim_{n\to\omega} \mu(A_n \cap B).$$

Proof. From (2),

$$\lim_{n\to\omega} \mu(A_n \cap B) = \lim_{n\to\omega} \int_X \chi_{A_n}\chi_B \, d\mu = \int_X \mu_\omega(\hat{A})\, \chi_B \, d\mu = \mu_\omega(\hat{A})\, \mu(B). \quad \text{Q.E.D.}$$

(2.3) Lemma. *Let $R \in \mathscr{N}[T]$ be such that for some $\hat{A} \in \mathscr{B}_\omega$, $\hat{A} \neq 0$,*

$$R_\omega \hat{B} = \hat{B}, \qquad \hat{B} \subset \hat{A}. \tag{4}$$

Then $R_\omega = 1$.

Proof. We assume that there is a $\hat{C} \subset 1 - \hat{A}$, $\hat{C} \neq 0$ such that $R_\omega \hat{C} \subset 1 - \hat{C}$, and we proceed to arrive at a contradiction. Let $(A_n)_{n\in\mathbb{N}} \in \hat{A}$ and let $(C_k)_{k\in\mathbb{N}} \in \hat{C}$ be such that

$$RC_k \subset X - C_k, \qquad k \in \mathbb{N}. \tag{5}$$

By lemma (2.2) there exists a subsequence $k(n)$, $n \in \mathbb{N}$, such that with

$$B_n = A_n \cap C_{k(n)}, \qquad n \in \mathbb{N},$$
$$\mu(B_n) > \tfrac{1}{2}\mu(A_n)\, \mu_\omega(\hat{C}).$$

$(B_n)_{n\in\mathbb{N}}$ represents an element $\hat{B}$ of $\mathscr{B}$, $\hat{B} \neq 0$, and $\hat{B} \subset \hat{A}$. From (5)

$$RB_n \subset X - B_n, \qquad n \in \mathbb{N}.$$

Hence $R_\omega \hat{B} \subset 1 - \hat{B}$, contradicting (4). Q.E.D.

(2.4) Lemma. *For $R \in \mathscr{N}[T]$, $R_\omega = 1$ if and only if $R \in [T]$.*

Proof. Let $\mathscr{G}(k)$, $k \in \mathbb{N}$, be an increasing sequence of finite subgroups of $[T]$ that have uniform orbit size, and whose union is uniformly dense in $[T]$. Let d be the uniform metric on $\mathscr{N}[T]$. If $R_\omega = 1$, then there is a $k_0 \in \mathbb{N}$ such that

$$\inf_{S \in \mathscr{G}(k_0)} d(R, S) < 1. \tag{6}$$

Indeed, otherwise one could find for every $k \in N$ a $\mathscr{G}(k)$-invariant set $B(k)$ such that

$$\mu(B(k) \triangle RB(k)) \geqslant \tfrac{1}{2}, \qquad k \in \mathbb{N}, \tag{7}$$

(see [7, pp. 137–139]). By Lemma (2.1), $(B_k)_{k \in \mathbb{N}}$ would represent an element $\hat{B} \neq 0$ of $\mathscr{B}_\omega$, and by (7) one would then have $R_\omega \hat{B} \neq \hat{B}$. That $R \in [T]$ follows from (6). Q.E.D.

The proofs of the following two theorems are quite similar.

(2.5) THEOREM. *Let T be an ergodic automorphism of (X, μ), and let $R \in \mathscr{N}[T]$ be outer aperiodic. Then T and R generate an approximately finite group.*

Proof. To prove the approximate finiteness of the group $\mathscr{G}$ that is generated by T and R it is enough to construct a sequence $\mathscr{G}(k)$, $k \in \mathbb{N}$, of finite subgroups of $[\mathscr{G}]$ such that

$$\mathscr{G}(k)\, x \subset \mathscr{G}(k+1)\, x, \qquad \mathscr{G}x = \bigcup_{k \in \mathbb{N}} \mathscr{G}(k)\, x, \qquad \text{f.a.a. } x \in X$$

(see [11]). A construction of such a sequence by an inductive procedure will be possible, once one has established that, for all $L \in \mathbb{N}$ and for all $\eta > 0$, there is a finite subgroup $\mathscr{H}$ of $[\mathscr{G}]$ such that

$$\mu\left(\bigcap_{-L<i,j<L} \{x \in X\colon T^i R^j x \in \mathscr{H}x\} \right) > 1 - \eta,$$

and for this it suffices to know that for every $\epsilon > 0$ the finite subgroup $\mathscr{H}$ of $[\mathscr{G}]$ can be chosen such that

$$\mu(\{x \in X\colon Tx \in \mathscr{H}x\}) > 1 - \epsilon, \tag{8}$$

$$\mu(\{x \in X\colon Rx \in \mathscr{H}x\}) > 1 - \epsilon. \tag{9}$$

To produce such an $\mathscr{H}$, let $M \in \mathbb{N}$,

$$M > 2\epsilon^{-1}. \tag{10}$$

By Lemmas (2.3) and (2.4) one can apply the Rohlin tower theorem to R_ω, and hence one has an $\hat{F} \in \mathscr{B}_\omega$ such that

