JOURNAL OF FUNCTIONAL ANALYSIS 28, 248–253 (1978)

On the Cohomology of Operator Algebras

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Communicated by I. Segal

Received November 17, 1976

Any amenable C^* algebra is nuclear. Injective von Neumann algebras are characterized by the vanishing of their cohomology with coefficients in dual normal Banach bimodules.

INTRODUCTION

As in [1] or [3], a Banach algebra A is called amenable when any norm continuous derivation δ of A in a dual Banach A-bimodule is a coboundary. In [5, 7.9, 7.6], Johnson showed that any C^* algebra A which is an inductive limit of type I C^* algebras is amenable as a Banach algebra. He left open the existence of nonamenable C^* algebras. We show below that any amenable C^* algebra is also nuclear. It is very likely that any nuclear C^* algebra is amenable but we leave this question open.

In their work on the cohomology of operator algebras [6–8], Johnson, Kadison, and Ringrose introduced the notion of *normal* dual Banach M-bimodule, where M is a von Neumann algebra. They showed that any von Neumann algebra M which is generated by an increasing sequence of finite-dimensional * algebras has the following property: Any derivation δ of M in a *normal* dual Banach M-bimodule is a coboundary.

We shall prove the converse: If M is a von Neumann algebra with separable predual and all (continuous) derivations of M in *normal* dual Banach bimodules are coboundaries, then M is generated by an increasing sequence of finite-dimensional * algebras. This class also coincides with the class of injective or of semidiscrete von Neumann algebras [3]. It seems that "amenable" is the best terminology to qualify this class.

We now fix the notation for the sequel of the paper. Let A be a Banach algebra. Then a Banach space X which is also an A-bimodule is called a Banach A-bimodule when for some $K > 0$ one has $\| a\xi b \|_X \leq K \| a \| \| \xi \|_X \| b \|$ for all $a, b \in A$, $\xi \in X$.

0022-1236/78/0282–0226$02.00/0

If furthermore X is (isometric to) the dual of a Banach space X_* and for each $a \in A$ the operators $\xi \to a\xi$, $\xi \to \xi a$ of X in X are $\sigma(X, X_*)$ continuous, then X is called a ***dual*** Banach A-bimodule.

Now if M is a von Neumann algebra and X is a dual Banach M bimodule, one says that X is normal when for any $\xi \in X$ and $\eta \in X_*$ the functionals $x \in M \to \langle \eta, x\xi \rangle$ or $\langle \eta, \xi x \rangle$ are normal functionals on M.

Since von Neumann algebras are in general not norm separable the requirement of normalcy for M-bimodules is a necessity to avoid pathological phenomena.

Let A be a Banach algebra, X a Banach A-bimodule. Then a derivation δ of A in X is a continuous linear map of A in X such that $\delta(ab) = \delta(a)b + a\delta(b)$, $\forall a, b \in A$.

We say that a von Neumann algebra is ***amenable*** when all derivations of M in ***normal*** dual Banach M-bimodules are coboundaries. By [6, Theorem 5.6], this is the same as asking that all ***normal*** derivations (i.e., $\eta \circ \delta$ is normal for any $\eta \in X_*$) be coboundaries.

THEOREM 1. *Let M be a von Neumann algebra with separable predual. Then M is amenable if and only if it is injective.*

COROLLARY 2. *Let A be a separable C* algebra which is amenable as a Banach algebra; then A is nuclear.*

Proof of the corollary. Using [2], to show that A is nuclear, one just needs to show that any representation π of A in a Hilbert space $\mathscr{H}_\pi$ generates an injective von Neumann algebra. By [4], one can assume that $\mathscr{H}_\pi$ is separable. Then let $M = \pi(A)''$ and X a dual normal Banach M-bimodule. Then any normal derivation δ of M in X defines by composition with π a derivation of A in X. Since A is amenable and X is a dual Banach A-bimodule one gets that $\delta \cdot \pi$ is a coboundary, and δ, being normal, is also a coboundary.

Proof of injective $\Rightarrow$ amenable. By [3], any injective von Neumann algebra M with separable predual is the weak closure of an increasing sequence of finite-dimensional * algebras. Hence the result of Johnson *et al.* [6, Corollary 6.4, p. 95] gives the implication.

Proof of amenable $\Rightarrow$ injective. In this proof we shall not make use of the hypothesis M_* separable.

