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PROPERTY T AND ASYMPTOTICALLY INVARIANT SEQUENCES

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ABSTRACT

A countable group Γ has property T of Kazhdan if and only if no measure preserving ergodic action of Γ has non-trivial asymptotically invariant sets.

A countable group Γ has property T of Kazhdan iff the trivial representation I of Γ is isolated in $\hat{\Gamma}$ ([1]). If Γ acts on a probability space (Ω, μ), a sequence of sets in Ω, $\{B_n\}$, is called asymptotically invariant iff for all $g \in \Gamma$, $\mu(B_n \Delta g B_n) \to 0$ (cf. [2]), and it is trivial if $\mu(B_n)\cdot(1-\mu(B_n)) \to 0$. K. Schmidt proved in [2, proposition 2.10] that if Γ has property T then any asymptotically invariant sequence for an ergodic measure preserving Γ action is trivial. When this happens let's say that the action is *strongly ergodic.* We shall establish the converse and thereby prove:

THEOREM. *A countable group Γ has property T iff any measure preserving ergodic action of Γ is strongly ergodic.*

PROOF. We assume that Γ does not have property T and construct an ergodic action of Γ which is not strongly ergodic. Let φ_n be a sequence of pure positive definite function on Γ such that for all $g \in \Gamma$, $\varphi_n(g) \to 1$, and let π_n denote the corresponding irreducible representations of Γ. We distinguish two cases:

(a) *All π_n are infinite dimensional*

For each n, let X_n^g, $g \in \Gamma$, be centered Gaussian random variables with covariance given by

$$E(X_n^g X_n^{g'}) = \varphi_n(g^{-1}g'), \qquad \forall g, g' \in \Gamma,$$

and let (Ω_n, μ_n) be a probability space on which the X_n^g's are defined and generate the σ-field. The action of Γ on X_n^g's given by $g_1(X_n^g) = X_n^{g_1 g}$ defines a

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measure preserving action of Γ on (Ω_m, μ_n). The corresponding unitary representation of Γ in $L^2(\Omega_n, \mu_n)$ is equivalent to $\Sigma_{k=0}^{\infty} \times \pi_n^k$. Let $(\Omega, \mu) = \Pi_1^{\infty}(\Omega_n, \mu_n)$ and take the product action of Γ. The corresponding unitary representation is now the direct sum of representations $\pi_1^{k_1} \times \pi_2^{k_2} \times \cdots \times \pi_n^{k_n}$ where $\pi_j^{k_j} = \times^{k_j} \pi_j$. Since each π_j is infinite dimensional irreducible, standard facts imply that for $n > 0$ the above tensor product doesn't contain I, the trivial representation of Γ, so that the action of Γ on (Ω, μ) is ergodic (even weakly mixing in the sense that the product action is also ergodic). We now show that the sequence $\{B_n\}$, $B_n = \{x \in \Omega;\, X_n^e(x) \geqq 0\}$ (e is the unit of Γ) is asymptotically invariant. Since by the assumption of centering, $\mu(B_n) = 1/2$, the sequence is non-trivial and so the theorem is proved in this case. An obvious two dimensional computation using the joint distribution of X_n^e and X_n^g gives that $\mu(B_n \Delta g B_n) = \alpha/\pi$ where $\varphi_n(g) = \cos\alpha$, and thus the assumption that $\varphi_n(g) \to 1$ for all $g \in \Gamma$ shows that $\{B_n\}$ is asymptotically invariant.

(b) Assume now that all π_n are finite dimensional. Since π_n is irreducible and does not contain the trivial representation there exists some $g_n \in \Gamma$ with $\varphi_n(g_n) \leqq 0$. (Indeed, letting K denote the compact group $\overline{\pi_n(\Gamma)}$, dk its Haar measure we have for any ξ in the Hilbert space H_{π_n}, $\int_K k\xi \cdot dk = 0$ so that $\varphi_n(g) = (\xi_0, g\xi_0)$ couldn't be >0 for all $g \in \Gamma$.) Now construct (Ω, μ) just as in (a), and define B_n just as before. Since the action is now not necessarily ergodic we take instead some ergodic component. All that we have to do is make sure that for some ergodic component the sequence B_n is not trivial. This can be done since $\varphi_n(g_n) \leqq 0$ implies that $(B_n \Delta g_n B_n) \geqq \frac{1}{2}$ and thus for some ergodic component (Ω_t, μ_t) we must have $\lim \mu_t(B_n^t \Delta g_n B_n^t) > 0$, while clearly that component can also be chosen so that one still has for fixed g, $\mu_t(B_n^t \Delta g B_n^t) \to 0$. Q.E.D.

References

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