$$\hat{F} \cap R_\omega^m \hat{F} = 0, \qquad 0 < m < M, \tag{11}$$

and

$$M\mu_\omega(\hat{F}) > 1 - \tfrac{1}{2}\epsilon. \tag{12}$$

Choose a representative $(F_n)_{n\in\mathbb{N}}$ of $\hat{F}$ such that

$$F_n \cap R^m F_n = \varnothing, \qquad 0 < m < M,\ n \in \mathbb{N}.$$

Let then F be one of the F_n such that

$$M\mu(R^m F) > 1 - \tfrac{1}{2}\epsilon, \qquad 0 \leqslant m < M, \tag{13}$$

and

$$4M\mu(TR^m F \triangle R^m F) < \epsilon, \qquad 0 \leqslant m < M. \tag{14}$$

For

$$D = \bigcup_{0 \leqslant m < M} R^m F,$$

by (13),

$$\mu(D) > 1 - \tfrac{1}{2}\epsilon. \tag{15}$$

Let $\delta > 0$ be such that for $A \subset X$, $\mu(A) < \delta$ implies that $4M\mu(R^m A) < \epsilon$, $0 < m < M$. Choose then a $\tilde{T} \in [T]$ that is the identity on $X - F$, that is periodic on F, and that is such that, with $\mathscr{T}$ the group generated by $\tilde{T}$,

$$\mu\left(\bigcap_{0 \leqslant m < M} \{x \in F: R^{-m} T_{R^m F} R^m x \in \mathscr{T}x\}\right) > \mu(F) - \delta. \tag{16}$$

Denote by $\tilde{R}$ the element of $[\mathscr{G}]$ that is the identity on $X - D$, that has period period M on D, and that is such that

$$\tilde{R}x = Rx, \qquad \text{f.a.a.}\quad x \in \bigcup_{0 \leqslant m < M-1} R^m F.$$

The finite subgroup $\mathscr{H}$ of $[\mathscr{G}]$ that is generated by $\tilde{T}$ and $\tilde{R}$ satisfies (8) by (14) and (15) in conjunction with (16) and the choice of δ. $\mathscr{H}$ also satisfies (9) by (10) and (13). Q.E.D.

(2.6) THEOREM. *Let T be an ergodic automorphism of (X, μ), and let Γ be a finite group. Let there be given for every $\gamma \in \Gamma$ an $R(\gamma) \in \mathscr{N}[T]$, such that $R(\gamma) \notin [T]$ for $\gamma \neq e$, and such that*

$$R(\gamma')\, R(\gamma) = R(\gamma\gamma') \bmod [T], \qquad \gamma, \gamma' \in \Gamma.$$

Then T generates together with the $R(\gamma)$, $\gamma \in \Gamma$, an approximately finite group.

Proof. In order to prove the approximate finiteness of the group $\mathscr{G}$ that is generated by T together with the $R(\gamma)$, $\gamma \in \Gamma$, it is enough to construct for every $\epsilon > 0$ a finite subgroup $\mathscr{H}$ of $[\mathscr{G}]$ such that

$$\mu(\{x \in X: Tx \in \mathscr{H}x\}) > 1 - \epsilon, \tag{17}$$

$$\mu(\{x \in X: R(\gamma)x \in \mathscr{H}x\}) > 1 - \epsilon, \qquad \gamma \in \Gamma. \tag{18}$$

For this, let $L \in \mathbb{N}$ be such that

$$\mu\left(\bigcap_{\gamma,\gamma' \in \Gamma} \bigcup_{-L<i<L} \{x \in X: R(\gamma^{-1}\gamma')\, R(\gamma)x = R(\gamma')T^i x\}\right) > 1 - |\Gamma|^{-1}\epsilon. \quad (19)$$

Lemmas (2.3) and (2.4) allow an exhaustion argument that yields an $\hat{F} \in \mathscr{B}_\omega$ such that

$$\hat{F} \cap R_\omega(\gamma)\hat{F} = 0, \qquad \gamma \in \Gamma, \gamma \neq e,$$

$$\bigcup_{\gamma \in \Gamma} R_\omega(\gamma)\hat{F} = 1.$$

Choose a representative $(F_n)_{n \in \mathbb{N}}$ of $\hat{F}$ such that

$$F_n \cap R(\gamma)F_n = \varnothing, \qquad \gamma \in \Gamma, \gamma \neq e, n \in \mathbb{N}.$$

Let then F be one of the F_n such that

$$|\Gamma|\, \mu(R(\gamma)F) > 1 - \tfrac{1}{2}\,\epsilon, \qquad \gamma \in \Gamma, \quad (20)$$

and such that

$$4\,|\Gamma|\, \mu(T^i R(\gamma)F \,\triangle\, R(\gamma)F) < \epsilon, \qquad -L < i < L, \gamma \in \Gamma. \quad (21)$$

For

$$D = \bigcup_{\gamma \in \Gamma} R(\gamma)\, F,$$

by (20),

$$\mu(D) > 1 - \tfrac{1}{2}\,\epsilon. \quad (22)$$

Let $\delta > 0$ be such that for $A \subset X$, $\mu(A) < \delta$ implies that $4\,|\Gamma|\,\mu(R(\gamma)A) < \epsilon$, $\gamma \in \Gamma$. Choose then a $\tilde{T} \in [T]$, that is the identity on $X - F$, that is periodic on F, and that is such that, with $\mathscr{T}$ the group generated by $\tilde{T}$,

$$\mu\left(\bigcap_{-L<i<L} \bigcap_{\gamma \in \Gamma} \{x \in F: R(\gamma)^{-1}\, T^i_{R(\gamma)F} R(\gamma)\, x \in \mathscr{T}x\}\right) > \mu(F) - \delta. \quad (23)$$

