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A FACTOR OF TYPE II_1 WITH COUNTABLE FUNDAMENTAL GROUP

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INTRODUCTION. Let M be a factor, $\mathrm{Out}M = \mathrm{Aut}M/\mathrm{Int}M$ will denote the quotient of the group of automorphisms of M by the normal subgroup of inner automorphisms. For $M = R$, the hyperfinite factor of type II_1, the group $\mathrm{Out}R$ is very large, in fact it contains any locally compact, separable group and is however simple as an abstract group. If M is of type II_1 and has no non trivial central sequences, then $\mathrm{Int}M$ is a closed subgroup of $\mathrm{Aut}M$, when $\mathrm{Aut}M$ is endowed with the topology of pointwise norm convergence in M_* (its predual), and then $\mathrm{Out}M$ inherits a topology which makes it into a Polish group. So in order to show that $\mathrm{Out}M$ is countable, it is enough to show that it is discrete. The next result is the first rigidity result pertaining to the theory of operator algebras.

THEOREM. *Let Γ be a countable discrete group with infinite conjugacy classes, satisfying property T of Kazhdan, then $M = \lambda(\Gamma)''$ is a factor of type* II_1 *such that*

1°) $\mathrm{Int}M$ *is closed in* $\mathrm{Aut}M$ *(for the topology of norm pointwise convergence in M_*).*

2°) $\mathrm{Out}M$ *is a discrete group.*

COROLLARY. $\mathrm{Out}M$ *is countable.*

Proof (of Corollary). Since $\mathrm{Out}M$ is a Polish topological space it has a dense countable subset, so being discrete it is countable.

Proof of 1°). Let M act in $l^2(\Gamma)$, and let J be the canonical antilinear involution, such that $J\xi_0 = \xi_0$, $Jx\xi_0 = x^*\xi_0$ $\forall x \in M$, where ξ_0 is the vector associated to the neutral element of Γ (i.e. $\xi_0(g) = 0$ if $g \neq e$ and $\xi_0(e) = 1$, if we consider ξ_0 as a function on Γ). For each $x \in M$, $JxJ \in M'$, the commutant of M; moreover the map $x \to JxJ$ is an antilinear isomorphism of M onto M'. Let λ be the left regular representation of Γ in $l^2(\Gamma)$, so for $g \in \Gamma$, $\lambda(g)$ is the operator of left translation by g; one has $\lambda(g) \in M$, and the equality $\rho(g) = \lambda(g)J\lambda(g)J$ defines a representation of Γ in $l^2(\Gamma)$.

By hypothesis the non-trivial conjugacy classes of Γ are infinite and Γ satisfies the Kazhdan property: ([3])

There exists a finite subset $F \subset \Gamma$ and an $\varepsilon > 0$, so that for any unitary representation π of Γ, if there exists $\xi \in H_\pi$ such that

$$\|\xi\| = 1, \ \|\pi(g)\xi - \xi\| \leqslant \varepsilon \quad \forall g \in F$$

then there exists $\xi' \in H_\pi$, $\pi(g)\xi' = \xi'$ $\forall g \in \Gamma$.

Now let π be the restriction of ρ to the orthogonal H_π of ξ_0 in $l^2(\Gamma)$. If $(x_n)_{n\in\mathrm{N}}$ is a central sequence in M, with $\tau(x_n) = 0$, one has $\xi_n = x_n\xi_0 \in H_\pi$ and $\pi(g)\xi_n = \rho(g)x_n\xi_0 = \lambda(g)x_n\lambda(g)^*\xi_0$ so that $\|\pi(g)\xi_n - \xi_n\| \to 0$ $\forall g \in \Gamma$. Now since M is a factor the equality $\pi(g)\xi = \xi$ $\forall g \in \Gamma$ implies that $\xi = 0$, thus one concludes from the above that $\|\xi_n\| \to 0$, i.e. all central sequences are trivial and hence Int$M$ is closed in Aut$M$ (cf. [2]). Q.E.D.

Proof of 2°). We shall construct a neighborhood $\mathscr{V}$ of the identity id $\in$ Aut$M$ so that $\mathscr{V} \subset$ Int$M$. This will show that Int$M$ is open and hence ends the proof. We take:

$$\mathscr{V} = \{\theta \in \mathrm{Aut}M, \|\theta(\lambda(g))J\lambda(g)J\xi_0 - \xi_0\| < \varepsilon \quad \forall g \in F\}.$$

$\mathscr{V}$ is open by construction, the only point is to show that it consists only of inner automorphisms. But notice that $\pi'(g) = \theta(\lambda(g))J\lambda(g)J$ defines a representation of Γ in $l^2(\Gamma)$, and hence by property T, there exists ξ so that:

$$\xi \neq 0, \ \theta(\lambda(g))J\lambda(g)J\xi = \xi, \qquad \forall g \in \Gamma$$

thus

$$\theta(\lambda(g))\,\xi - J\lambda(g)^*J\,\xi = 0 \qquad \forall g \in \Gamma.$$

This means that $\theta(x)\xi - Jx^*J\xi = 0$ $\forall x \in M$. By [1] this shows that $\theta \in$ Int$M$. Q.E.D.

COROLLARY. *The fundamental group $\mathfrak{S}(M)$ is countable. So there exist a denumerable subgroup D of $\mathbf{R}_+^*$ and factors M_λ, $\lambda \notin D$ which are of type* II$_1$ *pairwise non isomorphic but such that all $M_\lambda \otimes F_\infty$ are isomorphic to $M \otimes F_\infty$ where F_∞ is a factor of type I_∞.*

Proof. The discrete group $\Gamma \times \Gamma$ has property T because Γ has it, so Out$(M \otimes M)$ is countable. Let $\theta_\lambda \in$ Aut$(M \otimes F_\infty)$ with module λ then $\theta_\lambda \otimes \theta_\lambda^{-1}$ has module 1 and hence gives rise to a well defined automorphism class $\alpha_\lambda \in$ Out(M ⊗ M). For $\lambda_1 \neq \lambda_2$, $\theta_{\lambda_1}\theta_{\lambda_2}^{-1}$ is outer, so $\alpha_{\lambda_1} \neq \alpha_{\lambda_2}$. Q.E.D.

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