We first assume that M is semifinite, and let τ be a faithful semi-finite normal trace on M. Let M act in a Hilbert space $\mathscr{H}$. We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is of τ-finite rank when its initial and final supports are majorized by projections $e, f \in M$ with $\tau(e) < \infty$, $\tau(f) < \infty$.

Since the sup of projections $e, f \in M$ with $\tau(e) < \infty$, $\tau(f) < \infty$ also satisfies

$\tau(e \vee f) < \infty$, the set $\mathscr{F}$ of τ-finite rank operators is an M submodule of $\mathscr{L}(\mathscr{H})$.

For $x \in M$, put $\| x \|_2 = \tau(x^*x)^{1/2}$. (One can have $\| x \|_2 = +\infty$.) Let Y be the space of all linear functionals φ on $\mathscr{F}$ with $\exists K > 0$, $|\varphi(aTb)| \leqslant K \| a \|_2 \| T \|_\infty \| b \|_2$, $\forall a, b \in M \cap \mathscr{F}$, $T \in \mathscr{F}$. The smallest possible K is noted $\| \varphi \|_Y$ and defines a norm on Y. The corresponding unit ball of Y is compact for the weak topology $\sigma(Y, \mathscr{F})$, so Y is a Banach space and is the dual of some Banach space Y_* in which $\mathscr{F}$ is dense.

For $\varphi \in Y$, $x, y \in M$, define $x\varphi y \in Y$ by

$$x\varphi y(T) = \varphi(yTx) \qquad \forall T \in \mathscr{F},$$

then, since $\| ya \|_2 \leqslant \| y \|_\infty \| a \|_2$, $\| bx \|_2 \leqslant \| b \|_2 \| x \|_\infty$ for $a, b \in M \cap \mathscr{F}$, it follows that $\| x\varphi y \|_Y \leqslant \| x \|_\infty \| y \|_\infty \| \varphi \|_Y$. Hence Y is a Banach M bimodule. For $x \in M$ the map $\varphi \to x\varphi$ from Y to Y is $\sigma(Y, \mathscr{F})$ continuous because $\mathscr{F}$ is an M-bimodule. So Y is a dual Banach M-bimodule. Finally for fixed $T \in \mathscr{F}$ and $\varphi \in Y$ the map $x \in M \to \varphi(xT)$ is normal, because letting $e \in M$ be a projection, with $\tau(e) < \infty$, majorizing the two supports of T, one has

$$|\varphi(xT)| = |\varphi(xeTe)| \leqslant \| xe \|_2 \| T \|_\infty \| e \|_2 .$$

So Y is a dual normal M-bimodule.

Let X be the submodule of Y defined as follows:

$$X = \{\varphi \in Y, \varphi(x) = 0 \; \forall x \in M \cap \mathscr{F}\}.$$

It is a submodule because $M \cap \mathscr{F}$ is an M-submodule of $\mathscr{F}$. It is $\sigma(Y, \mathscr{F})$ closed in Y by construction. Hence it is also a dual normal Banach M-bimodule.

Now since the trace τ on M is normal, there exists in $\mathscr{H}$ a family $(\xi_\alpha)_{\alpha \in I}$ of unit vectors $\| \xi_\alpha \| = 1$, such that, for any $x \in M \cap \mathscr{F}$: $\tau(x) = \sum_{\alpha \in I} \langle x\xi_\alpha , \xi_\alpha \rangle$, where $\sum |\langle x\xi_\alpha , \xi_\alpha \rangle| < \infty$.

For any $T \in \mathscr{F}$ there is a projection $e \in M$, $\tau(e) < \infty$ such that, with $T = T_1 + iT_2$, one has $-\| T_j \| e \leqslant T_j \leqslant \| T_j \| e$, $j = 1, 2$. Hence the sum $\sum_{\alpha \in I} |\langle T\xi_\alpha , \xi_\alpha \rangle|$ is convergent and

$$\tilde{\tau}(T) = \sum_{\alpha \in I} \langle T\xi_\alpha , \xi_\alpha \rangle$$

defines a linear functional on $\mathscr{F}$ satisfying

(a) $\tilde{\tau}(T) \geqslant 0$, $\forall T \geqslant 0$, $T \in \mathscr{F}$,

(b) $\tilde{\tau}(x) = \tau(x)$, $\forall x \in M \cap \mathscr{F}$.