Order Γ,

$$\Gamma = \{\gamma_m : 0 \leqslant m < |\Gamma|\}.$$

Denote by $\tilde{R}$ the element of $[\mathscr{G}]$ that is the identity on $X - D$, that has period $|\Gamma|$ on D, and that is such that

$$\tilde{R}^m x = R(\gamma_m)x, \qquad \text{f.a.a. } x \in F, 0 \leqslant m < |\Gamma|.$$

For the finite subgroup $\mathscr{H}$ of $[\mathscr{G}]$ that is generated by $\tilde{T}$ and $\tilde{R}$, one has, by (21) and (22), in conjunction with (23), and by the choice at δ, that

$$\mu\left(\bigcap_{-L<i<L} \{x \in X: T^i x \in \mathscr{H}x\}\right) > 1 - \tfrac{1}{4}\,\epsilon. \quad (24)$$

Thus $\mathscr{H}$ satisfies (17). By (19), (22), and (23), $\mathscr{H}$ also satisfies (18). Q.E.D.

(2.7) COROLLARY. *Let T be an ergodic automorphism of (X, μ), and let $R \in \mathcal{N}[T]$. Then T and R generate an approximately finite group.*

Proof. This is Theorem (2.5) and a particular case of Theorem (2.6). Q.E.D.

3. OUTER CONJUGACY

We describe now unit systems (compare [12]). For this let (X, μ) be a Lebesgue measure space. A unit system of $A \subset X$, $\mu(A) > 0$, consists of a partition $(A_\omega)_{\omega \in \Omega}$ of A together with a collection $U(\omega', \omega)$, $\omega, \omega' \in \Omega$, of isomorphisms, that is, bi-measurable and nonsingular bijections,

$$U(\omega', \omega) : A(\omega) \to A(\omega'),$$

such that

$$U(\omega, \omega) = 1, \qquad U(\omega'', \omega')\, U(\omega', \omega) = U(\omega'', \omega), \qquad \omega, \omega', \omega'' \in \Omega.$$

We indicate such a unit system α by the notation

$$\alpha = (\Omega, A, A(\cdot), U(\cdot, \cdot)).$$

Assign to a permutation π of Ω the automorphism $U(\pi)$ of A that is given by

$$U(\pi)x = U(\pi\omega, \omega)x, \qquad \text{f.a.a. } x \in A(\omega),\ \omega \in \Omega,$$

and denote by $\mathcal{G}(\alpha)$ the group of these $U(\pi)$. With $\Omega_0 \subset \Omega$, and

$$B = \bigcup_{\omega \in \Omega_0} A(\omega),$$

we say that the unit system α is an extension of the unit system β,

$$\beta = (\Omega_0, B, A(\cdot), U(\cdot, \cdot)).$$

Let the unit system α be given, let $\omega_0 \in \Omega$, and let there also be given a unit system γ,

$$\gamma = (\Xi, A(\omega_0), C(\cdot), W(\cdot, \cdot)).$$

Then we define the refinement $\bar{\alpha}$ of α by γ as the unit system

$$\bar{\alpha} = (\Omega \times \Xi, A, \bar{A}(\cdot), \bar{U}(\cdot, \cdot))$$

where

$$\bar{A}(\omega, \xi) = U(\omega, \omega_0)\, C(\xi), \qquad \omega \in \Omega,\ \xi \in \Xi$$

and

$$\bar{U}((\omega, \xi'), (\omega, \xi))x = U(\omega, \omega_0)\, W(\xi', \xi)\, U(\omega_0, \omega)x,$$

$$\bar{U}((\omega', \xi), (\omega, \xi))x = U(\omega', \omega)x, \qquad \text{f.a.a. } x \in C(\omega, \xi),\ \omega, \omega' \in \Omega,\ \xi, \xi' \in \Xi.$$

Given a countable group $\mathscr{H}$ of automorphisms of (X, μ), we say that an isomorphism $U: A \to B$, $A, B \subset X$ is associated to $\mathscr{H}$, or is an $\mathscr{H}$-isomorphism, if $Ux \in \mathscr{H}x$, f.a.a. $x \in A$. We say that an isomorphism is a T-isomorphism if it is associated to the group that is generated by an automorphism T. A unit system is called a system of $\mathscr{H}$-units (resp., of T-units) if its isomorphisms are $\mathscr{H}$-isomorphisms (resp., T-isomorphisms).

