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Homogeneity of the State Space of Factors of Type III_1

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A factor M is of type III_1 if and only if the action of its unitary group on its state space by inner automorphisms is topologically transitive in the norm topology.

1. INTRODUCTION

The states on the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$ are classified up to unitary equivalence by their eigenvalue list $(\lambda_j)_{j=1,\dots,n}$, i.e., the list of eigenvalues of the associated density matrix. Let M be a factor with separable predual; then it is a natural problem to try and classify the normal states of M up to unitary equivalence, viz., $\phi_1 \sim \phi_2$ when ϕ_2 is in the norm closure of the orbit of ϕ_1 under inner automorphisms of M. If M is of type $\neq \text{III}_1$ it is an easy exercise, using the results on the flow of weights of M to get such a classification. If M is of type III_1 the flow of weights is trivial and one is led to conjecture (cf. [5]) that any two normal states are topologically equivalent.

We prove this fact below, and obtain some easy consequences. The first is that M has property L_λ of Powers [8] if and only if $\lambda/(1-\lambda) \in S(M)$. We then show that if M is of type II or III then there exists a faithful normal state on M whose centralizer contains the hyperfinite factor. This last result makes it very easy to prove a conjecture of Dell'Antonio [6] to the effect that type I factors are the only factors in which weak convergence of a sequence of normal states to a normal state implies norm convergence of the same sequence.

In order to prove the main result we study in Section 2 the skew information

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$I(\phi, x)$ of an operator x relative to a state ϕ. If M is a factor of type I with trace Tr and ϕ the state $\phi(x) = \mathrm{Tr}(hx)$ the skew information was introduced by Wigner and Yanase [9] as the quantity $-\mathrm{Tr}([h^{1/2}, x]^2)$ when x is self-adjoint, and is a measure of how far x is from commuting with ϕ.

2. Commutation of Operators with Normal States

Let M be a von Neumann algebra with separable predual. Let M act standardly on $\mathscr{H}$ and $\mathscr{P}^\natural$ be the natural cone associated with some cyclic and separating vector [1, 3, 10]. Let J be the involution associated with $\mathscr{P}^\natural$. If ϕ is a normal positive linear functional on M let ξ_ϕ be the unique vector in $\mathscr{P}^\natural$ representing ϕ: $\phi(x) = \langle x\xi_\phi, \xi_\phi\rangle$, $x \in M$. When ϕ is faithful we let Δ_ϕ be the modular operator for (M, ξ_ϕ). In order to estimate the commutativity of $x \in M$ with ϕ we use the quantity

$$I(\phi, x) = \tfrac{1}{2} \|(Jx^*J - x)\xi_\phi\|^2;$$

cf. [9]. By construction $0 \leqslant I(\phi, x) \leqslant (\|x\|_\phi^\#)^2$, where $(\|x\|_\phi^\#)^2 = \|x\xi_\phi\|^2 + \|x^*\xi_\phi\|^2$. We list a few properties of $I(\phi, x)$ which will be used below.

Proposition 1. *Let ϕ be as above. Then*

(a) $I(\phi, xy)^{1/2} \leqslant I(\phi, x)^{1/2} \|y\| + \|x\| I(\phi, y)^{1/2}$, $x, y \in M$.

(b) $\|[\phi, x]\| \leqslant 2^{3/2}\phi(1)^{1/2} I(\phi, x)^{1/2}$, $x \in M$.

(c) *If ϕ is faithful and $x \in M(\sigma^\phi, [(1-\delta)^2, (1+\delta)^2])$, where the spectral subspace is taken with respect to the group $\mathbb{R}_*^+$, then $I(\phi, x) \leqslant \frac{1}{2}\delta^2(\|x\|_\phi)^2$.*

(d) *Let e be a projection in the centralizer M_ϕ of ϕ, then $I(\phi^e, x) = I(\phi, x)$, where $\phi^e = \phi \mid M_e$* and $x \in M_e$.

(e) *Let e be a projection in M_ϕ, let $x \in eM$, $y \in (1-e)M$. Then $I(\phi, x+y) = I(\phi, x) + I(\phi, y)$.*

(f) $\big| \|x\xi_\phi\| - \|x^*\xi_\phi\| \big| \leqslant (2I(\phi, x))^{1/2}$.