As by construction $\mathscr{F}$ is a * subalgebra of $\mathscr{L}(\mathscr{H})$ one deduces from (a) that

$|\tilde{\tau}(T^*_1 T_2)|^2 \leqslant \tilde{\tau}(T^*_1 T_1)\,\tilde{\tau}(T^*_2 T_2)$, $\forall T_1, T_2 \in \mathscr{F}$. For $T \in \mathscr{F}$, $a, b \in M \cap \mathscr{F}$, let $T^*_1 = aT$, $T_2 = b$; then

$$|\tilde{\tau}(aTb)|^2 \leqslant \tilde{\tau}(aTT^*a^*)\,\tilde{\tau}(b^*b) \leqslant \tilde{\tau}(aa^*)\,\|T\|_\infty^2\,\tilde{\tau}(b^*b)$$

because by construction of $\tilde{\tau}$ one has $\tilde{\tau}(aTT^*a^*) = \sum \|T^*a^*\xi_\alpha\|^2 \leqslant \|T\|_\infty^2 \sum \|a^*\xi_\alpha\|^2 = \|T\|_\infty^2 \sum \|a^*\xi_\alpha\|^2 = \|T\|_\infty^2\,\tilde{\tau}(aa^*)$. Hence by (b), $|\tilde{\tau}(aTb)| \leqslant \|a\|_2 \|b\|_2 \|T\|_\infty$. Thus $\tilde{\tau} \in Y$, $\|\tilde{\tau}\|_Y \leqslant 1$.

Now let δ be the derivation of M in X defined by

$$\delta(x) = x\tilde{\tau} - \tilde{\tau}x \qquad \forall x \in M.$$

One has $\delta(x) \in X$ because $\tilde{\tau}(xy - yx) = \tau(xy - yx) = 0$ for any $y \in M \cap \mathscr{F}$.

One can also check directly that δ is normal. If M is amenable, δ must be a coboundary, so there is a $\varphi \in X$ such that $\delta(x) = x\varphi - \varphi x$, $\forall x \in M$.

Now $\tilde{\tau} - \varphi = \psi \in Y$ and one has

(1°) $\tau(x) = \psi(x)$, $\forall x \in M \cap \mathscr{F}$ (because $\varphi \in X$),

(2°) $x\psi = \psi x$, $\forall x \in M$ (because $x\tilde{\tau} - \tilde{\tau}x = x\varphi - \varphi x$).

Let e be a projection of M, $\tau(e) < \infty$. Let $\mathscr{K} = e\mathscr{H}$ be the range of e. Then M_e is a finite von Neumann algebra acting in the Hilbert space $\mathscr{K}$. Since τ is faithful, its restriction τ_1 to M_e is also faithful, and in particular by (1°) the restriction ψ_1 of ψ to $\mathscr{L}(\mathscr{K}) = \{T \in \mathscr{L}(\mathscr{H}),\ eT = Te = T\}$ is nonzero and satisfies (2)′: $x\psi_1 = \psi_1 x$ for any x in M_e.

Replacing ψ_1 by $\frac{1}{2}(\psi_1 + \psi^*_1)$ does not affect (2)′ and (1)′: $\psi_1(x) = \tau_1(x)$ for any $x \in M_e$. So one can assume that $\psi_1 = \psi^*_1$ as an element of $\mathscr{L}(\mathscr{K})^*$. Writing its unique Jordan decomposition $\psi_1 = \psi^+_1 - \psi^-_1$, with $\|\psi^+_1\| + \|\psi^-_1\| = \|\psi_1\|$ one gets:

(α) $\psi^+_1(x) \geqslant \tau_1(x)$, $\forall x \in M_e$, $x \geqslant 0$.

(β) $\psi^+_1(xT) = \psi^+_1(Tx)$, $\forall T \in \mathscr{L}(\mathscr{K})$, $\forall x \in M_e$.

Now let $K = \|\psi^+_1\|$, we shall prove that for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in P = M_e$ one has:

$$\left|\tau_1\left(\sum a_j b_j^*\right)\right| \leqslant K \left\|\sum a_j \otimes b_j{}^c\right\|_{\mathscr{K} \otimes \mathscr{K}^c},$$

where $\mathscr{K}^c$ is the conjugate Hilbert space of $\mathscr{K}$.