When constructing our models for outer conjugacy we need to consider product spaces. The cylinder sets in a product space $X = \prod_{n \in \mathbb{N}} \Omega_n$, where the Ω_n are finite sets, will be denoted by

$$Z(a) = \{x \in X: x_n = a_n, 1 \leqslant n \leqslant N\}, \qquad a \in \prod_{1 \leqslant n \leqslant N} \Omega_n, \qquad N \in \mathbb{N}.$$

Given such a product space the term odometer group will refer to the group of transformations of X that is generated by the U_m, where, with π_m a cyclic permutation of Ω_m,

$$\begin{aligned}(U_m x)_n &= \pi_m x_m, & &\text{if} \quad n = m,\\ &= x_n, & &\text{if} \quad n \neq m, \qquad n \in \mathbb{N},\ m \in \mathbb{N}.\end{aligned}$$

Denote by Φ_0 the set of functions $\varphi: \mathbb{Z}_+ \to \mathbb{Z}_+$ with finite support and such that $\varphi(0) \geqslant 1$. Set

$$\Gamma(\varphi) = \{(j, k)\,;\, 1 \leqslant k \leqslant \varphi(j), j \in \operatorname{supp} \varphi\}, \qquad \varphi \in \Phi_0,$$

and define for $\lambda > 0$ measures $q_{\varphi,\lambda}$ on $\Gamma(\varphi)$ by

$$q_{\varphi,\lambda}(j, k) = \frac{\lambda^j}{\sum_{j \in \mathbb{Z}_+} \varphi(j)\, \lambda^j}, \qquad (j, k) \in \Gamma_\varphi, \quad \varphi \in \Phi_0.$$

We shall have to consider product spaces

$$(Y, \nu) = \prod_{n \in \mathbb{N}} (\Gamma(\varphi(n)), q_{\varphi(n),\lambda}), \qquad (\varphi(n))_{n \in \mathbb{N}} \in \Phi_0^{\mathbb{N}}, \quad \lambda > 0.$$

For an isomorphism $U: A \to B$, $A, B \subset Y$, that is associated with the odometer group of such a product space (Y, ν), we set f.a.a. $x = (j_n, k_n)_{n \in \mathbb{N}} \in A$

$$h_U(x) = \sum_{n \in \mathbb{N}} (j_n' - j_n), \qquad Ux = (j_n', k_n')_{n \in \mathbb{N}}.$$

For isomorphisms $U: A \to B$, $V: B \to C$, $A, B, C \subset Y$, that are associated with the odometer group of the product space, one has a cocycle identity

$$h_{VU}(x) = h_U(x) + h_V(Ux), \qquad \text{f.a.a. } x \in A.$$

Let φ_0 denote the element of Φ_0 that is given by

$$\varphi_0(0) = \varphi_0(1) = 1, \qquad \varphi_0(j) = 0, \qquad j > 1.$$

(3.1) LEMMA. *Let φ_0 appear infinitely often as an entry in a sequence $(\varphi(n))_{n\in\mathbb{N}} \in \Phi_0^{\mathbb{N}}$. Then the elements U in the full group of the odometer group of*

$$(Y, \nu) = \prod_{n\in\mathbb{N}} (\Gamma(\varphi(n)), q_{\varphi(n),\lambda}), \qquad \lambda > 0,$$

such that $h_U = 0$, form an ergodic group.

Proof. Let $\mathscr{G}$ stand for the odometer group of (Y, ν). Let $B \subset Y$, $\nu(B) > 0$. We show that there exists a $D \subset B$, $\nu(D) > 0$, and a $\mathscr{G}$-isomorphism V that maps D into B such that $h_V = 1$. For this, we can assume that $\lambda \leqslant 1$. We first find an $N \in \mathbb{N}$ and an

$$a \in \prod_{1\leqslant n\leqslant N} \Gamma(\varphi(n))$$

such that one has, for $C = B \cap Z(a)$, that

$$\nu(C) > (1 + (\lambda/2))(1 + \lambda)^{-1} \nu(Z(a)). \tag{25}$$

Then we choose an $M > N$ such that $\varphi(M) = \varphi_0$. For

$$C(0) = \{x \in C\colon x_M = (0, 1)\}, \qquad C(1) = \{x \in C\colon x_M = (1, 1)\},$$

then, by (25),

$$\nu(C(0)) > \tfrac{1}{2}(1 + \lambda)^{-1} \nu(Z(a)), \qquad \nu(C(1)) > \tfrac{1}{2}\lambda(1 + \lambda)^{-1} \nu(Z(a)). \tag{26}$$

With $\bar{V}$ denoting the automorphism of (Y, ν) that changes the Mth coordinate, set

$$D = C(0) \cap \bar{V}C(1).$$

From (26), $\nu(D) > 0$, and we can choose for V the restriction of $\bar{V}$ to D.

If now $E \subset Y$, $0 < \nu(E) < 1$, then we have by the ergodicity of the odometer group sets $A \subset E$ and $B \subset Y - E$, $\nu(A)$, $\nu(B) > 0$, and a $J \in \mathbb{Z}$, together with a $\mathscr{G}$-isomorphism $W\colon A \to B$, such that $h_W = J$. A repeated application of the preceeding remark yields a set $D \subset B$ and a $\mathscr{G}$-isomorphism V that maps D into B and such that $h_V = -J$. For the restriction U of VW to $W^{-1}D$ one has then $h_U = 0$. Q.E.D.

