von Neumann Algebras

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For every selfadjoint operator T in the Hilbert space H, $^1 f(T)$ makes sense not only in the obvious case where f is a polynomial but also if f is just measurable, and if $f_n(x) \rightarrow f(x)$ for all $x \in \mathbb{R}$ (with (f_n) bounded) then $f_n(T) \rightarrow f(T)$ weakly, i.e. $\langle f_n(T)\xi, \eta \rangle \rightarrow \langle f(T)\xi, \eta \rangle \forall \xi, \eta \in H$. Moreover the set $\{f(T), f$ measurable} is the set of all operators S in H invariant under all unitary transformations of H which fix T. More generally, if $(T_i), i=1, ..., k$, are operators in H then the weak closure of the set of polynomials in T_i, T_i^* is the space of all operators in H invariant under all the unitaries fixing the T_i , as follows from the bicommutation theorem of von Neumann (1929):

A subset M of L(H) is the commutant of a subgroup G of the unitary group U(H) iff it is a weakly closed * subalgebra of L(H) (containing the identity 1).

Such an algebra is called a von Neumann algebra (or ring of operators). Any commutative one is of the form $\{f(T), f \text{ measurable}\}\$ for a selfadjoint T, and hence is the algebra of essentially bounded measurable functions: L^{∞} (Spectrum T, Spectral measure T). In general the center of M is a commutative von Neumann algebra and hence an $L^{\infty}(X, \mu)$ for some measure space X, then $M = \{(T(x))_{x \in X}, T(x) \in M(x) \ \forall x \in X\}\$ is the algebra of all essentially bounded measurable sections of a family $M(x), x \in X$, of von Neumann algebras with trivial centers, i.e. factors. If $M = \pi(G)'$ is, to start with, the commutant of the unitary representation π of the group G, by the above decomposition, π becomes the direct integral of factor

¹ With infinite countable orthonormal basis.

representations π_x , i.e. representations with $\pi_x(G)'$ a factor. As subrepresentations of π correspond bijectively to selfadjoint idempotents of $M = \pi(G)'$, to say that $\pi_x(G)'$ is a factor means that any two subrepresentations of π_x have a common subrepresentation. In finite dimension this says that π_x is a multiple of an irreducible subrepresentation, i.e. that $\pi_x(G)'$ is $M_n(C)$, with n=multiplicity of π_x , but in infinite dimension it is not always true that π_x has an irreducible subrepresentation, or equivalently that a factor always has a minimal projection. In fact it does iff it arises from an honest factorization of H as a tensor product: $H=H_1\otimes H_2$ with $M=\{T\otimes 1, T\in L(H_1)\}$. Murray and von Neumann discovered the existence of factors M not coming from the above trivial factorizations of H, and translating in terms of projections in M (i.e. selfadjoint idempotents $e=e^2=e^*\in M$) the comparison of subrepresentations they obtained the following multiplicity theory:

THEOREM Let M be a factor, then there exists a unique (up to normalization) injection of equivalence classes of projections of M in $[0, +\infty]$ such that:

$$\dim_{M} (e+f) = \dim_{M} (e) + \dim_{M} (f) \quad whenever \quad e \perp f,^{2}$$

and its range is

 $\{0, 1, ..., n\} \text{ then } M \text{ is of type } I_n, \\ \{0, 1, ..., \infty\} \text{ then } M \text{ is of type } I_\infty, \\ [0, 1] \text{ then } M \text{ is of type } II_1, \\ [0, +\infty] \text{ then } M \text{ is of type } II_\infty, \\ \{0, +\infty\} \text{ then } M \text{ is of type } III. \end{cases}$

The simplest example of a factor not of type I is the group algebra of an infinite discrete group Γ such that the normal subgroup of finite classes is trivial. One lets $R(\Gamma)$ be generated in $l^2(\Gamma)$ by the right translations, it is the commutant of the left translations, and is a factor. If ξ is the basis vector associated in $l^2(\Gamma)$ to the unit of Γ then the functional Trace_{Γ} $(A) = \langle A\xi, \xi \rangle$ on $R(\Gamma)$ satisfies:

 $\operatorname{Trace}_{\Gamma}(AB) = \operatorname{Trace}_{\Gamma}(BA) \quad \forall A, B,$

 $\operatorname{Trace}_{\Gamma}(1) = 1$

which is impossible if M was of type I_{∞} , i.e. isomorphic to $L(H_1)$ since every $A \in L(H_1)$ is a finite sum of commutators. What is amazing in case II_1 (or II_{∞}) is that the relative dimension of projection $e \in M$ (or equivalently the relative multiplicity of subrepresentations of π) can be any real number α , even irrational, in [0, 1]. Moreover, if one defines for any selfadjoint $T \in M$, its relative trace by $\operatorname{Trace}_M(T) = \int \lambda \dim_M (dE_{\lambda})$ (where $E_{\lambda} = 1_{1-\infty,\lambda}$)(T) is the spectral resolution of T), then, while it is easy to check that $\operatorname{Trace}_M(TT^*) = \operatorname{Trace}_M(T^*T) \ge 0 \ \forall T \in M$,

² I.e.
$$ef = fe = 0$$
.

the additivity of the trace, $\operatorname{Trace}_M(T_1+T_2) = \operatorname{Trace}_M(T_1) + \operatorname{Trace}_M(T_2)$, $\forall T_1, T_2$ was another striking result of Murray and von Neumann.

Around 1940 Gelfand and Naimark discovered a remarkable class of infinite dimensional algebras over C. Among all * algebras over C the C^* algebras are characterized by the very simple condition [1]:

$$||x|| = \sqrt{\text{Spectral Radius } x^*x}$$
 is a complete norm.

The commutative ones (with unit) are canonically isomorphic to the algebra of continuous functions on their compact spectrum. Every normed closed * subalgebra of L(H) is a C^* algebra and conversely every C^* algebra has a faithful representation in a Hilbert space. If A = C(X) is a commutative C^* algebra and π a representation of A in H then each coefficient $f \rightarrow \langle \pi(f)\xi, \xi \rangle$ is a positive linear functional on C(X), i.e. a Radon measure on X. In the noncommutative situation, positive linear functionals (i.e. elements φ of A^* with $\varphi(x^*x) > 0$) always exist in profusion (thanks to the convexity of $\{x^*x, x \in A\}$) and each determines a Hilbert space: the completion H_{φ} of A with the scalar product $\langle x, y \rangle_{\varphi} = \varphi(y^*x)$ and a representation π_{φ} of A in H_{φ} by left multiplication. This extends the usual construction of $L^2(X, \mu)$ for a Radon measure μ on the compact space X, and as in the commutative case the integral extends from continuous functions to measurable functions, i.e. here to the von Neumann algebra $\pi_{\varphi}(A)''$ generated by A in H_{φ} .

As an example let us describe the non commutative analogue of the construction of the probability space associated with the experiment of coin tossing. Instead of the Radon measure μ on the cantor set, $X = \prod_{1}^{\infty} X_{\nu}, X_{\nu} = \{a, b\}$, defined by

$$\mu(f_1 \otimes f_2 \otimes \ldots \otimes f_k \otimes 1) = \prod_1^k \mu(f_i)$$

one considers on the C^* algebra A, inductive limit of the $\bigotimes_1^k M_2(C)$, the positive linear functional Ψ such that

$$\Psi(x_1 \otimes x_2 \otimes \ldots \otimes x_k \otimes 1) = \prod_1^k \varphi(x_i)$$

where φ is a positive linear functional on $M_2(C)$ with $\varphi(1)=1$ (such a φ is called a state, because it corresponds to a state of a quantum mechanical system with $M_2(C)$ as algebra of observables). Up to unitary equivalence φ is always of the form

$$\varphi_{\lambda}(x) = \left(\frac{\lambda}{1+\lambda}\right) x_{11} + \left(\frac{1}{1+\lambda}\right) x_{22} \quad \forall x = [x_{ij}] \in M_2(C).$$

