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# On the Spatial Theory of von Neumann Algebras

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If $M$ is a von Neumann algebra in $\mathscr{H}$, each faithful weight $\psi$ on $M'$ defines an operator-valued weight $\psi^{-1}$ of $\mathscr{L}(\mathscr{H})$ on $M$. For each weight $\varphi$ on $M$ the positive unbounded operator $d\varphi/d\psi = \varphi \circ \psi^{-1}$ satisfies all the usual properties of a Radon–Nikodym derivative.

This paper is motivated by work of Woronowicz [8] and Haagerup [4]. In [8] Woronowicz gives a new proof of the main theorem of the Tomita–Takesaki theory and introduces as a tool a realization of the predual of a von Neumann algebra $M$, acting in the Hilbert space $\mathscr{H}$, by unbounded operators acting in $\mathscr{H}$ and forming a "phase system," i.e., very roughly, a linear space of operators $T$, stable under multiplication by elements of $M$, and whose polar decomposition $T = u \mid T \mid$ gives a phase $u \in M$. This realization of $M_*$ in $\mathscr{H}$ is not canonical and in [8] depends on the choice of a cyclic and separating vector for $M$. On the other hand Haagerup [4] develops a theory of operator-valued weights and (extending results of Combes and Delaroche [1]) sets up in Theorem 6.13 of [4] an order-reversing bijection of the set $P(A, B)$ (of operator-valued weights from the von Neumann algebra $A$ acting in the Hilbert space $\mathscr{H}$ on the von Neumann subalgebra $B$) on $P(B', A')$.

If one applies this with $A = \mathscr{L}(\mathscr{H})$, $B = M$, one gets an order-reversing bijection from $P(\mathscr{L}(\mathscr{H}), M)$ on the space $P(M')$ of weights on the commutant $N = M'$ of $M$ in $\mathscr{H}$. Thus choosing a faithful weight $\psi$ on $N$ will determine a natural operator-valued weight,[1] let us call it $\psi^{-1}$, from $\mathscr{L}(\mathscr{H})$ to $M$, which in turn associates to each weight $\varphi$ on $M$ the weight $\varphi \circ \psi^{-1}$ on $\mathscr{L}(\mathscr{H})$.

As weights on $\mathscr{L}(\mathscr{H})$ correspond to positive self-adjoint operators with dense domain in $\mathscr{H}$, one thus gets such an operator $T$ associated with any pair of (faithful normal semifinite) weights $\varphi$ on $M$ and $\psi$ on $N = M'$. We set $T = d\varphi/d\psi$. When $\psi$ is the state on $\mathscr{L}$ associated with the cyclic and separating vector $\xi_0 \in \mathscr{H}$ then the map $\varphi \mapsto d\varphi/d\psi$ from $M_*^+$ to operators in $\mathscr{H}$ is exactly the map

[1] This does not quite follow from [4, Theorem 6.13] because the bijection $\alpha$ constructed there is not uniquely chosen.



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defined by Woronowicz discussed above. It turns out that this "Radon–Nikodym derivative" $d\varphi/d\psi$ has many remarkable properties and can be defined in a very simple manner. We shall see for instance that it is additive in $\varphi$ for fixed $\psi$, that $d\psi/d\varphi = (d\varphi/d\psi)^{-1}$, that $T = d\varphi/d\psi$ satisfies

$$T^{it}xT^{-it} = \sigma_t^{\varphi}(x)\ \forall x \in M, \qquad T^{it}yT^{-it} = \sigma_{-t}^{\psi}(y)\ \forall y \in N,\ \forall t \in \mathbb{R},$$

and that the operators of the form $T = d\varphi/d\psi$ for fixed $\psi$ are characterized by the last equality above. As a simple application of this tool we show that if $(\varphi_n)$ is an increasing sequence of weights on $M$ with sup $\varphi$ then the $\sigma_t^{\varphi_n}$ converge to $\sigma_t^{\varphi}$ in Aut $M$. This notion will also be used in a crucial manner in [3].

## Preliminaries on $N$-Modules

Let $N$ be a von Neumann algebra. By an $N$-module $\mathscr{H}$ we mean a Hilbert space $\mathscr{H}$ together with a nondegenerate normal representation of $N$ in $\mathscr{H}$. We let $x\xi$ be the result of the action of $x \in N$ on $\xi \in \mathscr{H}$. The commutant of the action of $N$ in $\mathscr{H}$ will be denoted by $\mathscr{L}_N(\mathscr{H})$; it is a von Neumann algebra in $\mathscr{H}$.

Let $\Psi$ be a faithful semifinite normal weight on $N$. We let $\mathscr{H}_\Psi$ be the $N$-module corresponding to the Gelfand–Segal construction for $\Psi$ so that we have a canonical injection $\eta_\Psi$ of $\{y \in N, \Psi(y^*y) < \infty\}$ in $\mathscr{H}_\Psi$ such that $\eta_\Psi(xy) = x\eta_\Psi(y)$, $\forall x \in N$.

Definition 1. Let $\mathscr{H}$ and $\Psi$ be as above. A vector $\xi \in \mathscr{H}$ is called $\Psi$-bounded iff there exists $C < \infty$ such that

$$\| y\xi \| \leqslant C \| \eta_\Psi(y)\| \qquad \forall y \in N, \qquad \Psi(y^*y) < \infty.$$

We let $D(\mathscr{H}, \Psi)$ be the subspace of $\mathscr{H}$ consisting of all $\Psi$-bounded vectors.