(g) *Let $\theta = \left(\begin{smallmatrix} \phi & 0 \\ 0 & \phi \end{smallmatrix}\right)$ be the functional $\phi \otimes$* Trace *on $M \otimes M_2(\mathbb{C})$. Then $I(\theta, \left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)) = 2I(\phi, x)$, $x \in M$.*

(h) *Given $k = k^* \in M$ there exists a positive finite measure μ on $\mathbb{R}^2$ such that for any bounded real Borel function f on $\mathbb{R}$ we have*

$$I(\phi, f(k)) = \frac{1}{2} \int |f(x) - f(y)|^2 \, d\mu(x, y),$$

$$\phi(f(k)) = \int f(x)\, d\mu = \int f(y)\, d\mu.$$

Proof. (a) Since $JMJ = M'$ we have

$$\|(J(xy)^*J - xy)\xi_\phi\| \leqslant \|y\| \|(Jx^*J - x)\xi_\phi\| + \|x\| \|(Jy^*J - y)\xi_\phi\|.$$

(b) If $y \in M$ we have

$$\begin{aligned} |[\phi, x](y)| &= |\langle xy\xi_\phi , \xi_\phi\rangle - \langle yx\xi_\phi , \xi_\phi\rangle| \\ &\leqslant |\langle y\xi_\phi , x^*\xi_\phi\rangle - \langle y\xi_\phi , JxJ\xi_\phi\rangle| + |\langle yJx^*J\xi_\phi , \xi_\phi\rangle - \langle yx\xi_\phi , \xi_\phi\rangle| \\ &\leqslant 2\|y\| \|\xi_\phi\| \|(Jx^*J - x)\xi_\phi\|. \end{aligned}$$

(c) Let $S = [(1-\delta)^2, (1+\delta)^2]$ and $x \in M(\sigma^\phi, S)$. Then $x\xi_\phi$ belongs to the range of the spectral projection of Δ_ϕ corresponding to S, so that

$$(2I(\phi, x))^{1/2} = \| \Delta_\phi^{1/2} x\xi_\phi - x\xi_\phi \| \leqslant \sup_{\lambda \in S} | \lambda^{1/2} - 1 | \| x\xi_\phi \|.$$

(d) Let $x \in M_e$ and identify x with exe in M. Since the involution of $e\xi_\phi$ with respect to M_e is eJe [3], and $Je\xi_\phi = e\xi_\phi$, we have

$$\begin{aligned} I(\phi^e, x) &= \tfrac{1}{2} \|((eJe)\, x^*(eJe) - x)\xi_\phi\|^2 \\ &= \tfrac{1}{2} \| Jex^*eJe\xi_\phi - x\xi_\phi \|^2 \\ &= \tfrac{1}{2} \| Jx^*\xi_\phi - x\xi_\phi \|^2 \\ &= I(\phi, x). \end{aligned}$$

(e) As in (d) we have $eJx^*\xi_\phi = Jx^*Je\xi_\phi = Jx^*e\xi_\phi = Jx^*\xi_\phi$ when $x \in eM$. Thus $e(x - Jx^*J)\xi_\phi = (x - Jx^*J)\xi_\phi$. Similarly $(1-e)(y - Jy^*J)\xi_\phi = (y - Jy^*J)\xi_\phi$ for $y \in (1-e)M$, and (e) follows.

(f) Use that $| \|x\xi_\phi\| - \|x^*\xi_\phi\| | = | \|x\xi_\phi\| - \|Jx^*J\xi_\phi\| | \leqslant \|x\xi_\phi - Jx^*J\xi_\phi\|$.

(g) We have $\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ commutes with θ, so that $I(\theta, \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}) = I(\theta, \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix}) = 2I(\phi, x)$, using (d) and (e).

(h) Let A and B be Borel subsets of $\mathbb{R}$. Let l_A, l_B be their characteristic functions. Put $\mu(A \times B) = \langle l_A(k)\xi_\phi , Jl_B(k)\xi_\phi\rangle$. Then $\mu(A \times B) \geqslant 0$ since $Jl_B(k)\, Jl_A(k) \geqslant 0$, so there exists a unique positive measure μ on $\mathbb{R}^2$ determined by these values on rectangles. If f and g are bounded real Borel functions on $\mathbb{R}$ we get

$$\int f(x)\, g(y)\, d\mu(x, y) = \langle f(k)\, \xi_\phi , Jg(k)\, \xi_\phi\rangle,$$

$$\int f(x)\, d\mu(x, y) = \langle f(k)\, \xi_\phi , \xi_\phi\rangle = \langle \xi_\phi , Jf(k)\, \xi_\phi\rangle = \int f(y)\, d\mu(x, y).$$

Hence we have

$$\begin{aligned} \| Jf(k)\, \xi_\phi - f(k)\, \xi_\phi \|^2 &= 2\|f(k)\, \xi_\phi\|^2 - 2\langle f(k)\, \xi_\phi , Jf(k)\, \xi_\phi\rangle \\ &= \int (f(x)^2 + f(y)^2 - 2f(x)f(y))\, d\mu(x, y) \quad \text{Q.E.D.} \end{aligned}$$

Next, as in [4], we let E_a be the characteristic function of the interval $[a, +\infty) \subset \mathbb{R}$ for each $a > 0$. For $x \in M$ we put $u_a(x) = u(x)\, E_a(|\, x \,|)$, where $x = u(x) \mid x \mid$ is the polar decomposition of x. Let da be the Lebesgue measure on $\mathbb{R}$.