The corresponding von Neumann algebras $R_{\lambda} = (\pi_{\varphi_{\lambda}}(A))''$ are factors of type III and R. Powers (motivated by quantum field theory) proved in 1967 that they *are mutually nonisomorphic*. Previously only finitely many nontype I factors were known. The problem of classification of von Neumann algebras up to *spatial*

isomorphism (i.e. as pairs (H, M)) was since the beginning of the theory reduced to the problem of algebraic isomorphism. (If M is a factor, then the isomorphisms of M with von Neumann algebras in H are parametrized up to equivalence by an integer $n \in \{1, \dots, \infty\}$ in the type I case, a real $\lambda \in [0, +\infty]$ in the type II case and are all equivalent in the type III case.) Moreover an abstract * algebra M is a von Neumann algebra iff (1) it is a C^* algebra (2) as a Banach space it is a dual [31]. Moreover the predual of a C^* algebra M is unique, if it exists, and is the space of σ -additive linear functionals φ on M (i.e. $\varphi(\Sigma E_{\sigma}) = \Sigma \varphi(E_{\sigma})$ for any family of pairwise orthogonal projections). A foliated manifold f gives rise in a natural way to such an abstract von Neumann algebra $R(\mathfrak{f})$. Let Ω be the set of leaves of f, a random operator $T = (T_f)_{f \in \Omega}$ is a bounded measurable family of operators, T_f acting in $L^2(f)$ for all f. Sums, product and * are defined pointwise, and as in usual measure theory, one neglects any set of leaves whose union in V is negligible (here for the smooth measure class) and any random operator T with $T_{c}=0$ for almost all leaves. Thus R(f) plays the role of the algebra of all bounded operators in "L² (generic leaf of f)". It is not of type I in general, it is a factor iff f is ergodic (i.e. any measurable function on V, constant on the leaves, is a.e. constant), and can be of type II_{∞} or III. If A is a holonomy invariant transverse measure for f one can give a meaning to $\varphi(T) = \int \operatorname{Trace} (T_f) d\Lambda(f)$ for every positive random operator T, and this defines on the von Neumann algebra M of random operators (modulo equality Λ almost everywhere) a functional φ satisfying:

(1) φ is a weight on M i.e. φ is a linear map from M_+ to $[0, +\infty]$, $\varphi(\operatorname{Sup} T_{\alpha}) = \operatorname{Sup} \varphi(T_{\alpha})$ for any increasing bounded family, and there are enough T with $\varphi(T) < \infty$ to generate M.

(2) φ is faithful: $\varphi(T) > 0, \forall T > 0$ in M.

(3) φ is a *trace* i.e. is unitarily invariant, $\varphi(UTU^{-1}) = \varphi(T)$.

Here (3) is the translation of the holonomy invariance of Λ .

Every von Neumann algebra M has a faithful weight; those which possess a faithful trace are called *semifinite*. The *additivity of the trace* of Murray and von Neumann shows that a factor fails to be semifinite iff it is of type III. Around 1950, Dixmier and Segal showed many important consequences of semifiniteness. One can define, as in usual integration theory, the L^p spaces by the norms

$$||x||_p = (\operatorname{Trace}_M |x^p|)^{1/p}$$
 where $x \in M$, $|x| = \sqrt{x^* x}$.

Then L^1 is the predual M_* , and the representation π of M by left multiplication in L^2 satisfies the commutation theorem:

$$\pi(M)' = J\pi(M)J, J: L^2 \to L^2, J^2 = 1$$

where J is the isometric involution $x \to x^*$ in L^2 . As a corollary one gets the commutation theorem for tensor products $((M_1 \otimes M_2)' = M_1' \otimes M_2')$ for M_1 and

³ This can be finite even if Trace $T_f = +\infty$ for all $f \in \Omega$, see [8] for more details.