For $\xi \in D(\mathscr{H}, \Psi)$, there exists a unique bounded operator $R^\Psi(\xi)$ from $\mathscr{H}_\Psi$ to $\mathscr{H}$ such that

$$R^\Psi(\xi)\, \eta_\Psi(y) = y\xi \qquad \forall y \in N, \qquad \Psi(y^*y) < \infty.$$

By construction $R^\Psi(\xi)$ is $N$-linear, i.e., $R^\Psi(\xi) \in \mathrm{Hom}_N(\mathscr{H}_\Psi, \mathscr{H})$. The adjoint $T^\Psi(\xi) = (R^\Psi(\xi))^*$ is an $N$-linear operator from $\mathscr{H}$ to $\mathscr{H}_\Psi$ and $\langle T^\Psi(\xi)\alpha, \eta_\Psi(y)\rangle = \langle \alpha, y\xi\rangle\ \forall \alpha \in \mathscr{H}$. Note that $R^\Psi(\xi) = 0$ implies $\xi = 0$, since $\Psi$ is semifinite.

Lemma 2. *$D(\mathscr{H}, \Psi)$ is dense in $\mathscr{H}$.*

*Proof.* For $x \in N$ let $U(x)$ be the corresponding operator in $\mathscr{H}$ and let $c \in N$ be the central projection corresponding to the kernel of the representation $U$, so that $U$ is an isomorphism of $N_c$ on $U(N)$. Let $\Psi_c$ be the restriction of $\Psi$ to $N_c$ and $(\xi_\alpha)_{\alpha \in I}$ be a family of vectors in $\mathscr{H}$ such that $\Psi_c(x) = \Sigma\langle U(x)\, \xi_\alpha, \xi_\alpha\rangle$

$\forall x \in (N_c)_+$ [7]. For every $\alpha \in I$ we have $\| y\xi_\alpha \|^2 \leqslant \Psi(y^*y)\ \forall y \in N$, and hence $\xi_\alpha$ is $\Psi$-bounded. Now let $E$ be the orthogonal projection of $\mathscr{H}$ on the closure of $D(\mathscr{H}, \Psi)$. By construction it commutes with $\mathscr{L}_N(\mathscr{H})$ and hence is of the form $E = U(e)$, $e \in N_c$. If $e \neq c$, as $\Psi(c - e) > 0$ there exists $\alpha$ such that $U(c - e) \cdot \xi_\alpha \neq 0$, which contradicts the equality $E\xi_\alpha = \xi_\alpha$. Q.E.D.

PROPOSITION 3. (a) *Let $\mathscr{H}$ and $\Psi$ be as above. Then the vector subspace of $\mathscr{L}_N(\mathscr{H})$ generated by the operators*

$$\theta^\Psi(\xi, \eta) = R^\Psi(\xi)\, T^\Psi(\eta), \qquad \xi, \eta \in D(\mathscr{H}, \Psi),$$

*is a weakly dense two-sided ideal of $\mathscr{L}_N(\mathscr{H})$.*

(b) *Let $\mathscr{J}_\Psi$ be the above ideal. Then every positive element $T$ of $\mathscr{J}_\Psi$ is a finite sum*

$$T = \sum_{i=1}^{n} \theta^\Psi(\xi_i, \xi_i), \qquad \xi_i \in D(\mathscr{H}, \Psi).$$

(c) *There exists a family $(\xi_\alpha)_{\alpha\in I}$ of $\Psi$-bounded vectors such that $\Sigma\theta_\Psi(\xi_\alpha, \xi_\alpha) = 1$.*

*Proof.* (a) For $A, B \in \mathscr{L}_N(\mathscr{H})$ we have $\theta^\Psi(A\xi, B\eta) = A\theta^\Psi(\xi, \eta)\, B^*$ so that $\mathscr{J}_\Psi$ is a two-sided ideal of $\mathscr{L}_N(\mathscr{H})$. Let $A \in \mathscr{L}_N(\mathscr{H})$, $A \neq 0$. Then there exists (Lemma 2) a $\xi \in D(\mathscr{H}, \Psi)$, with $A\xi \neq 0$. So $\theta^\Psi(A\xi, A\xi) \neq 0$, since $R^\Psi(A\xi) \neq 0$, and $A\theta^\Psi(\xi, \xi)\, A^*$ is a nonzero element of $\mathscr{J}_\Psi$ smaller than $AA^* \| \theta^\Psi(\xi, \xi)\|$. This shows that $\mathscr{J}_\Psi$ is weakly dense in $\mathscr{L}_N(\mathscr{H})$.

(b) Let $A \in \mathscr{J}_\Psi$ so that $A = \sum_{i=1}^{n} \theta^\Psi(\xi_i, \eta_i)$. If $A \geqslant 0$ then $A = \frac{1}{2}(A + A^*) = \frac{1}{2} \sum R^\Psi(\xi_i)\, T^\Psi(\eta_i) + R^\Psi(\eta_i)\, T^\Psi(\xi_i)$ so that $A \leqslant \frac{1}{2} \sum R^\Psi(\xi_i + \eta_i)\, T^\Psi(\xi_i + \eta_i) = B$. Hence, let $C \in \mathscr{L}_N(\mathscr{H})$ so that $A = CBC^*$, one gets

$$A = \tfrac{1}{2} \sum \theta^\Psi(C(\xi_i + \eta_i), C(\xi_i + \eta_i)).$$

Condition (c) follows from (a) and (6). Q.E.D.

If we take $\mathscr{H} = \mathscr{H}_\Psi$, then $D(\mathscr{H}_\Psi, \Psi)$ is exactly the set of right bounded vectors $\xi \in \mathscr{H}_\Psi$ for the canonical left Hilbert algebra of $\Psi$. Also, $R^\Psi(\xi)$ is the operator of right multiplication by $\xi$:

$$\eta_\Psi(y) \to y\xi, \qquad \forall y \in N, \qquad \Psi(y^*y) < \infty.$$

Then let $\Psi'$ be the canonical weight on the right von Neumann algebra $\mathscr{L}_N(\mathscr{H}_\Psi)$ [7]. By construction of $\Psi'$ one has $\Psi'(R^\Psi(\xi)^* R^\Psi(\xi)) = \| \xi \|^2$ for every $\xi \in D(\mathscr{H}_{\Psi,\Psi})$.