We set

$$\Phi_p = \{\varphi \in \Phi_0 : \varphi(j) = 0, j \geqslant p\}, \qquad p \in \mathbb{N},$$

and we fix now for all $p \in \mathbb{Z}_+$ sequences $(\varphi(p, n))_{n\in\mathbb{N}} \in \Phi_p^{\mathbb{N}}$ such that every element of Φ_p appears infinitely often as an entry in $(\varphi(p, n))_{n\in\mathbb{N}}$. Then we build the product spaces

$$(X_p, \nu_p) = \prod_{n\in\mathbb{N}} (\Gamma(\varphi(p, n)), q_{\varphi(p,n),1}), \qquad p \in \mathbb{Z}_+,$$

$$(X_\lambda, \nu_\lambda) = \prod_{n\in\mathbb{N}} (\Gamma(\varphi(0, n)), q_{\varphi(0,n),\lambda}), \qquad \lambda > 0,$$

and, with κ the counting measure on $\mathbb{Z}$,

$$(X_{p,1,\infty}, \nu_{p,1,\infty}) = (\mathbb{Z}, \kappa) \times (X_p, \mu_p), \qquad p \in \mathbb{Z}_+,$$
$$(X_{0,\lambda,\infty}, \nu_{0,\lambda,\infty}) = (\mathbb{Z}, \kappa) \times (X_\lambda, \mu_\lambda), \qquad \lambda > 0.$$

Let $\mathscr{G}_p$ stand for the odometer group on (X_p, ν_p), $p \in \mathbb{Z}_+$, and let $\mathscr{G}_\lambda$ stand for the odometer group on (X_λ, ν_λ), $\lambda > 0$. With τ the translation of $\mathbb{Z}$ by one, set

$$\mathscr{G}_{p,1,\infty} = \{\tau^i \times G : i \in \mathbb{Z}, G \in \mathscr{G}_p\}, \qquad p \in \mathbb{Z}_+,$$
$$\mathscr{G}_{0,\lambda,\infty} = \{\tau^i \times G : i \in \mathbb{Z}, G \in \mathscr{G}_\lambda\}, \qquad \lambda > 0.$$

Let also

$$\mathscr{H}_p = \{U \in [\mathscr{G}_p] : h_U(x) = 0 \bmod p, \text{ f.a.a. } x \in X_p\}, \qquad p \in \mathbb{Z}_+.$$

As is seen from Lemma (3.1), the $\mathscr{H}_p$ are ergodic. We make on the $\mathscr{G}_{p,\lambda,\infty}$ the analog definition of h, and set

$$\mathscr{H}_{p,1,\infty} = \{U \in [\mathscr{G}_{p,1,\infty}] : h_U(x) = 0 \bmod p, \text{ f.a.a. } x \in X_{p,1,\infty}\}, \qquad p \in \mathbb{Z}_+,$$
$$\mathscr{H}_{0,\lambda,\infty} = \{U \in [\mathscr{G}_{0,\lambda,\infty}] : h_U(x) = 0, \text{ f.a.a. } x \in X_{0,\lambda,\infty}\}, \qquad \lambda > 0.$$

These $\mathscr{H}_{p,\lambda,\infty}$ are ergodic as well. Further, set

$$\mathscr{R}_p = \{R \in [\mathscr{G}_p] : h_R(x) = 1 \bmod p, \text{ f.a.a. } x \in X_p\},$$
$$\mathscr{R}_{p,1,\infty} = \{R \in [\mathscr{G}_{p,1,\infty}] : h_R(x) = 1 \bmod p, \text{ f.a.a. } x \in X_{p,1,\infty}\}, \qquad p \in \mathbb{Z}_+,$$
$$\mathscr{R}_{0,\lambda,\infty} = \{R \in [\mathscr{G}_{0,\lambda,\infty}] : h_R(x) = 1, \text{ f.a.a. } x \in X_{0,\lambda,\infty}\}, \qquad \lambda > 0.$$

An exhaustion argument based on Lemma (3.1) shows that these $\mathscr{R}_p$ and $\mathscr{R}_{p,\lambda,\infty}$ are not empty. From the cocycle identity, one further has that

$$\begin{aligned} &R'R^{-1} \in \mathscr{H}_p, && R, R' \in \mathscr{R}_p \\ &R'R^{-1} \in \mathscr{H}_{p,\lambda,\infty}, && R, R' \in \mathscr{R}_{p,\lambda,\infty} \qquad p \in \mathbb{Z}_+, \quad \lambda > 0. \end{aligned} \tag{27}$$

Given an ergodic ν-preserving automorphism T of (X, ν), and given $R \in \mathscr{N}[T]$, let, for $A, B \subset X$ of positive measure and for $k \in \mathbb{Z}$, $\mathscr{I}_{A,B}([T]R^k)$ be the set of isomorphisms $U: A \to B$ such that for some T-isomorphism $V: A \to R^{-k}B$, $U = R^k V$. If here A and B are understood then we write simply $\mathscr{I}([T]R^k)$.

(3.2) Lemma. *Let T be an ergodic ν-preserving automorphism of (X, ν), let $R \in \mathscr{N}[T]$ and let $A, B \subset X$, $i \in \mathbb{Z}$, such that $\nu(B) = (\operatorname{mod} R)^i \nu(A) > 0$. Then $\mathscr{I}_{A,B}([T]R^i) \neq \varnothing$.*

Proof. An exhaustion argument based on the ergodicity of T yields this. Q.E.D.