THEOREM 2. *For any $\phi \in M_*{}^+$ and $x \in M$ we have:*

(a) $\int_0^\infty (\| u_{a^{1/2}}(x)\|_\phi^\#)^2\, da = (\| x \|_\phi^\#)^2$,

(b) $\int_0^\infty I(\phi, u_{a^{1/2}}(x))\, da \leqslant 6I(\phi, x)^{1/2} \| x \|_\phi^\#$.

Proof. (a) We have $u_{a^{1/2}}(x)^* u_{a^{1/2}}(x) = E_{a^{1/2}}(|\, x \,|) = E_a(x^*x)$ and $\int_0^\infty E_a(x^*x)\, da = x^*x$. Since $u_{a^{1/2}}(x^*) = u_{a^{1/2}}(x)^*$ (a) is immediate.

(b) First replace ϕ by $\theta = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$ and x by $\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$. We have $\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} = \begin{pmatrix} x^*x & 0 \\ 0 & xx^* \end{pmatrix}$, and hence

$$\left|\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right| = \begin{pmatrix} |\, x \,| & 0 \\ 0 & |\, x^* \,| \end{pmatrix}, \qquad u\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & u(x)^* \\ u(x) & 0 \end{pmatrix}.$$

Thus we have

$$u_a\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & u(x^*) \\ u(x) & 0 \end{pmatrix}\begin{pmatrix} E_a(|\, x \,|) & 0 \\ 0 & E_a(|\, x^* \,|) \end{pmatrix} = \begin{pmatrix} 0 & u_a(x^*) \\ u_a(x) & 0 \end{pmatrix}.$$

Using Proposition 1(g) and the computation of $\|\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\|_\theta^\#$ we see that to prove (b) we can assume that $x = x^*$. Then by Proposition 1(h) we just have to show that with the Borel functions $F_a\colon \mathbb{R} \to \mathbb{R}$, $F_a(t) = \operatorname{sign}(t)\, E_a(|\, t \,|)$, we have

$$\int_0^\infty \left(\int |\, F_{a^{1/2}}(x) - F_{a^{1/2}}(y)|^2\, d\mu(x, y)\right) da$$
$$\leqslant 4 \left(\int |\, x - y \,|^2\, d\mu(x, y)\right)^{1/2} \left[\left(\int |\, x \,|^2\, d\mu(x, y)\right)^{1/2} + \left(\int |\, y \,|^2\, d\mu(x, y)\right)^{1/2}\right]$$

for any positive finite measure μ on $\mathbb{R}^2$. For sign x = sign y we have, since $E_{a^{1/2}}(x) = E_a(x^2)$,

$$\int |\, F_{a^{1/2}}(x) - F_{a^{1/2}}(y)|^2\, da = \int |\, E_a(x^2) - E_a(y^2)|\, da = |\, x - y \,|\, (|\, x \,| + |\, y \,|).$$

For sign x = $-$sign y we have $|\, F_{a^{1/2}}(x) - F_{a^{1/2}}(y)|^2 \leqslant 2(E_a(x^2) + E_a(y^2))$, so that

$$\int |\, F_{a^{1/2}}(x) - F_{a^{1/2}}(y)|^2\, da \leqslant 2(x^2 + y^2) \leqslant 4(x - y)^2 = 4 \,|\, x - y \,|\, (|\, x \,| + |\, y \,|).$$

The desired inequality now follows easily from the Schwarz inequality. Q.E.D.

COROLLARY 3. *Let $\phi \in M_*^+$ and $x \in M$ satisfy $x \neq 0$, $I(\phi, x) \leqslant \epsilon(\| x \|_\phi^\#)^2$. Then there exists $a > 0$ such that $v = u_a(x) \neq 0$ and $I(\phi, v) \leqslant 7\epsilon^{1/2}(\| v \|_\phi^\#)^2$.*

Proof. By hypothesis $I(\phi, x)^{1/2} \leqslant \epsilon^{1/2} \| x \|_\phi^\#$; hence by Theorem 2

$$\int I(\phi, u_{a^{1/2}}(x))\, da \leqslant 6\epsilon^{1/2}(\| x \|_\phi^\#)^2$$

$$= 6\epsilon^{1/2} \int (\| u_{a^{1/2}}(x) \|_\phi^\#)^2\, da.$$

As $x \neq 0$ it is impossible that

$$I(\phi, u_{a^{1/2}}(x)) \geqslant 7\epsilon^{1/2}(\| u_{a^{1/2}}(x) \|_\phi^\#)^2 \qquad \text{for all} \quad a > 0,$$

hence the conclusion.