LEMMA 4. *Let $\Psi'$ be as above, and let $\mathscr{H}$ be an $N$-module. Then for any $\xi \in D(\mathscr{H}, \Psi)$ one has*

$$\Psi'(R^\Psi(\xi)^* R^\Psi(\xi)) = \| \xi \|^2.$$

*Proof.* Let $U \in \mathrm{Hom}_N(\mathscr{H}, \mathscr{H}_\Psi)$ be a partial isometry such that $U^*U\xi = \xi$. Then $U\xi \in D(\mathscr{H}_\Psi, \Psi)$ and $| R^\Psi(U\xi)|^2 = | R^\Psi(\xi)|^2$ because $R^\Psi(U\xi) = UR^\Psi(\xi)$, $R^\Psi(\xi) = U^*R^\Psi(U\xi)$ so $\Psi'(| R^\Psi(\xi)|^2) = \Psi'(| R^\Psi(U\xi)|^2) = \| A\xi \|^2 = \| \xi \|^2$. Q.E.D.

## Preliminaries on Closable Forms

Let $D$ be a dense subspace of a Hilbert space $\mathscr{H}$. By a positive form on $D$ we mean a map $q$ from $D$ to $[0, +\infty]$ such that

(1) $q(\lambda\xi) = | \lambda |^2 q(\xi)\ \forall \lambda \in \mathbb{C}, \forall \xi \in D$.

(2) $q(\xi + \eta) + q(\xi - \eta) = 2q(\xi) + 2q(\eta)\ \forall \xi, \eta \in D$.

It follows that Domain $q = \{\xi \in D, q(\xi) < \infty\}$ is a subspace of $D$ on which $q$ defines a positive quadratic form in the sense of [6], provided that Dom $q$ is dense.

The next lemma follows easily from [6, Theorem 3], but we include a proof for the convenience of the reader.

Lemma 5. (1) *Let $q$ be a positive form, with dense domain, on $D$. Then the following conditions are equivalent:*

(a) *There exists a closed positive quadratic form $p$ on $\mathscr{H}$ with* (Dom $p$) $\cap$ $D =$ Dom $q$ *and* $q(\xi) = p(\xi, \xi)\ \forall \xi \in$ Dom $q$.

(b) *The map $\xi \to q(\xi) \in [0, +\infty]$ is lower semicontinuous.*

(2) *Under the above conditions, the positive operator $T$ associated with the closure of $q/$*Dom $q$ *is the largest positive self-adjoint operator such that* $\| T^{1/2}\xi \|^2 = q(\xi)\ \forall \xi \in$ Dom $q$.

*Proof.* (1) (a) $\Rightarrow$ (b) Let $T$ be the positive operator associated with the closed quadratic form $p$ (cf. [6, Theorem VIII 15]). Then by the spectral theorem, the map $q_1$, $q_1(\xi) = \| T^{1/2}\xi \|^2$ if $\xi \in$ Dom $T^{1/2}$, $q_1(\xi) = +\infty$ otherwise, is lower semicontinuous from $\mathscr{H}$ to $[0, +\infty]$, so its restriction $q$ to $D$ is lower semicontinuous.

(b) $\Rightarrow$ (a) For $\xi \in \mathscr{H}$, let $q_1(\xi)$ be the inf of the limits $\lim_{n\to\infty} q(\xi_n)$, for all sequences $\xi_n \in D$, $\xi_n \to \xi$ such that $q(\xi_n)$ converges. Then $q_1$ satisfies (1) and (2) on $\mathscr{H}$ because to prove (2) one just needs an inequality instead of the equality. Moreover $q_1$ is also lower semicontinuous. Then $\mathscr{H}_1 =$ Dom $q_1$ becomes a Hilbert space with the norm $(\| \xi \|_1)^2 = \| \xi \|^2 + q_1(\xi)$. In fact if $(\xi_n)$ is a Cauchy sequence in $\mathscr{H}_1$ then there exists $\xi \in \mathscr{H}$ such that $\xi_n \to \xi$ in $\mathscr{H}$ and as $q_1(\xi_n)$ is bounded one gets $\xi \in \mathscr{H}_1$, and finally $\xi_n \to \xi$ in $\mathscr{H}_1$ by the semi-continuity of $q_1$. So $q_1$ defines a closed quadratic form $p$, and as $q_1(\xi) = q(\xi)$ $\forall \xi \in D$ one gets (a).

(2) Let $T$ be any positive operator, self-adjoint and such that $\| T^{1/2}\xi \|^2 = q(\xi)$, $\forall \xi \in \operatorname{Dom} q$. Then with $q_1$ constructed as above, one has for any $\xi \in \mathscr{H}$, $q_1(\xi) < \infty$ the inequality $\| T^{1/2}\xi \|^2 \leqslant q_1(\xi)$, since for any sequence $\xi_n \in D$, $\xi_n \to \xi$ one has $\| T^{1/2}\xi \|^2 \leqslant \lim_{n\to\infty} \| T^{1/2}\xi_n \|^2$. So one has $T \leqslant T_1$, where $T_1$ is the positive operator associated with $q_1$. It also follows that $T_1^{1/2}$ is the closure of its restriction to the domain of $q$.

*Notation.* We shall use the following two abuses of notation in the rest of the text:

(1) $\langle T\xi, \xi\rangle = \| T^{1/2}\xi \|^2$ if $\xi \in \operatorname{Dom} T^{1/2}$ and equals $+\infty$ otherwise, where $T$ is any positive self-adjoint operator.

(2) $T_1 + T_2$ is the positive self-adjoint operator associated with the positive quadratic form $\xi \to \langle T_1\xi, \xi\rangle + \langle T_2\xi, \xi\rangle$, where $T_1$ and $T_2$ are positive self-adjoint operators such that $\operatorname{Dom} T_1^{1/2} \cap \operatorname{Dom} T_2^{1/2}$ is a core for both $T_1^{1/2}$ and $T_2^{1/2}$.

## Definition of the Spatial Derivative $d\phi/d\Psi$

Let $M$ be a von Neumann algebra in $\mathscr{H}$, $\phi$ a semifinite normal weight on $M$, and $\Psi$ a semifinite faithful normal weight on the commutant $N$ of $M$.