(3.3) LEMMA. *Let T be an ergodic ν-preserving automorphism of (X, ν), and let $R \in \mathcal{N}[T]$. If $U \in \mathcal{J}_{A,B}([T]R^i)$, $V \in \mathcal{J}_{B,C}([T]R^j)$, then $VU \in \mathcal{J}_{A,C}([T]R^{i+j})$.*

Proof. Use $R \in \mathcal{N}[T]$. Q.E.D.

In the sequel we use, for $a = (j_m, k_m)_{1 \leqslant m \leqslant n}$, $a' = (j_m', k_m')_{1 \leqslant m \leqslant n} \in \Gamma(\varphi)^n$, the notation

$$\sum (a' - a) = \sum_{1 \leqslant m \leqslant n} (j_m' - j_m).$$

(3.4) LEMMA. *Let T be an ergodic ν-preserving automorphism of (X, ν), and let $R \in \mathcal{N}[T]$, $A \subset X$, $\nu(A) > 0$, $\varphi \in \Phi_0$. Then there exists a unit system $(\Gamma(\varphi), A, A(\cdot), U(\cdot, \cdot))$, where*

$$U(a', a) \in \mathcal{J}([T]R^{\Sigma(a'-a)}), \qquad a, a' \in \Gamma(\varphi).$$

Proof. One first chooses an appropriate partition $(A(a))_{a \in \Gamma(\varphi)}$ of A and then applies Lemmas (3.2) and (3.3). Q.E.D.

(3.5) LEMMA. *Let T be an ergodic ν-preserving automorphism of (X, ν), let $R \in \mathcal{N}[T]$, and let there be given $A \subset X$, $0 < \nu(A) < \infty$, $A_k \subset A$, $1 \leqslant k \leqslant K$, and $\epsilon > 0$. Then there exists, with some $N \in \mathbb{N}$, a unit system $(\Gamma(\varphi_0)^N, A, A(\cdot), U(\cdot, \cdot))$ and subsets $\Delta_k \subset \Gamma(\varphi_0)^N$ with*

$$\nu\left(A_k \triangle \left(\bigcup_{a \in \Delta_k} A(a)\right)\right) < \epsilon, \qquad 1 \leqslant k \leqslant K.$$

and

$$U(a', a) \in \mathcal{J}([T]R^{\Sigma(a'-a)}), \qquad a, a' \in \Gamma(\varphi_0)^N.$$

Proof. One first chooses an appropriate $N \in \mathbb{N}$ and a partition $(A(a))_{a \in \Gamma(\varphi_0)^N}$ of A and then applies Lemmas (3.2) and (3.3). Q.E.D.

(3.6) LEMMA. *Let T be an ergodic ν-preserving automorphism of (X, ν), $\nu(X) = 1$, and let $R \in \mathcal{N}[T]$. Set $p = p(R)$. Let Q be an automorphism of (X, ν) such that*

$$\{T^m R^l x : m, l \in \mathbb{Z}\} = \{Q^i x : i \in \mathbb{Z}\}, \qquad \text{f.a.a. } x \in X. \tag{28}$$

Then there exists for all $A \subset X$, $\nu(A) > 0$, and for all $\epsilon > 0$, $I \in \mathbb{N}$, a $\varphi \in \Phi_p$ and a unit system

$$\alpha = (\Gamma(\varphi), A, A(\cdot), U(\cdot, \cdot))$$

such that

$$U(a', a) \in \mathcal{J}([T]R^{\Sigma(a'-a)}), \qquad a, a' \in \Gamma(\varphi)$$

and

$$\nu\left(\bigcap_{-I<i<I} \{x \in A : Q_A{}^i x \in \mathcal{G}(\alpha)\, x\}\right) > (1 - \epsilon)\, \nu(A). \tag{29}$$

Proof. With some $L_1 \in \mathbb{N}$ there is a system β of Q-units,

$$\beta = ([1, L_1], A, B(\cdot), V(\cdot, \cdot))$$

such that

$$\nu\Big(\bigcap_{-I<i<I} \{x \in A: Q_A{}^i x \in \mathscr{G}(\beta)\, x\}\Big) > (1 - \tfrac{1}{2}\epsilon)\, \nu(A). \tag{30}$$

We exploit (28) to obtain with some $L_2 \in \mathbb{N}$ disjoint sets

$$C(l_2) \subset B(1), \qquad 1 \leqslant l_2 \leqslant L_2\,,$$

and

$$j(l_2\,, l_1', l_1) \in \mathbb{Z}, \qquad 1 \leqslant l_2 \leqslant L_2\,, 1 \leqslant l_1\,, l_1' \leqslant L_1\,,$$

such that, with some $L_3 \in \mathbb{N}$,

$$L_3 > 2\epsilon^{-1} L_1 L_2\,, \tag{31}$$

and

$$\nu(C(l_2)) = L_3^{-1} \nu(A), \qquad 1 \leqslant l_2 \leqslant L_2\,,$$

and such that, with $V(l_2\,, l_1', l_1)$, the restriction of $V(l_1', l_1)$ to $V(l_1\,, 1)\, C(l_2)$,

$$V(l_2\,, l_1', l_1) \in \mathscr{I}([T]\, R^{j(l_2, l_1', l_1)}), \qquad 1 \leqslant l_2 \leqslant L_2\,, \quad 1 \leqslant l_1\,, l_1' \leqslant L_1\,.$$