3. HOMOGENEITY OF THE STATE SPACE OF FACTORS OF TYPE III_1

It was shown in [5] that if M is a factor of type III_1 with unitary group U then the action of U by inner automorphisms on the space of weights with infinite multiplicity, gifted with a natural topology, is topologically transitive, i.e., each weight has a dense orbit. Moreover it was conjectured that the same is true for the action of U on the set of normal states of M. We shall prove:

THEOREM 4. *Let M be a factor type III_1 with separable predual. Then for any $\epsilon > 0$ and normal states ϕ and ψ there exists a unitary u in M such that $\| \phi_u - \psi \| < \epsilon$, ($\phi_u(x) = \phi(u^*xu)$, $x \in M$).*

Before we give the proof we include some applications. Our von Neumann algebras will always have separable preduals. Note first that if $M \neq \mathbb{C}$ is a factor satisfying the conclusion of the theorem then it is easy to see that $S(M) = \mathbb{R}_+$; cf. the proof of the next corollary.

COROLLARY 5. *Let $\lambda \in [0, \frac{1}{2})$. Then a factor M has property L_λ of Powers if and only if $\lambda/(1 - \lambda) \in S(M)$.*

Proof. By [8, p. 157], M has property L_0 if and only if M is of infinite type, hence if and only if $0 \in S(M)$. If $\lambda \in (0, \frac{1}{2})$ by [2, Théorème 3.5.4 and Corollaire 3.7.7], all that remains is to show that if M is of type III_1 then M has property L_λ. Using a state of the form $\omega_\lambda \otimes \phi$ on $M_2(\mathbb{C}) \otimes M$, where $\omega_\lambda = \mathrm{Tr}((\begin{smallmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{smallmatrix})\cdot)$, it is clear that condition L_λ is satisfied for some state on M for all $\lambda \in (0, \frac{1}{2}]$, hence by all states using the homogeneity (Theorem 4).

COROLLARY 6. *A von Neumann algebra M acting standardly on an infinite-*

dimensional Hilbert space $\mathscr{H}$ is a factor of type III_1 *if and only if the product $\mathscr{U}\mathscr{U}'$ of the unitary groups of M and M', respectively, acts topologically transitively on the unit sphere in $\mathscr{H}$.*

Proof. The "if" part is easy; cf. the proof of Corollary 5. Conversely, to show that given unit vectors ξ and η in $\mathscr{H}$ and $\epsilon > 0$, there exist $u \in \mathscr{U}$, $v \in \mathscr{U}'$ with $\| uv\xi - \eta \| < \epsilon$, one can assume that ξ and η are separating and cyclic by [7] and that $\eta \in \mathscr{P}^{\natural}\xi$ by [2, Lemma 3.5.5]. Then one applies Theorem 4 to the vector states ω_ξ and ω_η.

COROLLARY 7. *Let $(M_\nu)_{\nu\in A}$ be a denumerable family of factors of type* III_1. *Then the infinite tensor product $\otimes_{\nu\in A}(M_\nu, \phi_\nu)$ is up to isomorphism, independent of the choice of the sequence (ϕ_ν), ϕ_ν normal state of M_ν.*

Proof. Immediate by homogeneity.

COROLLARY 8. *Let R be the hyperfinite factor and let M be a non-type-*I *factor. Then there exists a faithful normal state ϕ on M whose centralizer contains R.*

Proof. When M is semifinite or of type III_μ, $0 \leqslant \mu < 1$, the conclusion is easy (cf. [2]), so we assume M is of type III_1. From the proof of Corollary 5, M has property $L_{1/2}$, so if ϕ_0 is a normal state of M then there exists a subfactor K_0 of type I_2 of M such that $\| \phi_0 - \phi_0 \mid K'_0 \otimes \tau_{K_0} \| < \epsilon$ for $\epsilon > 0$, where τ_{K_0} is the normalized trace on K_0, and $\phi \mid K'_0$ the restriction of ϕ_0 to the commutant of K_0 in M. Repetition of this procedure gives a sequence (ϕ_n, K_n), where ϕ_n is a normal state of M, the K_j are pairwise commuting I_2 subfactors of M, and $K_1, ..., K_n$ belong to the centralizer of ϕ_n. Moreover, we can assume $\| \phi_n - \phi_{n+1} \| < 2^{-n}$; hence the ϕ_n converge in norm to a normal state ϕ. By construction the von Neumann algebra K generated by the K_j is contained in the centralizer of ϕ and hence is the hyperfinite factor R. Now ϕ can fail to be faithful, but as its support e belongs to the relative commutant of K in M we obtain the conclusion of the corollary for M_e. Since M is isomorphic to M_e we are through.