We can consider $\mathscr{H}$ as an $N$-module and let $D(\mathscr{H}, \Psi)$ be the subspace of $\Psi$-bounded vectors; by Lemma 1 it is a dense subspace of $\mathscr{H}$.

Lemma 6. *The equality $q(\xi) = \phi(\theta^\Psi(\xi, \xi)$, $\forall \xi \in D(\mathscr{H}, \Psi)$, defines a lower semicontinuous positive form on $D(\mathscr{H}, \Psi)$ with dense domain.*

*Proof.* Let us first show that Domain $q$ is dense in $\mathscr{H}$. As $\phi$ is semifinite the union of the ranges $a(\mathscr{H})$, $a \in M$, $\phi(aa^*) < \infty$ is total in $\mathscr{H}$. So as $D(\mathscr{H}, \Psi)$ is dense in $\mathscr{H}$, it is enough to check that for such an $a \in M$ and $\xi \in D(\mathscr{H}, \Psi)$ one has $q(a\xi) < \infty$. But

$$\theta^\Psi(a\xi, a\xi) = a\theta^\Psi(\xi, \xi)\, a^* \leqslant aa^* \| \theta^\Psi(\xi, \xi)\|, \qquad \text{so} \qquad \phi(\theta^\Psi(a\xi, a\xi)) < \infty.$$

To prove that $q$ is lower semicontinuous we can assume that there exists a $\xi_0 \in \mathscr{H}$ such that $\phi(x) = \langle x\xi_0, \xi_0\rangle$, $\forall x \in M$. Then $q(\xi) = \| T^\Psi(\xi)\, \xi_0 \|^2$, $\forall \xi \in D(\mathscr{H}, \Psi)$, and we have

$$q(\xi)^{1/2} = \operatorname*{Sup}_{y \in N,\, \Psi(y^*y) \leqslant 1} |\langle T^\Psi(\xi)\, \xi_0, \eta_\Psi(y)\rangle|.$$

As $\langle T^\Psi(\xi)\, \xi_0, \eta_\Psi(y)\rangle = \langle y^*\xi_0, \xi\rangle$ is a continuous function of $\xi$ we get the lower semicontinuity of $q$. Q.E.D.

DEFINITION. The spatial derivative $d\phi/d\Psi$ is the positive self-adjoint operator associated with the form $\xi \to \phi(\theta^\Psi(\xi, \xi))$ as in Lemma 5.

Hence $d\phi/d\Psi$ is the largest positive s.a. operator $T$ such that

$$\langle T\xi, \xi\rangle = \phi(\theta^\Psi(\xi, \xi)) \qquad \forall \xi \in D(\mathscr{H}, \Psi).$$

In particular, if $\phi(1) < \infty$ then $D(\mathscr{H}, \Psi)$ is contained in the domain of $(d\phi/d\Psi)^{1/2}$ and is a core for this operator. Hence if $\phi_j(1) < \infty, j = 1, 2$, then

$$d(\phi_1 + \phi_2)/d\Psi = d\phi_1/d\Psi + d\phi_2/d\Psi.$$

PROPOSITION 8. (a) *One has* $\phi_1 \leqslant \phi_2$ *iff* $d\phi_1/d\Psi \leqslant d\phi_2/d\Psi$.

(b) *For any invertible operator* $a \in M$ *one has*

$$d(a\phi a^*)/d\Psi = a(d\phi/d\Psi)\, a^*.$$

*Proof.* (a) One has $\langle d\phi_1/d\Psi\xi, \xi\rangle \leqslant \langle d\phi_2/d\Psi\xi, \xi\rangle$ for every $\xi \in D(\mathscr{H}, \Psi)$ and hence every $\xi \in \mathscr{H}$ by Lemma 5. Conversely if $d\phi_1/d\Psi \leqslant d\phi_2/d\Psi$, then $\phi_1(A) \leqslant \phi_2(A)$ for every $A \in \mathscr{J}_\Psi$, $A \geqslant 0$ by Proposition 3(b), so that as $\phi_1$ and $\phi_2$ are normal, we have for every $A \in M_+$, $A = \mathrm{Sup}_{\alpha\in I} A_\alpha$ with $A_\alpha \in \mathscr{J}_\Psi$, $A_\alpha \geqslant 0$, the inequality

$$\phi_1(A) = \mathrm{Sup}\, \phi_1(A_\alpha) \leqslant \mathrm{Sup}\, \phi_2(A_\alpha) = \phi_2(A).$$

(b) Let $T = d\phi/d\Psi$. Then $aTa^*$ is a positive self-adjoint operator and $\langle aTa^*\xi, \xi\rangle = \langle Ta^*\xi, a^*\xi\rangle$ $\forall \xi \in \mathscr{H}$, so that $\langle aTa^*\xi, \xi\rangle = (a\phi a^*)(\theta^\Psi(\xi, \xi))$ $\forall \xi \in D(\mathscr{H}, \Psi)$. As $a$ is invertible, $aTa^*$ is the largest positive operator satisfying the last equality and we get $aTa^* = d(a\phi a^*)/d\Psi$. Q.E.D.

THEOREM 9. *Let* $M$, $N$, $\phi$, *and* $\Psi$ *be as above, with* $\phi$ *faithful.*

(1) *The operator* $T = d\phi/d\Psi$ *is nonsingular, and for all* $t \in \mathbb{R}$:

$$T^{it} x T^{-it} = \sigma_t^\phi(x), \forall x \in M; \qquad T^{it} y T^{-it} = \sigma_{-t}^\Psi(y), \forall y \in N.$$

(2) *If* $\phi_1$, $\phi_2$ *are faithful weights on* $M$, *then*

$$(d\phi_2/d\Psi)^{it} = (D\phi_2 ; D\phi_1)_t (d\phi_1/d\Psi)^{it} \qquad \forall t \in \mathbb{R}.$$

(3) $d\Psi/d\phi = (d\phi/d\Psi)^{-1}$.

