THE von NEUMANN ALGEBRA OF A FOLIATION

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Abstract :

Every smooth foliation $\mathfrak{F}$ of a manifold V gives rise very naturally to a von Neumann algebra $M = L^{\infty}(V/\mathfrak{F})$. The weights on M correspond exactly to operator valued forms on the "manifold" of leaves of $\mathfrak{F}$. We compute their modular automorphism group, this yields the continuous decomposition of M in terms of another foliation $\mathfrak{F}_o$ of V and a one parameter group of automorphisms of $\mathfrak{F}_o$. We then illustrate this decomposition with a few examples.

Let V be a (smooth, finite dimensional real) manifold and $\mathfrak{F}$ a smooth foliation of V . We assume for simplicity that the union of the leaves of $\mathfrak{F}$ which have non-trivial holonomy is negligible (for the smooth measures on V). This is always the case if $\mathfrak{F}$ is analytic.

Definition 1 - *A random operator* $T = (T_f)$ *a measurable family* $(T_f)_{f \in \mathcal{L}}$ *where for each leaf* $f \in \mathcal{L}$ *of* $\mathfrak{F}$, T_f *is an operator in* $L^2(f)$. (*)

As an example, let $Y \in C_o^{\infty}(V, T(\mathfrak{F}))$ be a section (smooth with compact support) of the tangent bundle of $\mathfrak{F}$. Then the flow $F_t = \exp tY$ induces on each leaf f a one parameter group of unitaries U_f^t of $L^2(f)$. For each t , $U^t = (U_f^t)_{f \in \mathcal{L}}$ is a random operator.

The random operators are added and composed as follows :

$$(T_1+T_2)_f = T_{1f}+T_{2f} \quad \forall f \in \mathcal{L} \quad , \quad (T_1 T_2)_f = T_{1f} T_{2f} \quad \forall f \in \mathcal{L} \quad .$$

The natural norm is $\|T\|_{\infty} = \text{Ess Sup } \|T_f\|$, defined as the smallest $\lambda \geqq 0$ such that $\|T_f\| \leqq \lambda$ holds almost everywhere (i.e. the union of leaves where this fails is negligible).

(*) Given a manifold X , we let $L^2(X)$ be the hilbert space completion of the vector space $C_o^{\infty}(X, |\Lambda|^{1/2})$ of smooth half densities with compact support, with the canonical scalar product : $< \omega_1, \omega_2 > = \int \omega_1 \bar{\omega}_2$.

We did not state the measurability condition, up to equality almost everywhere it means simply that the family $(T_f)_{f \in \mathcal{L}}$ can be exhibited. For instance it is in general impossible to construct a random operator $T = (T_f)_{f \in \mathcal{L}}$ such that T_f is of rank one for almost all f, because this would give a measurable subset S of V which meets almost all leaves in one and only one point, which is impossible if for instance $\mathcal{F}$ is ergodic (*) and $\dim \mathcal{F} < \dim V$. As in ordinary measure theory we shall say "random operator" instead of "class of random operators modulo equality almost everywhere".

<u>Proposition</u> 2 - <u>With the above algebraic operations and norm the random operators form a von Neumann algebra</u> $M = L^\infty(V/\mathcal{F})$.

The notation $L^\infty(V/\mathcal{F})$ indicates that M depends only on the couple $(V/\mathcal{F})$ it also suggest that M generalises the L^∞ space of a manifold. If $\mathcal{F}$ has dimension 0 then it coincides with $L^\infty(V)$ and is commutative. In general it will be type I only if the above measurable cross section exists, i.e. very scarcely.

In non commutative integration theory, the basic object is a von Neumann algebra M (it reduces to L^∞(X, class of the measure) in the commutative situation) and the role of the measure (possibly infinite) is played by the <u>weights</u> φ on M.

If $M = L^\infty$(X, class of measure) then weights on M correspond exactly to positive measures μ in the class. If in particular X is a smooth manifold and the class is the smooth one, then μ is entirely described by a 1-density, positive and measurable (or equivalently an odd form, positive with measurable coefficients).

We now describe how this correspondance extends to our situation. Let us consider the space $\mathcal{L}$ of leaves as a "manifold" (see [1]) of dimension $q = \operatorname{Codim} \mathcal{F}$. The tangent bundle to $\mathcal{L}$ is obtained as follows : for each $f \in \mathcal{L}$ the holonomy gives a natural trivialisation of the transverse bundle $x \longrightarrow T_x(V)/T_x(\mathcal{F})$ along f. Thus a tangent vector $\xi \in \tau_f(\mathcal{L})$ should be a constant section $x \longrightarrow \xi_x$ of the above bundle. We can then speak of $\Lambda^q\tau$ where $q = \operatorname{Codim} \mathcal{F} = \dim \tau_f \quad \forall f \in \mathcal{L}$.