Following Dell'Antonio [6], a factor M has *property* U if each sequence of normal states of M which converges weakly to a normal state already converges in the norm topology. Dell'Antonio showed that every factor of type I has property U and conjectured the converse [6].

COROLLARY 9. *A factor has property U if and only if it is of type* I.

Proof. Let M be a factor not of type I. By Corollary 8, there is a faithful normal state ϕ on M whose centralizer M_ϕ contains the hyperfinite factor R. Composing the canonical expectations of M onto M_ϕ and of M_ϕ onto R we get a normal expectation Φ of M onto R. Since R does not have property U [6] there is a sequence (ϕ_n) of normal states on R which converges weakly to a normal state but not in norm. Then the sequence $(\phi_n \cdot \Phi)$ has the same properties in M_*, so M does not have property U.

Note that since we might consider a reduced algebra M_e, Corollary 9 remains true without the hypothesis that M_* is separable.

We now prove two lemmas which will be important for the proof of Theorem 4.

LEMMA 10. *Let $\phi \in M_*^+$, where M is a factor of type III_1. Let $e', f' \in M_\phi$ be nonzero projections smaller than the support of ϕ. For any $\epsilon > 0$ there exists a partial isometry $u \neq 0$ in M such that $u^*u = e \leqslant e'$, $uu^* = f \leqslant f'$, and*

(α) $I(\phi, u) \leqslant \epsilon(\| u \|_\phi^\#)^2$,

(β) $I(\phi, e) \leqslant \epsilon\phi(e)$, $I(\phi, f) \leqslant \epsilon\phi(f)$.

Proof. We can assume that ϕ is faithful [7]. Since M is a factor of type III_1, for any $\delta > 0$, there exists $x \neq 0$, $x \in M(\sigma^\phi, [1 - \delta, 1 + \delta])$, $x \in f'Me'$; see [2, Sect. 2.1]. Now by Corollary 3 and Proposition 1(c), we can find a partial isometry $u \in f'Me'$ such that $I(\phi, u) \leqslant \epsilon(\| u \|_\phi^\#)^2$, $u \neq 0$. Next, by Proposition 1(f) we get, with $u^*u = e$, $uu^* = f$, that

$$\begin{aligned} |\phi(e) - \phi(f)| &\leqslant |\phi(e)^{1/2} - \phi(f)^{1/2}| \, |\phi(e)^{1/2} + \phi(f)^{1/2}| \\ &\leqslant 2I(\phi, u)^{1/2}(\phi(e) + \phi(f))^{1/2} \\ &\leqslant 2\epsilon^{1/2}(\phi(e) + \phi(f)). \end{aligned}$$

Hence by assuming $\epsilon^{1/2} < \frac{1}{8}$ we get $\frac{1}{2}\phi(e) \leqslant \phi(f) \leqslant 2\phi(e)$. By Proposition 1(a) we have $I(\phi, u^*u) \leqslant 4I(\phi, u) \leqslant 4\epsilon(\phi(e) + \phi(f))$. Hence ($\beta$) follows. Q.E.D.

LEMMA 11. *Let $\xi \in \mathscr{P}^\natural$, and $e \in M$ be a projection. Put $\xi' = eJeJ\xi + (1 - e) J(1 - e) J\xi$. Then*

(a) *ξ' belongs to $\mathscr{P}^\natural$ and with obvious notation, $\phi'(e) = \phi(e) - I(\phi, e)$.*

(b) *Let u be a partial isometry in M such that $ue = u$, $eu = 0$. Then $I(\phi', u) \leqslant I(\phi, u)$.*

Proof. (a) Both $eJeJ\xi$ and $(1 - e) J(1 - e) J\xi$ belong to $\mathscr{P}^\natural$ [3], so $\xi' \in \mathscr{P}^\natural$. We have $\langle e\xi', \xi' \rangle = \langle eJeJ\xi, eJeJ\xi \rangle = \langle e\xi, JeJ\xi \rangle$, but $I(\phi, e) = \phi(e) - \langle e\xi, JeJ\xi \rangle$.

(b) We have $(u - Ju^*J) eJeJ = uJeJ$, since $u^*e = 0$. Also, $(u - Ju^*J) \times (1 - e) J(1 - e)J = -(1 - e) Ju^*J$. Thus $(u - Ju^*J)\xi' = JeJu\xi - (1 - e) \times Ju^*J\xi = (1 - e) JeJ(u - Ju^*J)\xi$ since $(1 - e)u = u$ and $eu^* = u^*$. Since $\|(1 - e) JeJ\| \leqslant 1$, (b) follows. Q.E.D.