We can then build a suitable system of Q-units over the index set $[1, L_1] \times [1, L_2]$ with the partition $(V(l_1\,, 1)\, C(l_2))$, $1 \leqslant L_1\,, 1 \leqslant l_2 \leqslant L_2$. However, we enlarge the index set, and then relabel its elements, and we then extend the unit system in order to arrive at a unit system

$$\alpha = (\Gamma, A, A(\cdot), U(\cdot, \cdot)),$$

where, with some $J \in \mathbb{N}$, $J = p$ in case $p \in \mathbb{N}$, and with some $K_j \in \mathbb{Z}_+\,, 1 \leqslant j < J$,

$$\Gamma = \{(j, k): 1 \leqslant k \leqslant K_j\,, 0 \leqslant j < J\}, \qquad \sum_{0 \leqslant j < J} K_j = L_3\,,$$

and where

$$U((j', k'), (j, k)) \in \mathscr{I}([T] R^{j'-j}), \qquad (j, k), (j', k') \in \Gamma.$$

Set $\varphi(j) = K_j\,, 0 \leqslant j < J$, $\varphi(j) = 0$, $j \geqslant J$. By (30) and (31), we have satisfied (29). Q.E.D.

(3.7) Theorem. *Let T be an ergodic ν-preserving automorphism of (X, ν), $\nu(X) = 1$, and let $R \in \mathscr{N}[T]$. Set $p = p(R)$. There exists an isomorphism*

$$V: (X, \nu) \to (X_p\,, \nu_p), \qquad V\nu = \nu_p\,,$$

such that

$$V[T]V^{-1} = \mathscr{H}_p, \tag{32}$$

and

$$VRV^{-1} \in \mathscr{R}_p. \tag{33}$$

Proof. We denote

$$\Gamma_p(n) = \prod_{1 \leqslant m \leqslant n} \Gamma(\varphi(p, m)).$$

On (X_p, ν_p), one has the unit systems

$$(\Gamma_p(n), X_p, Z(\cdot), P(\cdot, \cdot)), \qquad n \in \mathbb{N},$$

where for $a, a' \in \Gamma_p(n)$ the $P(a', a)$ leave all coordinates beyond the nth coordinate unchanged, $n \in \mathbb{N}$. The proof relies on an inductive construction of a sequence α_n of unit systems

$$\alpha_n = (\Gamma_p(n), X, A_n(\cdot), U_n(\cdot, \cdot))$$

such that

$$U_n(a', a) \in \mathscr{I}([T]R^{\Sigma(a'-a)}), \qquad a, a' \in \Gamma_p(n), n \in \mathbb{N}, \tag{34}$$

and where α_{n+1} refines α_n. One requires also that the σ-algebra of (X, ν) is generated by the $A_n(a)$, $a \in \Gamma_p(n)$, $n \in \mathbb{N}$, and that

$$\{T^m R^l x: m, l \in \mathbb{Z}\} = \bigcup_{n \in \mathbb{N}} \mathscr{G}(\alpha_n)\, x, \qquad \text{f.a.a.} \quad x \in X. \tag{35}$$

To obtain such a sequence of unit systems one constructs inductively the unit systems $\alpha_{n(k)}$ with some increasing sequence $n(k)$, $\in \mathbb{N}$, $k \in \mathbb{N}$, where the $\alpha_{n(k+1)}$ is produced by refining $\alpha_{n(k)}$, by means of a suitably chosen unit system β_k. To make a suitable choice of the β_k one appeals now to Lemma (3.5) in order to ensure the generation property, and one appeals to Lemma (3.6) and Corollary (2.7) in order to ensure the validity of (35). Both lemmas are used in conjunction with Lemmas (3.2) and (3.4). One uses at this point the hypothesis that every $\varphi \in \Phi_p$ appears infinitely often as an entry in $(\varphi(p, n))_{n \in \mathbb{N}}$.

The generation property yields an isomorphism

$$V: (X, \nu) \to (X_p, \nu_p), \qquad V\nu = \nu_p,$$

such that

$$VA_n(a) = Z(a), \qquad a \in \Gamma_p(n), n \in \mathbb{N},$$

and by the refining property of the α_n one has

$$VU_n(a', a)V^{-1} = P(a', a), \qquad a, a' \in \Gamma_p(n), n \in \mathbb{N}. \tag{36}$$

By (34) and (35) we can cover X by an increasing sequence $E(n) \subset X$, $n \in \mathbb{N}$, such that one has a decomposition of the sets $A_n(a) \cap E(n)$,

$$A_n(a) \cap E(n) = \bigcup_{\{a' \in \Gamma_p(n) : \Sigma(a'-a)=0 \bmod p\}} E(n, a', a), \qquad a \in \Gamma_p(n), \quad n \in \mathbb{N}$$

such that

$$Tx = U_n(a', a)\, x, \qquad \text{f.a.a. } x \in E(n, a', a), \quad a, a' \in \Gamma_p(n),$$
$$\textstyle\sum (a' - a) = 0 \bmod p, \quad n \in \mathbb{N}.$$