<u>Definition 3</u> - <u>An operator density</u> $T = (T_v)_{v \in \Lambda^q\tau}$ <u>is a measurable map</u> $v \longrightarrow T_v$ <u>which to each</u> $f \in \mathcal{L}$ <u>and</u> $v \in \Lambda^q\tau_f$ <u>associates a positive self adjoint (**) operator</u> T_v <u>in the hilbert space</u> $L^2(f)$, <u>with</u> $T_{\lambda v} = |\lambda| T_v$, $\forall \lambda \in \mathbb{R}$.

(*) i.e. any measurable union of leaves A which is not negligible has a negligible complement A^c in V.

(**) In general unbounded.

Thus, for each $f \in \mathcal{L}$, one has a ray of positive operators, $T_v = \lambda T_{v_0}$, $v \in \Lambda^q \tau_f$, $\lambda > 0$. One cannot speak of T_f but only of T_f up to multiplication by a positive scalar. The simplest example of an operator density comes from a choice of a positive measurable transversal density ρ on V : for each $x \in V$, ρ_x maps $\Lambda^q(T_x(V)/T_x(\mathcal{F}))$ to $\mathbb{R}_+$ and $\rho_x(\lambda v) = |\lambda| \rho_x(v)$, $\forall \lambda$. Given such a ρ let for each $f \in \mathcal{L}$ and $v \in \Lambda^q \tau_f$, the operator T_v be the multiplication in $L^2(f)$ by the function $x \longrightarrow \rho_x(v)$ on f . One checks that it defines an operator density.

We shall now describe the correspondence between operator densities and weights on $M = L^\infty(V/\mathcal{F})$. In the ordinary case $(\dim \mathcal{F} = 0)$ an operator density is just a density on V and one can integrate it over V to get a scalar. Our first aim is to show the existence of a canonical trace on operator densities, i.e. to give a meaning to $\int \mathrm{Trace}(T_v)$ in general. Let T be an operator density, in general T_v is not trace class in $L^2(f)$, however it will happen (*) that T_v is locally of trace class i.e. that $\chi T_v \chi$ is of trace class for every characteristic function of compact support χ on f .

Lemma 4 - Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be a locally finite partition of V where each U_α is contained in the domain of a foliation chart and is a measurable union $U_\alpha = \bigcup_{x \in S_\alpha} P_x$ where S is a transversal to $\mathcal{F}$ and P_x is relatively compact in the leaf through x , then : $\sum_{\alpha \in I} \int_{S_\alpha} \mathrm{Trace}(P_x^\alpha T_v P_x^\alpha)$ is independent of the choice of $\mathcal{U}$.

We noted P_x^α the orthogonal projection in $L^2(f_x)$ associated to the subset P_x of the leaf f_x through f . Note that the integral makes sense because $\mathrm{Trace}\,(P_x^\alpha T_v P_x^\alpha)$ is a one density on S_α .

This lemma defines the quantity $\int \mathrm{Trace}\, T$ for every operator density T . It is unitarily invariant under $M = L^\infty(V/\mathcal{F})$ that is $\int \mathrm{Trace}(UTU^*) = \int \mathrm{Trace}(T)$ for every random unitary operator $U \in L^\infty(V/\mathcal{F})$.

Theorem 5 - For every operator density T the equality $\varphi(A) = \int \mathrm{Trace}(TA)$ (**) defines a (semi finite, normal) weight φ on $L^\infty(V/\mathcal{F})$; every weight on $L^\infty(V/\mathcal{F})$ occurs exactly once in this way and the modular automorphism group σ^φ of φ is given by :

$$(\sigma_t^\varphi(A))_f = T(v)^{it} A_f\, T(v)^{-it} \quad , \quad v \in \Lambda^q \tau_f \,,\ v \neq 0 \quad .$$

(*) This will be automatic for all $f \in \mathcal{L}$ if $\mathrm{Trace}\ T_v < \infty$.
(**) More precisely $\mathrm{Trace}(T^{1/2} A T^{1/2})$.

Note that the choice of $v \neq 0$, $v \in \Lambda^q \tau_f$ does not affect this equality, since $T(\lambda v) = |\lambda| T(v)$, $\forall \lambda \in \mathbb{R}$.