Proof of Theorem 4. Let $\delta > 0$, $\delta \leqslant 1$, ϕ_0, ψ_0 be faithful normal states of M and ξ_0, η_0 be the corresponding unit vectors in $\mathscr{P}^\natural$. Let R be the set of all triples $r = (w, \alpha, \beta)$, where w is a partial isometry in M, $\alpha, \beta \in \mathscr{H}$, and:

(a) With $a = w^*w$, $b = ww^*$, we have $a\alpha = \alpha$, $b\beta = \beta$.

(b) $\|\alpha\|^2 \leqslant \delta\phi_0(a)$, $\|\beta\|^2 \leqslant \delta\psi_0(b)$.

(c) $\xi = \xi_0 - \alpha - J\alpha$ and $\eta = \eta_0 - \beta - J\beta$ belong to $\mathscr{P}^\natural$ and $(a - JaJ)\xi = 0$, $(b - JbJ)\eta = 0$, $\|\xi\| \leqslant 1$, $\|\eta\| \leqslant 1$.

(d) Let $\phi, \psi \in M_*^+$ correspond to ξ, η, let $\theta = \left(\begin{smallmatrix}\phi & 0\\ 0 & \psi\end{smallmatrix}\right)$; then $\bar{w} = \left(\begin{smallmatrix}0 & 0\\ w & 0\end{smallmatrix}\right)$ satisfies $I(\theta, \bar{w}) \leqslant \delta(\|\bar{w}\|_\theta^\#)^2$.

We define a partial ordering on R by setting $r \leqslant r'$ when

(1) w' is an extension of w (i.e., $w'a = w$, $w'^*b = w^*$),

(2) $a(\alpha' - \alpha) = 0$, $b(\beta' - \beta) = 0$,

(3) $\|\alpha' - \alpha\|^2 \leqslant \delta\phi_0(a' - a)$, $\|\beta' - \beta\|^2 \leqslant \delta\psi_0(b' - b)$.

This relation is transitive, in fact if $r \leqslant r'$ and $r' \leqslant r''$ then $a(\alpha'' - \alpha) = 0$ because $a'(\alpha'' - \alpha') = 0$ and $a \leqslant a'$. Also $\|\alpha'' - \alpha\|^2 = \|\alpha'' - \alpha'\|^2 + \|\alpha' - \alpha\|^2$ because $a'(\alpha' - \alpha) = \alpha' - \alpha$ while $a'(\alpha'' - \alpha') = 0$. It follows that $\leqslant$ is indeed a partial ordering on R.

To prove the existence of a maximal element in R we note that the map $r = (w, \alpha, \beta) \to \phi_0(a)$ $(a = w^*w)$ is injective on any totally ordered subset of R since $a = a'$ implies $r = r'$ whenever $r \leqslant r'$. Hence we just have to show that any increasing sequence $(r_n)_{n\in N}$ in R is majorized. To see that, let $a = \lim_n a_n$, $b = \lim_n b_n$ in the strong topology. Furthermore, since the w_n's are extensions of each other they converge in the strong $*$-topology to a partial isometry w such that $w^*w = a$, $ww^* = b$. By (3), $\|\alpha_n - \alpha_m\|^2 \leqslant \delta\phi_0(a_m - a_n)$ whenever $n \leqslant m$. Thus $\alpha = \lim_n \alpha_n$, $\beta = \lim_n \beta_n$ exist, and by continuity we have $r = (w, \alpha, \beta) \in R$. The relation $r_n \leqslant r$ for all n also follows by continuity.

Now let by Zorn's lemma $r = (w, \alpha, \beta)$ be a maximal element of R. We assume that $a = w^*w \neq 1$, $b = ww^* \neq 1$ and shall obtain a contradiction.

Let $e' = 1 - a$, $f' = 1 - b$, $\bar{e}' = \left(\begin{smallmatrix}e' & 0\\ 0 & 0\end{smallmatrix}\right)$, $\bar{f}' = \left(\begin{smallmatrix}0 & 0\\ 0 & f'\end{smallmatrix}\right)$, and $\theta = \left(\begin{smallmatrix}\phi & 0\\ 0 & \psi\end{smallmatrix}\right)$, where ϕ and ψ are as in (d). By construction $\bar{e}'$ and $\bar{f}'$ commute with θ, moreover $\theta^{\bar{e}'}$ is identical with $\phi^{e'}$; $\phi^{e'}$ is represented in $\mathscr{P}^\natural$ by the vector $e'Je'J\xi = (1 - a)J(1 - a)J\xi = (1 - a)J(1 - a)J\xi_0$ because $\xi = \xi_0 - \alpha - J\alpha$ and $(1 - a)\alpha = 0$. Since ξ_0 is separating and cyclic for M, it follows that $\phi^{e'}$, which corresponds to $e'Je'J\xi_0$, is faithful on $M_{e'}$. In particular $\bar{e}'$ is smaller than support θ. Similarly $\bar{f}' \leqslant$ support θ. Thus Lemma 10 gives us a partial isometry of the form $\bar{u} = \left(\begin{smallmatrix}0 & 0\\ u & 0\end{smallmatrix}\right)$ with support $\bar{u} = \left(\begin{smallmatrix}e & 0\\ 0 & 0\end{smallmatrix}\right)$, support $\bar{u}^* = \left(\begin{smallmatrix}0 & 0\\ 0 & f\end{smallmatrix}\right)$, where $e \leqslant e'$, $f \leqslant f'$, $u^*u = e$, $uu^* = f$, and