For the proof we consider $\mathscr{H}$ as an $N$-module and form the sum $\mathscr{H}' = \mathscr{H} \oplus \mathscr{H}_\Psi$, hence considering $\mathrm{Hom}_N(\mathscr{H}_\Psi, \mathscr{H})$ as a subspace of $\mathscr{L}_N(\mathscr{H}')$, namely, the space of all operators $T \in \mathscr{L}_N(\mathscr{H}')$ with $T = Te = (1 - e)T$, where $e$ is the orthogonal projection of $\mathscr{H}'$ on $\mathscr{H}_\Psi$. For every $\phi$ we let $\theta$ be the weight on $\mathscr{L}_N(\mathscr{H}')$ such that $\theta(x) = \phi((1 - e)\, x(1 - e)) + \Psi'(exe)$, it is faithful, and its modular automorphism group $\sigma_t^\theta$ defines a one-parameter group of isometries of $\mathrm{Hom}_N(\mathscr{H}_\Psi, \mathscr{H})$.

LEMMA 10. (a) *There exists a unique one-parameter group of unitaries* $U_t^\phi$ *of* $\mathscr{H}$ *such that* ($\alpha$) $U_t^\phi x U_{-t}^\phi = \sigma_t^\phi(x)$, $\forall x \in \mathscr{L}_N(\mathscr{H})$, ($\beta$) $U_t^\phi y U_{-t}^\phi = \sigma_{-t}^\Psi(y)$ $\forall y \in N$, ($\gamma$) $R^\Psi(U_t^\phi(\xi)) = \sigma_t^\theta(R^\Psi(\xi))$ $\forall \xi \in D(\mathscr{H}, \Psi)$.

(b) *For every* $t \in \mathbb{R}$ *one has* $U_t^\phi = (d\phi/d\Psi)^{it}$.

*Proof.* (a) The uniqueness of $U_t^\phi$ is clear from ($\gamma$) since $\xi \mapsto R^\Psi(\phi)$ is injective. Assume that the existence has been proved for some $\phi_1$; put $U_t^\phi = (D\phi : D\phi_1)_t U_t^{\phi_1}$ $\forall t \in \mathbb{R}$. Then ($\alpha$) and ($\beta$) are satisfied and we have for $\xi \in D(\mathscr{H}, \Psi)$:

$$R^\Psi(U_t^\phi(\xi)) = R^\Psi((D\phi : D\phi_1)_t U_t^{\phi_1}(\xi)) = (D\phi : D\phi_1)_t R^\Psi(U_t^{\phi_1}(\xi))$$

$$= (D\phi : D\phi_1)_t \sigma_t^{\theta_1}(R^\Psi(\xi)) = \sigma_t^\theta(R^\Psi(\xi)).$$

So the existence follows for all $\phi$. Thus the existence problem depends only on the $N$-module $\mathscr{H}$. It is clear also that we need only solve it for the special case $\mathscr{H} = \mathscr{H}_\Psi$ and we choose $\Psi' = \phi$. In this case we let $\Delta$ be the modular operator of $\Psi$ and put $U_t^{\Psi'} = \Delta^{-it}$. Then $R^\Psi(\Delta^{-it}\xi) = \Delta^{-it} R^\Psi(\xi)\, \Delta^{it} = \sigma_t^\phi(R^\Psi(\xi))$ so that ($\alpha$) and ($\beta$) are checked.

(b) Let $\phi' = \phi \oplus \Psi'$ on the von Neumann algebra $P = \mathscr{L}_N(\mathscr{H}')$. For every $x \in P$ such that $\phi'(x^*x) < \infty$ the function $t \mapsto \phi'(x^*\sigma_t^{\phi'}(x))$ is equal to $\langle \Delta_{\phi'}^{it} \eta_{\phi'}(x), \eta_{\phi'}(x)\rangle$.

Hence there exists a positive measure $\mu$ on $R$ such that

$$\phi'(x^*\sigma_t^{\phi'}(x)) = \int e^{its}\, d\mu(s), \qquad \forall t \in R,$$

and

$$\phi'(xx^*) = \int e^s\, d\mu(s) \in [0, +\infty].$$

Take $x = R^\Psi(\xi)$; then by Lemma 4 we have $\Psi'(x^*x) = \Psi'(|R^\Psi(\xi)|^2) = \|\xi\|^2$ and from (a) we get for all $t \in \mathbb{R}$

$$\phi'(x^*\sigma_t^{\phi'}(x)) = \Psi(R^\Psi(\xi)^* R^\Psi(U_t^\phi(\xi))) = \langle U_t^\phi \xi, \xi\rangle.$$

Hence we get, with $U_t^\phi = e^{itH_\phi}$ $\forall t \in \mathbb{R}$, the equality

$$\phi(R^\Psi(\xi)\, R^\Psi(\xi)^*) = \langle e^{H_\phi}\xi, \xi\rangle \qquad \forall \xi \in D(\mathscr{H}, \Psi).$$

To conclude that $d\phi/d\Psi = e^{H_\phi}$ we need only show that those two positive self-adjoint operators commute. But for $t \in \mathbb{R}$, $U_t^\phi$ is a unitary in $\mathscr{H}$ which preserves $M, N, \phi, \Psi$, using ($\alpha$), ($\beta$), and ($\gamma$) and hence commutes with $d\phi/d\Psi$. Q.E.D.

*Proof of Theorem* 9. Condition (1) follows clearly from Lemma 10, and (2) follows from the proof of (a) in Lemma 10.