In particular T will define a trace on $L^\infty(V/\mathfrak{F})$ iff $T(v)$ commutes with all A_f and thus is a scalar, for almost all f , as is seen by the following lemma :

Lemma 6 - Let $\mathfrak{A}$ be any norm separable C^* algebra generating the von Neumann algebra $L^\infty(V/\mathfrak{F})$, then for almost all $f \in \mathcal{L}$ the natural representation of $\mathfrak{A}$ in $L^2(f)$ is irreducible.

So the (normal) traces on $L^\infty(V/\mathfrak{F})$ correspond exactly to the measurable scalar densities on the "manifold" $\mathcal{L}$. Those scalar densities which are locally integrable, are exactly the absolutely continuous holonomy invariant transverse measures on $\mathfrak{F}$, by [7] . The Reeb foliation of S^3 is an example where there is no (non-zero) absolutely continuous holonomy invariant transverse measure, while, since $L^\infty(V/\mathfrak{F})$ is of type I , there are lots of measurable scalar densities on $\mathcal{L}$.

In general there will be no (non-zero normal) trace on $L^\infty(V/\mathfrak{F})$ and the next step is to determine the spectrum of the modular automorphism groups σ^φ. Theorem 5 and lemma 6, have the following corollary :

Corollary 7 - Let T be an operator density, φ the associated weight (*) , and $E \in \mathbb{R}$, then $\mathrm{Exp}(E) \in \mathrm{Sp}\,\sigma^\varphi$ iff for every $\varepsilon > 0$ there exists a non negligible set of leaves $f \in \mathcal{L}$ such that :

$$\forall v \in \Lambda^q \tau_f \quad , \quad \exists\, E_1, E_2 \in \text{Spectrum } \log T_v \quad , \quad E_1 - E_2 \in [E-\varepsilon, E+\varepsilon] \quad .$$

The computation of $S(L^\infty(V/\mathfrak{F})) = S(M)$ [3] where

$$S(M) = \bigcap_{\varphi \text{ weight}} \text{Spectrum } \sigma^\varphi \quad ,$$

identifies it as the ratio set of W. Krieger (See [6] and [2]). R. Bowen has computed this invariant for Anosov foliations [2] and obtained that generally these will be of type III_1 so that $S(M) = [0,+\infty]$. We shall give below examples of type III_λ , $\lambda \in]0,1[$ i.e. $S(M) = \{0\} \cup Z$ and III_0 : $S(M) = \{0,1\}$ for analytic foliations.

(*) We assume here that φ is faithful i.e. that for $v \neq 0$, $T(v)$ is non singular.

So we see that the spectrum of $\mathcal{L}og\ T_v$ (which is self adjoint and is shifted by an additive constant when v is multiplied by λ) is far from arbitrary. (One can check that the existence of an operator density T with $\mathcal{L}og\ T_v$ bounded below for almost all v is equivalent to the semi finiteness of $L^\infty(V/\mathfrak{F})$) . In the type III situation (i.e. no non-zero scalar valued density on $\mathcal{L}$) one is interested in the continuous decomposition of the von Neumann algebra $L^\infty(V/\mathfrak{F})$. See [8] and [5] . Recall that a weight φ on M is called integrable when the following left ideal of M is generating :

$$\{x \in M, \exists c > 0,\ \left\| \int_{-K}^{K} \sigma_t^\varphi(x^* x)dt \right\| \leqq c,\ \forall K \in \mathbb{R}_+\} \quad .$$

This condition is equivalent to the smoothness of the map $\lambda \longrightarrow \lambda\varphi$ from $\mathbb{R}^+$ to the flow of weights on M ([5]) .

<u>Theorem 8</u> - <u>Let</u> ρ <u>be a</u> C^∞ <u>transversal density and</u> φ <u>the corresponding weight on</u> $L^\infty(V/\mathfrak{F})$. <u>Then</u> φ <u>is integrable iff the set of critical points of</u> ρ <u>is negligible</u>.

We have to define what we mean by a critical point of ρ . For each $v \in \Lambda^q \tau_f$, $f \in \mathcal{L}$ the function $\mathcal{L}og\ \rho_x(v)$ on f has a certain set C_f of critical points. It does not depend on the choice of $v \neq 0$ since changing v adds a constant to $\mathcal{L}og\ \rho$, thus we can speak of $C = \cup C_f$.

This theorem allows to determine the continuous decomposition of $L^\infty(V/\mathfrak{F})$ by means of the codimension $q+1$ "foliation" (with critical points) $\mathfrak{F}_\rho$ whose leaves partition each leaf of $\mathfrak{F}$ in the (not necessarily connected) level manifolds of ρ .