(α) $I(\theta, \bar{u}) \leqslant (\delta/2)(\|\bar{u}\|_\theta^\#)^2$,

(β) $I(\phi, e) \leqslant (\delta/2)\,\phi(e)$, $I(\psi, f) \leqslant (\delta/2)\,\psi(f)$.

As e is orthogonal to a and f to b we let $w' = w + u$ and obtain a partial isometry in M extending w. Let $a' = a + e$, $b' = b + f$ be, respectively, the support and range of w'. Let $\alpha' = \alpha + eJ(1-e)J\xi$, $\beta' = \beta + fJ(1-f)J\eta$. We assert $r' = (w', \alpha', \beta') \in R$ and $r \leqslant r'$.

Clearly (a) holds. To show (b) note that $e(\alpha' - \alpha) = \alpha' - \alpha$, and hence $a(\alpha' - \alpha) = 0$, and α is orthogonal to $\alpha' - \alpha$. But $\| \alpha' - \alpha \|^2 = \| eJ(1-e)J\xi \|^2 = \| e(e - JeJ)\xi \|^2 \leqslant 2I(\phi, e)$. Since $\phi(e) = \phi^{e'}(e) = \langle e(1-a)J(1-a)J\xi_0, (1-a)J(1-a)J\xi_0 \rangle = \langle J(1-a)Je\xi_0, J(1-a)Je\xi_0 \rangle \leqslant \phi_0(e)$, we get by (β) $\| \alpha' - \alpha \|^2 \leqslant 2I(\phi, e) \leqslant \delta\phi(e) \leqslant \delta\phi_0(e) = \delta\phi_0(a' - a)$. Since α and $\alpha' - \alpha$ are orthogonal, we get (b). To show (c) let $\xi' = \xi_0 - \alpha' - J\alpha'$ and $\eta' = \eta_0 - \beta' - J\beta'$. Then $\xi' = \xi - eJ(1-e)J\xi - (1-e)JeJ\xi = eJeJ\xi + (1-e)J(1-e)\xi$, so $\xi' \in \mathscr{P}^\natural$ and e commutes with ξ', i.e., $(e - JeJ)\xi' = 0$. As a commutes with ξ we have $Ja\xi = a\xi$ and $a(1-e)JeJ\xi = Jea\xi = 0$. Thus $a\xi = a\xi'$ and $(JaJ - a)\xi' = 0$, so that $(Ja'J - a')\xi' = 0$, and (c) follows.

To show (d) let ϕ' and ψ' in M_*^+ correspond to ξ' and η', respectively, and let $\theta' = \begin{pmatrix} \phi' & 0 \\ 0 & \psi' \end{pmatrix}$. From the preceding paragraph a commutes with ϕ'; thus $\bar{a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ commutes with θ' as well as $\bar{b} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. As the support of $\bar{w} = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$ is contained in $\bar{a}$ and that of $\bar{u} = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}$ in $1 - \bar{a}$ we get by Proposition 1(e) that

$$I(\theta', \bar{w} + \bar{u}) = I(\theta', \bar{w}) + I(\theta', \bar{u}).$$

Now by construction $\bar{c} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ commutes with $\theta = \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}$, and $\theta^{\bar{c}} = (\theta')^{\bar{c}}$, as can be seen using $\phi^a = (\phi')^a$, $\psi^b = (\psi')^b$. As $\bar{w} \in (M \otimes M_2(\mathbb{C}))_{\bar{c}}$ we hence get by Proposition 1(d) and (d),

$$I(\theta', \bar{w}) = I(\theta, \bar{w}) \leqslant \delta(\| \bar{w} \|_\theta^\#)^2 = \delta(\| \bar{w} \|_{\theta'}^\#)^2.$$

We claim that $I(\theta', \bar{u}) \leqslant I(\theta, \bar{u})$. Indeed, we can apply Lemma 11(b) twice, since in $\mathscr{H} \otimes \mathscr{H}_4$, where $\mathscr{H}_4$ is the Hilbert space of 2×2 Hilbert–Schmidt matrices, the vector $\begin{pmatrix} \xi' & 0 \\ 0 & \eta' \end{pmatrix}$ is obtained from $\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}$ by applying the operators $\bar{e}\bar{J}\bar{e}\bar{J} + (1 - \bar{e})\bar{J}(1 - \bar{e})\bar{J}$ and $\bar{f}\bar{J}\bar{f}\bar{J} + (1 - \bar{f})\bar{J}(1 - \bar{f})\bar{J}$, where $\bar{J} = J \otimes$ complex conjugation.