(3) Let $\Psi_1$, $\Psi_2$ be faithful weights on $N$; then for all $t \in \mathbb{R}$, $(d\phi/d\Psi_1)^{-it}(d\phi/d\Psi_2)^{it} = (D\Psi_1 : D\Psi_2)_t$. This follows from Theorem 9(1) applied in $\mathscr{H} \oplus \mathscr{H} = \mathscr{H} \otimes \mathbb{C}^2$ to the quadruple $M \otimes \mathbb{C}$, $\phi$, $N \otimes M_2(\mathbb{C})$, $\Psi_1 \oplus \Psi_2$, for which one has $d\phi/d(\Psi_1 \oplus \Psi_2) = d\phi/d\Psi_1 \oplus d\phi/d\Psi_2$. Hence it is enough to check (3) in the special case $\mathscr{H} = \mathscr{H}_\Psi$ and $\phi = \Psi'$, where it is immediate.

COROLLARY 11. *Let $M$, $\phi$, $N$, $\Psi$ be as above, let $\xi$ be a $\Psi$-bounded vector, and let $\eta$ be a $\phi$-bounded vector. Then*

$$|\langle \xi, \eta \rangle|^2 \leqslant \phi(\theta^\Psi(\xi, \xi))\, \Psi(\theta^\phi(\eta, \eta)).$$

*Moreover for fixed $\xi$, one has*

$$\phi(\theta^\Psi(\xi, \xi)) = \text{Sup}\, |\langle \xi, \eta \rangle|^2, \qquad \Psi(\theta^\phi(\eta, \eta)) \leqslant 1.$$

*Proof.* Let $T = d\phi/d\Psi$. Then by Theorem 9 we have $T^{-1} = d\Psi/d\phi$. So for $\xi \in D(\mathscr{H}, \Psi)$, $\eta \in D(\mathscr{H}, \phi)$ we have

$$|\langle \xi, \eta \rangle|^2 = |\langle T^{1/2}\xi, T^{-1/2}\eta \rangle|^2 \leqslant \| T^{1/2}\xi \|^2 \, \| T^{-1/2}\eta \|^2,$$

which proves the first inequality.

COROLLARY 12. *For every weight $\phi$ on $M$ the support of $d\phi/d\Psi$ is equal to the support of $\phi$.*

*Proof.* Let $e =$ support $d\phi/d\Psi$. Then $e \leqslant s(\phi)$ follows immediately from the definition of $d\phi/d\Psi$; moreover, $\phi$ is faithful on $M_{s(\phi)}$ so that $d\phi/d\Psi$ is nonsingular on $s(\phi)\mathscr{H}$ by Theorem 9.

## THE WEIGHT ASSOCIATED WITH A HOMOGENEOUS OPERATOR

Let $M$ be a von Neumann algebra in $\mathscr{H}$, $N$ its commutant, and $\phi$ a faithful weight on $N$. We now characterize the operators in $\mathscr{H}$ of the form $d\phi/d\Psi$ for some weight $\phi$ on $M$.

THEOREM 13. *Let $T$ be a positive self-adjoint operator on $\mathscr{H}$. The following conditions are equivalent*:

(1) *There exists a semifinite normal weight $\phi$ on $M$ such that $T = d\phi/d\Psi$.*

(2) *For every $t \in \mathbb{R}$ and $y \in N$ one has $T^{it}\sigma_t^\Psi(y) = yT^{it}$.*

(3) *$D(\mathscr{H}, \Psi) \cap$ Domain $T^{1/2}$ is a core for $T^{1/2}$ and the scalar $\sum_{i=1}^n \langle T\xi_i, \xi_i \rangle$ depends only on $\sum_{i=1}^n \theta^\Psi(\xi_i, \xi_i)$, for $\xi_i \in D(\mathscr{H}, \Psi)$.*

*Proof.* Let $e$ be the support of $T$. Then conditions (1) and (2) both imply that $e \in M$. So to prove (1) $\Leftrightarrow$ (2) we can assume that $e = 1$. Then (1) $\Rightarrow$ (2) follows from Theorem 9. Assuming (2), let $\phi_1$ be some faithful weight on $M$ and put $T_1 = d\phi_1/d\Psi$, $u_t = T^{it}T_1^{-it}$. Then $u_t$ is a strongly continuous family of unitaries of $M$ such that $u_{t_1+t_2} = u_{t_1}\sigma_{t_1}^{\phi_1}(u_{t_2})$ $\forall t_1, t_2 \in \mathbb{R}$. So by the converse of the Radon–Nikodym theorem [2] there exists a faithful $\phi$ on $M$ such that $u_t = (D\phi: D\phi_1)_t$ $\forall t \in R$ and hence that $d\phi/d\Psi = T$.

It is clear that (1) $\Rightarrow$ (3). We now show that (3) $\Rightarrow$ (1). We need only construct a semifinite normal weight $\phi$ on $M$ such that $\phi(\theta^\Psi(\xi, \xi)) = \langle T\xi, \xi\rangle$ $\forall \xi \in D(\mathscr{H}, \Psi)$. This will follow from the next lemma, the semifiniteness of $\phi$ being automatic since $D(\mathscr{H}, \Psi) \cap$ Domain $T^{1/2}$ is dense in $\mathscr{H}$.

LEMMA 14. (a) *Let $T$ be as in Theorem 13(3). Then there is a weight $\phi_1$ on $\mathscr{J}_\Psi$ such that $\phi_1(\theta^\Psi(\xi, \xi)) = \langle T\xi, \xi\rangle$ $\forall \xi \in D(\mathscr{H}, \Psi)$ and having the property that for any family $(x_\alpha)_{\alpha\in I}$ of elements of $M$, $x_\alpha \to 1$, $\alpha \to \infty$, strongly, one has*

$$\underline{\operatorname{Lim}}\, \phi_1(x_\alpha y x_\alpha^*) \geq \phi_1(y) \qquad \forall y \in \mathscr{J}_\Psi^+.$$

(b) *Any weight $\phi_1$ on $\mathscr{J}_\Psi$ with the above property extends to a normal weight on $M$.*