To treat our examples we make the further hypothesis that $C = \emptyset$ and that V is compact. The foliation $\mathfrak{F}_\rho$ does not have singular points in this situation and letting ω be the 1-form on $\mathfrak{F}$ (*) which is the gradient of $\mathcal{L}og\ \rho$, we can find a smooth vector field Y on V , $Y_x \in T_x(\mathfrak{F})$ $\forall x \in V$ such that $\langle Y,\omega \rangle$ is the constant function 1 . Let U^t be the one parameter group of random operators associated with Y as above, then as $\exp tY$ multiplies ρ by e^t one gets that $U^t \varphi U^{t*} = e^t \varphi$. Hence in this situation φ is a dominant weight, the semi finite von Neumann algebra of the continuous decomposition of $L^\infty(V/\mathfrak{F})$ identifies with the centralizer of φ i.e. with $L^\infty(V/\mathfrak{F}_\rho)$ and the one parameter group of automorphisms θ_t of this algebra is defined by the action on $L^\infty(V/\mathfrak{F}_\rho)$ of the flow $\exp Y$ of automorphisms of the foliation $\mathfrak{F}_\rho$. In particular the flow of weights of $L^\infty(V/\mathfrak{F})$ is the action of $\exp Y$ on the ergodic decomposition of the foliation $\mathfrak{F}_\rho$. We now describe examples showing how to construct a foliation of a compact manifold with given flow of weights. As a tool we use the simplest example of an Anasov foliation namely we let V_o be the quotient $SL(2,\mathbb{R})/\Gamma$ of $SL(2,\mathbb{R})$ by the discrete cocompact subgroup Γ and $\mathfrak{F}_o$ be the foliation on V_o coming from the action (on the left) of the subgroup of lower triangular matrices. This foliation is of type III_1 (cf. [2]) and from the above discussion it is easy to check that its continuous decomposition (with respect to a left invariant transverse density ρ_o) yields as $\mathfrak{F}_{\rho_o}$ the foliation associated to the horocycle flow (i.e. the matrices $\begin{bmatrix} o & o \\ b & o \end{bmatrix} = X_b$) and that $\exp Y_o$ is the geodesic flow (i.e. the matrices $\begin{bmatrix} e^t & o \\ o & e^{-t} \end{bmatrix} = Y_t$). Now let K be an auxiliary compact manifold and $F = (F_t)_{t \in \mathbb{R}}$ a smooth flow on K . We construct now a foliation $\mathfrak{F}$ on $K \times V_o$ as follows. It comes from an action of the group of matrices $\begin{bmatrix} e^t & o \\ b & e^{-t} \end{bmatrix}$ where $t,b \in \mathbb{R}$, where $\begin{bmatrix} 1 & o \\ b & 1 \end{bmatrix}$ act by $\text{identity}_K \times \text{horocycle}_b$, while the matrix $\begin{bmatrix} e^t & o \\ o & e^{-t} \end{bmatrix}$ acts by $F_t \times \text{Geodesic}_t$. It is clear that this gives an action of the above group. We assume for simplicity that F has a smooth invariant measure (it is easy to modify the above construction so that it works in general), let α be the corresponding 1-density on K . Then $\alpha \times \rho_o$ defines a transverse density ρ on $\mathfrak{F}$; and the corresponding foliation $\mathfrak{F}_\rho$ is just given by the flow : identity $\times$ horocycle , so that its ergodic decomposition gives us K back. The flow $\exp Y$ is simply $F_t \times \exp tY_o$ and thus its action on the ergodic decomposition of $\mathfrak{F}_\rho$. i.e. the flow of weights of $L^\infty(V/\mathfrak{F})$ is the flow F_t on K .

If in particular we take K to be a circle of length L while F_t acts by rotations with speed 1 we get a foliation $\mathfrak{F}$ of the compact manifold $V = S^1 \times V_o$, which is of type III_λ , $\lambda = \exp(-L)$. As soon as F acting on K is ergodic, with $\dim K > 1$, we get a factor of type III_o as $L^\infty(V/\mathfrak{F})$.

This shows that all types of factors occur from simple examples. The problem "when is $L^\infty(V/\mathfrak{F})$ approximately finite dimensional" is very interesting and examples will be discussed in [4], see also [2]. For instance an analytic (one dimensional) complex foliation on a 2 dimensional complex compact manifold can fail to be a.f.d., while all real flows are a.f.d.

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