It follows from this and from (34) and (35) that (32) holds. Similarly, we have from (34) and (35) that we can cover X by an increasing sequence $D(n) \subset X$, $n \in \mathbb{N}$, such that one has a decomposition of the sets $A_n(a) \cap D(n)$,

$$A_n(a) \cap D(n) = \bigcup_{\{a' \in \Gamma_p(n) : \Sigma(a'-a)=1 \bmod p\}} D(n, a', a), \qquad a \in \Gamma_p(n), \quad n \in \mathbb{N}$$

such that

$$Rx = U_n(a', a)\, x, \qquad \text{f.a.a. } x \in D(n, a', a), \quad a, a' \in \Gamma_p(n),$$
$$\textstyle\sum (a' - a) = 1 \bmod p, \quad n \in \mathbb{N}.$$

It follows from this and from (34) and (35) that

$$h_{VRV^{-1}} = 1,$$

and (33) is shown. Q.E.D.

(3.8) THEOREM. *Let T be an ergodic ν-preserving automorphism of (X, ν), $\nu(X) = \infty$, and let $R \in \mathcal{N}[T]$. Set $p = p(R)$, $\lambda = \operatorname{mod} R$. Then there exists an isomorphism*

$$V\colon (X, \nu) \to (X_{p,\lambda,\infty}, \nu_{p,\lambda,\infty}), \qquad V\nu = \nu_{p,\lambda,\infty},$$

such that

$$V[T]V^{-1} = \mathscr{H}_{p,\lambda,\infty}, \qquad VRV^{-1} \in \mathscr{R}_{p,\lambda,\infty}.$$

Proof. The proof is achieved by the same means as the proof of Theorem (3.7) Q.E.D.

(3.9) COROLLARY. *The outer period and the module form a complete set of invariants for the outer conjugacy of the elements in the normalizer of the full group of an ergodic measure preserving automorphism.*

Proof. This is from Theorems (3.7) and (3.8) and (27). Q.E.D.

4. Applications

We want to describe next some examples for Corollary (3.9). There are first the examples of infinite product type. Let $N_k \in \mathbb{N}$, $k \in \mathbb{N}$, and set

$$(X, \nu) = \prod_{k\in\mathbb{N}} (\{1,..., N_k\}, p_k), \qquad p_k(n) = N_k^{-1}, \qquad 1 \leqslant n \leqslant N_k, \quad k \in \mathbb{N}.$$

Let $\mathscr{F}$ be the odometer group on (X, ν). $\mathscr{F}$ is an ergodic hyperfinite group that leaves ν invariant. The infinite products are defined by

$$(Rx)_k = x_k + 1 \bmod N_k, \qquad k \in \mathbb{N}, x = (x_k)_{k\in\mathbb{N}} \in X,$$

and we have that $R \in \mathscr{N}[\mathscr{F}]$. If $\sup_{k\in\mathbb{N}} N_k = \infty$, then R is outer aperiodic, and hence by Corollary (3.9) all such R are outer conjugate. One can prove directly in this instance that R and $\mathscr{F}$ generate a hyperfinite group. Infinite products induce infinite product automorphisms of the hyperfinite II_1 factor.

Further, there are the Bernoulli shifts and the Markov shifts. Take, e.g., the n-shift. Set

$$(Y_n, \mu_n) = \prod_{i\in\mathbb{Z}} (\{1,..., n\}, p_n), \qquad p_n(m) = n^{-1}, \qquad 1 \leqslant m \leqslant n, \quad n \geqslant 2,$$

and define an $\mathscr{F}_n$ as the odometer group on (Y_n, μ_n). The n-shift S, $Sy = (y_{i+1})_{i\in\mathbb{Z}}$, $y \in Y_n$ is then in $\mathscr{N}[\mathscr{F}_n]$. By a similar construction, all Bernoulli shifts and all mixing Markov shifts can be viewed as elements of $\mathscr{N}[T]$, where T is ergodic type II_1 (compare [4, 5]). All these shifts are outer aperiodic, and therefore they are all outer conjugate by Corollary (3.9), and also outer conjugate to the outer aperiodic infinite products. Furthermore, the noncommutative Bernoulli shifts and the noncommutative Markov shifts that these shifts induce on the hyperfinite II_1 factor are all outer conjugate, and also outer conjugate to the outer aperiodic infinite products. These noncommutative shifts are not conjugate if their entropy differs [4, 5].

(4.2) Corollary. *For every $h > 0$ there exists a $P \in [\mathscr{F}_n]$ such that PS is Bernoulli with entropy h.*

(4.3) Corollary. *There exists a $P \in [\mathscr{F}_n]$ such that PS has discrete spectrum.*

We conclude with an application to the problem of approximate finiteness, which has been prominent in this theory since its creation by Dye [7, 8]. Let us say that a countable group is approximately finite if all its action by automorphisms of a Lebesgue measure space are approximately finite. For the proof of the following corollary one must, when distinguishing cases, in addition to Theorems (2.5) and (2.6), also use an argument of the sort as given in [13, Sect. 5, Lemma 5.4], and one must carry out an ergodic decomposition. It was known before that countable abelian groups are approximately finite [9]. Also it was known that every countable solvable group has a free ergodic hyperfinite finite action [10].

(4.4) COROLLARY. *Solvable groups are approximately finite. Extensions of approximately finite groups by solvable groups are approximately finite.*

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