 M_2 semifinite and for any unimodular locally compact group G the fact that the right regular representation generates the von Neumann algebra R(G) of left invariant operators in $L^2(G)$. The natural weight $\varphi_G(f) = f(e)$ (e the unit of G) on the convolution algebra R(G) is a trace iff G is unimodular. J. Dixmier obtained the above result also for nonunimodular G, and it was Tomita who succeeded in proving the two other results (existence of (π, J) and commutation theorem for tensor products) for arbitrary von Neumann algebras — this theory, once supplemented by the general theory of weights (Takesaki, Combes, Pedersen, Haagerup) can be summarized as follows:

Instead of a trace, one starts with a faithful weight φ on M. The lack of tracial property for φ creates two natural scalar products $\varphi(x^*x)$ and $\varphi(xx^*)$ and hence a positive (unbounded) operator Δ_{φ} in the Hilbert space H_{φ} of the first scalar product. In the group algebra situation H_{φ} is identical with $L^2(G)$ and Δ_{φ} is the multiplication by the module Δ_G of G. In this special case, since Δ_G is a homomorphism (from G to \mathbb{R}^*_+) it follows that the one parameter group of unitaries Δ_{φ}^{il} normalizes R(G). The most remarkable result of Tomita is that this is a general fact:

THEOREM. Let M act in H_{φ} by left multiplications, then $\Delta_{\varphi}^{il}M\Delta_{\varphi}^{-il}=M \quad \forall t \in \mathbf{R}.$

This result became central when Takesaki discovered that the corresponding one parameter group of automorphisms of M ($\sigma_t^{\varphi}(x) = \Delta_{\varphi}^{it} \times \Delta_{\varphi}^{-it} \forall t \in \mathbf{R}$) is characterized (in its link with φ) by an algebraic form⁴ of the condition long known in quantum statistical physics as the Kubo Martin Schwinger condition. (If A is the algebra of observables, φ a statistical state, and σ_t the time evolution, a group of automorphisms of A, then (φ, σ) satisfies the Kubo Martin Schwinger condition at inverse temperature β iff $\varphi(x\sigma_{-i\beta}(y)) = \varphi(yx) \forall x, y \in A$. When A = L(H)and $\sigma_t(x) = e^{itH}xe^{-itH}$ where H is the hamiltonian, the unique φ satisfying this condition is the Gibbs state $x \to \text{Trace}(e^{-\beta H}x)/\text{Trace}(e^{-\beta H})$.) After the discovery of Powers in 1967 of the non isomorphism of the factors $R_\lambda, \lambda \in [0, 1[$, Araki and Woods analyzed the infinite tensor products of finite dimensional factors by means of two invariants, computable in terms of the eigenvalue list,

$$r_{\infty}(M) = \{\lambda \in \mathbb{R}_{+}^{i} | M \otimes R_{\lambda} \text{ is isomorphic to } M\},\$$
$$\varrho(M) = \{\lambda \in \mathbb{R}_{+}^{i} | M \otimes R_{\lambda} \text{ is isomorphic to } R_{\lambda}\}.$$

My point of departure was the existence of simple formulae relating, in the special case considered by Araki and Woods, those invariants and the Tomita—Takesaki theory, namely:

$$r_{\infty}(M) = \bigcap_{\varphi} \operatorname{Sp} \Delta_{\varphi}, \ \varrho(M) = \left\{ e^{2\pi/T}, T \in \bigcup_{\varphi} \operatorname{Ker} \sigma^{\varphi} \right\}.$$

This suggested that one ought to study for their own sake the invariants S(M) =

⁴ Identified by Haag Hugenholtz and Winnink in 1966.