In fact, let $\varphi_1 = (1/K)\,\psi^+_1 \in \mathscr{L}(\mathscr{K})^*$. Then, as φ_1 is a weak limit of normal states on $\mathscr{L}(\mathscr{K})$, there is for any $\epsilon > 0$, a normal state φ on $\mathscr{L}(\mathscr{K})$ such that

$$\|b_j\varphi - \varphi b_j\| \leqslant \epsilon, \qquad j = 1, \ldots, n, \qquad \left|\varphi\left(\sum a_j b_j^*\right) - \varphi_1\left(\sum a_j b_j^*\right)\right| \leqslant \epsilon.$$

Hence there exists, for each $\epsilon > 0$, a Hilbert–Schmidt operator ρ in $\mathscr{K}$ with $\|\rho\|_{\mathrm{HS}} = 1$, $\|\rho b_j - b_j\rho\|_{\mathrm{HS}} \leqslant \epsilon$ and:

$$\left\|\sum a_j\rho b_j^*\right\|_{\mathrm{HS}} \geqslant \left|\varphi_1\left(\sum a_j b_j^*\right)\right| - \epsilon.$$

But the canonical identification of $\mathscr{K} \otimes \mathscr{K}^c$ with the Hilbert–Schmidt operators of $\mathscr{K}$ intertwines $a \otimes b^c$ with the operator $\rho \to a\rho b^*$, so we have shown that $|\varphi_1(\sum a_j b^*_j)| \leqslant \|\sum a_j \otimes b_j{}^c\|$. Now we know that the linear functional ψ_0 on $P \odot P^c$, defined by $\psi_0(\sum a_i \otimes b_i{}^c) = \tau_1(\sum a_i b^*_i)$ is continuous for the minimal norm of the tensor product [2]. Letting P act in the Hilbert space $\mathscr{K}_1$ of the Gelfand–Segal construction of τ_1 with canonical vector ξ_1 and involution J, one gets:

$$\left|\left\langle\sum a_i J b_i J\xi_1, \xi_1\right\rangle\right| \leqslant K\left\|\sum a_i \otimes J b_i J\right\|_{\min}.$$

It follows then from the cyclicity of ξ_1 for P that the canonical homomorphism η of $P \odot P'$ in $\mathscr{L}(\mathscr{K}_1)$ is bounded. Then by [4] we know that P is semidiscrete and hence injective. It follows that M itself is injective.

The General Case

Since the finite case is already treated it is easy to see that we can assume M to be properly infinite.

Let then $(N, (\theta_t)_{t\in\mathbb{R}})$ be a continuous decomposition of M [9], where N is semifinite and M is isomorphic to the cross product of N by the one parameter group $(\theta_t)_{t\in\mathbb{R}}$ of automorphisms of N. We just have to show that if M is amenable then N is amenable. In fact, then N is injective and so is M by [3].

But by construction N is generated by a von Neumann subalgebra P isomorphic to M and a one parameter group of unitaries $(v_s)_{s\in\mathbb{R}}$ such that $v_s P v^*_s = P$, $\forall s \in \mathbb{R}$. Now let X be a dual normal Banach N-bimodule. Then it is also a P bimodule. So given a normal derivation δ of N in X we can assume that $\delta(x) = 0$, $\forall x \in P$, since P is amenable. For each $s \in \mathbb{R}$, let $\xi_s = v_s\delta(v_s^{-1})$. As, for $y \in P$, one has $yv_s\delta(v_s^{-1}) = v_s(v_s^{-1}yv_s)\,\delta(v_s^{-1}) = v_s\delta(v_s^{-1}y) = v_s\delta(v_s^{-1})y$ we get $y\xi_s = \xi_s y$, $s \in \mathbb{R}$, $y \in P$. Also, for any $t \in \mathbb{R}$ and $s \in \mathbb{R}$: $v_t\xi_s = v_{t+s}\delta(v_{t+s}^{-1}v_t) = (v_{t+s}\delta(v_{t+s}^{-1}))v_t + \delta(v_t)$.

As X is a dual of a Banach space X_*, let ξ be a $\sigma(X, X_*)$ limit point of the $(1/F^{\#})\sum_F \xi_s$, where F runs through a summing family for $\mathbb{R}$ as a discrete amenable group. Then by the above equality one gets $v_t\xi = \xi v_t + \delta(v_t)$, for any $t \in \mathbb{R}$, and $y\xi = \xi y$, $\forall y \in P$. Q.E.D.

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