Next $(\| \bar{u} \|_\theta^\#)^2 = \phi(e) + \psi(f)$ and $(\| \bar{u} \|_{\theta'}^\#)^2 = \phi'(e) + \psi'(f)$ and by Lemma 11(a) and (β) we have $\phi'(e) \geqslant \frac{1}{2}\phi(e)$, $\psi'(f) \geqslant \frac{1}{2}\psi(f)$, since $\delta \leqslant 1$. Thus we have

$$I(\theta', \bar{u}) \leqslant I(\theta, \bar{u}) \leqslant (\delta/2)(\| \bar{u} \|_\theta^\#)^2 \leqslant \delta(\| \bar{u} \|_{\theta'}^\#)^2$$

Since $(\| \bar{w}' \|_{\theta'}^\#)^2 = (\| \bar{w} \|_{\theta'}^\#)^2 + (\| \bar{u} \|_{\theta'}^\#)^2$, we therefore have

$$I(\theta', \bar{w}') \doteq I(\theta', \bar{w}) + I(\theta', \bar{u}) \leqslant \delta(\| \bar{w} \|_{\theta'}^\#)^2 + \delta(\| \bar{u} \|_{\theta'}^\#)^2 = \delta(\| \bar{w}' \|_{\theta'}^\#)^2,$$

and (d) follows.

Thus $r' \in R$ as asserted. From the above discussion it is clear that $r \leqslant r'$ and $r \neq r'$. This contradicts the maximality of r, so that either w is an isometry or a coisometry. By symmetry we may assume $w^*w = 1$ and let $\phi_1 = \phi$ and ψ_1 be the reduced of ψ by $b = ww^*$. By (b) and (c) we have

$$\| \phi_0 - \phi_1 \| \leqslant 2 \| \xi - \xi_0 \| \leqslant 4\delta^{1/2};$$

in particular, $\theta(\bar{w}^*\bar{w}) = \phi(1) \geqslant 1 - 4\delta^{1/2}$. By Proposition 1(f), $| \theta(\bar{w}\bar{w}^*)^{1/2} - \theta(\bar{w}^*\bar{w})^{1/2} | \leqslant (2I(\theta, \bar{w}))^{1/2} \leqslant 2^{1/2}\delta^{1/2} \| \bar{w} \|_\theta^\# \leqslant 2\delta^{1/2}$. Thus $\psi(b)^{1/2} = \theta(\bar{w}\bar{w}^*)^{1/2} \geqslant \theta(\bar{w}^*\bar{w})^{1/2} - 2\delta^{1/2} \geqslant 1 - 6\delta^{1/2}$, and we have

$$\| \psi - \psi_1 \| \leqslant 2 \| b\eta - \eta \| = 2\psi(1 - b)^{1/2} < 14\delta^{1/2}.$$

In particular,

$$\| \psi_0 - \psi_1 \| \leqslant \| \psi_0 - \psi \| + \| \psi - \psi_1 \| < 4\delta^{1/2} + 14\delta^{1/2} = 18\delta^{1/2}.$$

As $I(\theta, \bar{w}) \leqslant 2\delta$ we have from Proposition 1(b) that $| \theta(\bar{w}y - y\bar{w})| \leqslant 8\delta^{1/2} \| y \|$ for any $y \in M \otimes M_2(\mathbb{C})$. It follows that $| \phi(xw) - \psi(wx)| \leqslant 8\delta^{1/2} \| x \|$ for any $x \in M$. In particular, since $ww^* = b \in M_\psi$ we have

$$| \phi_1(w^*xw) - \psi_1(x)| \leqslant 8\delta^{1/2} \| x \|, \qquad x \in M.$$

Since M is of type III standard arguments show that we can find a sequence of unitaries (v_n) in M converging strongly to w. Then $v_n\xi \to w\xi$ in $\mathscr{H}$, so that for large enough n we have

$$| \phi_1(v_n{}^*xv_n) - \psi_1(x)| \leqslant 9\delta^{1/2} \| x \|, \qquad x \in M.$$

Thus $| \phi_0(v_n{}^*xv_n) - \psi_0(x)| \leqslant 31\delta^{1/2} \| x \|$, $x \in M$, and the proof is complete.

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