*Proof.* (a) By Proposition 3(b) every element $x$ of $\mathscr{J}_\Psi^+$ can be written as $\sum_{i=1}^n \theta^\Psi(\xi_i, \xi_i)$ and $\phi_1(x) = \sum \langle T\xi_i, \xi_i\rangle$ is independent of the choice of the $\xi_i$ by Theorem 13(3). It follows easily that $\phi_1$ is a weight. Let $y \in \mathscr{J}_\Psi^+$, $y = \sum_{i=1}^n \theta^\Psi(\xi_i, \xi_i)$, and $(x_\alpha)_{\alpha\in I}$, $x_\alpha \in M$, $x_\alpha \to 1$ strongly; then $x_\alpha y x_a^* = \sum_{i=1}^n \theta^\Psi(x_\alpha\xi_i, x_\alpha\xi_i)$ and $\phi_1(x_\alpha y x_a^*)$ is the sum of the $\langle Tx_\alpha\xi_i, x_\alpha\xi_i\rangle$. For each $i$, $x_\alpha\xi_i$ converges to $\xi_i$ and by the semicontinuity of the positive form associated to $T$ (Lemma 5), we know that $\underline{\lim}_\alpha \langle Tx_\alpha\xi_i, x_\alpha\xi_i\rangle \geq \langle T\xi_i, \xi_i\rangle$ so we get (a).

(b) Put $\phi(x) = \operatorname{Sup}_{y\in\mathscr{J}^+, y\leq x} \phi_1(y)$. Then $\phi$ coincides with $\phi_1$ on $\mathscr{J}_\Psi$ and to show that it is a weight it is enough to show that it is normal, and its additivity will follow. Let $(x_\alpha)$ be an increasing family of elements of $M^+$ with sup equal to $x$. We can write $x_\alpha = y_\alpha x y_a^*$, with $y_\alpha \in M$, $y_\alpha \to 1$ strongly. For any $y \in \mathscr{J}_\Psi^+$, $y \leq x$ we have to show that $\operatorname{Sup}_\alpha \phi(x_\alpha) \geq \phi_1(y)$. But $\underline{\lim}\, \phi_1(y_\alpha y y_a^*) \geq \phi_1(y)$ and $y_\alpha y y_a^* \leq y_\alpha x y_a^* = x_\alpha$. Q.E.D.

COROLLARY 15. *Let $\phi_n$ be an increasing sequence of faithful normal weights on $M$ and assume that $\phi = \operatorname{Sup} \phi_n$ is semifinite. Then $d\phi_n/d\psi$ increases to $d\phi/d\Psi$ and for every $t \in \mathbb{R}$,*

$$\sigma_t^{\phi_n} \xrightarrow[n\to\infty]{} \sigma_t^{\phi} \quad \textit{in} \quad \operatorname{Aut} M.^2$$

[2] In the topology of norm-pointwise convergence in the predual.

*Proof.* Let $T_n = d\phi_n/d\Psi$. Then $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive s.a. nonsingular operators, and as $T_n \leqslant d\phi/d\Psi$ for all $n$, we know that there exists a positive s.a. operator $T$ such that $(1 + T_n)^{-1} \to (1 + T)^{-1}$ in the strong topology. Assume that $T_n^{it} \to T^{it}$ strongly for all $t \in \mathbb{R}$. This shows that $T$ is homogeneous of degree $-1$ in $\Psi$. Then Theorem 13 shows that the weight $\Psi'$ on $M$ associated with $T$ is larger than all $\phi_n$ and smaller than $\phi$, so finally $T = d\phi/d\Psi$. Finally for all $x \in M$ we have

$$\sigma_t^{\phi}(x) = T^{it}xT^{-it} = \operatorname*{Lim}_{n\to\infty} T_n^{it}xT_n^{-it} = \lim_{n\to\infty} \sigma_t^{\phi_n}(x).$$

So we need only to show that $T_n^{it} \to T^{it}$ for all $t \in \mathbb{R}$. Let $f$ be the continuous function on $[0, 1]$ such that

$$f(\lambda) = (\log(\lambda^{-1} - 1) + i)^{-1} \qquad \text{for } \lambda \neq 1 \qquad \text{and} \qquad f(1) = 0.$$

From a result of Kaplansky [6] this function is strong operator continuous, which shows that $f((1 + T_n)^{-1}) \to f((1 + T)^{-1})$ strongly when $n \to \infty$. The same is true for $\bar{f}$, and shows that the sequence of self-adjoint operators $H_n = \log T_n$ converge in the strong resolvent sense to operator $H = \log T$. Hence from a theorem of Trotter [6, Theorem VIII 21] that $\sigma_t^{\phi_n}(x)$ tends ${}_*$-strongly to $\sigma_t^{\phi}(x)$ for fixed $t$ and $x$. Applying this to $1 \otimes e_{21}$ in $M \otimes M_2(\mathbb{C})$ we get $(D\phi_n : D\phi)_t \to 1$ $*$-strongly. Q.E.D.

COROLLARY 16. (a) *For every faithful weight $\Psi$ on $N$ there exists a unique normal operator-valued weight $\Psi^{-1}$ from $\mathscr{L}(\mathscr{H})$ to $M = N'$ such that $\Psi^{-1}(\xi \otimes \xi^c) = \theta^{\Psi}(\xi, \xi)$ $\forall \xi \in D(\mathscr{H}, \Psi)$. ($\xi \otimes \xi^c$ is the rank one operator associated to $\xi$.)*

(b) *Every faithful normal operator-valued weight from $\mathscr{L}(\mathscr{H})$ to $M = N'$ is of the form $\Psi^{-1}$for a unique $\Psi$.*

*Proof.* (a) Let $\phi$ be a faithful weight on $M$. Then let $\tilde{\phi}$ be the weight on $\mathscr{L}(\mathscr{H})$ such that $\tilde{\phi} = \mathrm{Tr}(d\phi/d\Psi)$. By Theorem 9 the restriction of $\sigma^{\tilde{\phi}}$ to $M$ is equal to $\sigma^{\phi}$ so there exists [4] an operator-valued weight $E$ from $\mathscr{L}(\mathscr{H})$ to $M$ such that $\phi \circ E = \tilde{\phi}$ and $(D(\phi_1 \circ E){:}\ D(\phi \circ E)) = (D\phi_1 : D\phi)$ for every faithful weight $\phi_1$ on $M$. Hence by Theorem 9(2), $\phi_1 \circ E = \mathrm{Tr}(d\phi_1/d\Psi.)$ for every weight $\phi_1$ on $M$. This proves that $E$ does not depend on the choice of $\phi$. For $\xi \in D(\mathscr{H}, \Psi)$ one has $\phi(E(\xi \otimes \xi^c)) = \mathrm{Tr}(d\phi/d\Psi(\xi \otimes \xi^c)) = \langle d\phi/d\Psi\xi, \xi\rangle$ for every weight $\phi$ on $M$, with proves $E(\xi \otimes \xi^c) = \theta^{\Psi}(\xi, \xi)$.

(b) Let $E$ be such an operator-valued weight and let $\phi$ be a faithful weight on $M$. Then $\tilde{\phi} = \phi \circ E$ is a faithful weight on $\mathscr{L}(\mathscr{H})$ and hence of the form $\tilde{\phi} = \mathrm{Tr}(T.)$. Moreover $T^{it}xT^{-it} = \sigma_t^{\phi}(x)$ $\forall x \in M$, so by Theorems 9(3) and 13 there exists a faithful weight $\phi$ on $N$ with $T = d\phi/d\Psi$. As in the proof of (a) we get $\phi_1 \circ E = \mathrm{Tr}(d\phi/d\Psi)$ for every weight $\phi_1$ on $M$, using $(D(\phi_1 \circ E) : D(\phi \circ E)) = (D\phi_1 : D\phi)$. This shows that $E = \Psi^{-1}$.

DEFINITION 17. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be $N$-modules and let $T$ be a closed (densely defined) operator from $\mathscr{H}_1$ to $\mathscr{H}_2$, $T = u \mid T \mid$ its polar decomposition. We say that $T$ is homogeneous in $\Psi$ of degree $\alpha$ iff $u$ commutes with $N$ and for all $y \in N$, $t \in \mathbb{R}$:

$$| T |^{it} y = \sigma_{\alpha t}^{\Psi}(y) \mid T \mid^{it}.$$

Thus by Theorem 13 the positive operators of the form $d\phi/d\Psi$ are exactly the positive self-adjoint operators which are homogeneous in $\Psi$ of degree $-1$.

COROLLARY 18. *Let $T$ be a positive self-adjoint operator on $\mathscr{H}$. Then the following conditions are equivalent*:

(a) $\exists \phi \in M_*^+$ *with* $T = d\phi/d\Psi$.

(b) *$T$ is homogeneous of degree $-1$ in $\Psi$ and for some family* $(\xi_\alpha)$, $\xi_\alpha \in D(\mathscr{H}, \Psi)$, $\sum \theta^{\Psi}(\xi_\alpha, \xi_\alpha) = 1$ *one has* $\sum \langle T\xi_\alpha, \xi_\alpha\rangle < \infty$.

(c) $D(\mathscr{H}, \Psi) \subset$ Domain $T^{1/2}$ *is a core for* $T^{1/2}$ *and there is a $C < \infty$ such that*

$$\left| \sum_{i=1}^{n} \langle T\xi_i, \eta_i\rangle \right| \leqslant C \left\| \sum \theta^{\Psi}(\xi_i, \eta_i) \right\| \qquad \forall \xi_i, \eta_i \in D(\mathscr{H}, \Psi).$$

*Proof.* It is clear that (a) $\Rightarrow$ (c). Assume (c) and let $\phi$ be the corresponding weight on $M$ (by Theorem 13). Then $\phi(x) \leqslant C \| x \|$ for any $x \in \mathscr{I}_\Psi^+$ and hence $\phi(1) < \infty$. Finally, (a) $\Rightarrow$ (b) is easy. Note that $\phi(1) = \sum \langle T\xi_\alpha, \rangle_\alpha\rangle$ for any family $(\xi_\alpha)$, with $\sum \theta^{\Psi}(\xi_\alpha, \xi_\alpha) = 1$. This justifies the notation $\int T d\Psi$, for positive self-adjoint operators $T$ on $\mathscr{H}$ which are homogeneous in $\Psi$ of degree $-1$:

$$\int T d\Psi = \sum \langle T\xi_\alpha, \xi_\alpha\rangle \text{ independently of } (\xi_\alpha) \text{ with } \sum \theta^{\Psi}(\xi_\alpha, \xi_\alpha) = 1.$$

We shall say that $T$ is $\Psi$-integrable when it satisfies the equivalent conditions of Corollary 18. One can then define the spaces $L^p(\mathscr{H}, \Psi)$ as spaces of operators homogeneous in $\Psi$ of degree $\alpha = -1/p$ such that $| T |^p$ is integrable. It would be interesting to work out their theory, the Hölder inequality, and to compare them with the $L^p$ spaces of Haagerup. For [3] we shall need the following corollary:

COROLLARY 19. *Let $\mathscr{H}_1$, $\mathscr{H}_2$, and $T$ be as in Definition* 17 *with* $\alpha = -\frac{1}{2}$. *Then $T^*T$ is integrable iff $TT^*$ is, and the two integrals are equal.*

*Proof.* Let $T = u \mid T \mid$ be the polar decomposition of $T$. Then for every family $\xi_\alpha$ in $\mathscr{H}_1 \cap$ Support $| T |$ such that $\sum \theta_\Psi(\xi_\alpha, \xi_\alpha) =$ Support $| T |$ the

$\xi'_\alpha = u\xi_\alpha \in \mathscr{H}_2$ satisfy $\sum \theta_\Psi(u\xi_\alpha, u\xi_\alpha) = \text{Support} \mid T^* \mid$; thus the statement follows by

$$\sum \langle \mid T^* \mid u\xi_\alpha, u\xi_\alpha \rangle = \sum \langle \mid T \mid \xi_\alpha, \xi_\alpha \rangle. \qquad \text{Q.E.D.}$$

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