 \bigcap_{φ} Sp Δ_{φ} and $T(M) = \bigcup_{\varphi}$ Ker σ^{φ} . The first question was computability. In the semifinite case, all weights φ are of the form $\varphi(x) = \operatorname{Trace}_{M}(\varrho x)$ where ϱ is a positive operator and the spectrum of Δ_{φ} is the closure of the set of ratios $\lambda_{1}/\lambda_{2}, \lambda_{i} \in \operatorname{Spectrum} \varrho$ while $\sigma_{i}(x) = \varrho^{it} x \varrho^{-it}$ for $x \in M$, taking $\varphi = \operatorname{Trace}_{M}, S(M) = \{1\}, T(M) = R$. In the type III case, the one parameter group σ^{φ} is never inner but the following result solved completely the problem of computability of S and T.

THEOREM. Let M be a von Neumann algebra, Aut M its automorphism group, ε : Aut $M \rightarrow \text{Out } M = \text{Aut } M/\text{Int } M$ the canonical quotient map, and φ a weight on M. Then a one parameter group of automorphisms of M, $(\alpha_t)_{t \in \mathbb{R}}$ is of the form σ^{Ψ} for a suitable Ψ iff $\varepsilon(\alpha_t) = \varepsilon(\sigma_t^{\varphi}) \forall t \in \mathbb{R}$.

In particular together with a type III factor there is a canonical homomorphism $\delta: \mathbf{R} \rightarrow \text{Out } M$, with $\delta(t) = \varepsilon(\sigma_t^{\varphi})$ for any weight φ . Moreover with a suitable notion of spectrum for δ one has:

S(M) =Spectrum δ , T(M) =Kernel of δ .

In particular both are subgroups (of R_+^* and R). As S is closed and as closed subgroups of R_+^* form a compact interval [0, 1], one gets a finer classification of type III factors:

М	is of type ΠI_{λ} ,	λ∈]0, 1[if	$S(M) = \{0\} \cup \lambda^Z,$
М	is of type III ₀		if	$S(M) = \{0, 1\},\$
М	is of type III ₁		if	$S(M) = [0, +\infty[.$

In the case of foliations the invariant S of the von Neumann algebra coincides with the *ratio set* introduced by W. Krieger in ergodic theory as a generalization of the Araki-Woods ratio set.

Roughly speaking to evaluate the ratio set of a foliation, one travels on the generic leaf from the point a to a point b which is close to a in V (but at any distance on the leaf) and one compares a unit of transversal volume in a with its transformed under holonomy at b; the set of all essential such ratios coincides with S, and is thus a natural obstruction to the existence of a holonomy invariant choice of unit of volume in the transverse bundle.

Exactly as in noncommutative algebra where one uses the cross product of an algebra by a group of automorphisms, one defines the cross product of a factor N by an automorphism θ (it is characterized as a von Neumann algebra M generated by N and a unitary U with $UxU^* = \theta(x) \ \forall x \in N$, so that the equality $\sigma_t(x) = x \ \forall x \in N, \sigma_t(U) = e^{it}U$ defines an automorphism of M for all $t \in \mathbf{R}$).

The general theory of factors of type III₁, $\lambda \in]0, 1[$ is summarized as follows:

(a) Let N be a factor of type II_{∞} and θ an automorphism with $mod(\theta) = \lambda$ (i.e. $Trace_N \circ \theta = \lambda Trace_N$); then the cross product $N \otimes_{\theta} Z$ is a factor of type III_{λ} .

(b) Any factor of type III₁ is of the form (a), and in a unique way (i.e. if (N_i, θ_i) give the same M there exists an isomorphism $N_1 \rightarrow N_2$ carrying θ_1 on θ_2 .

In case III_0 we proved an analogue discrete description but the definitive understanding and solution of the III_1 case is contained in the following result of Takesaki:

Any factor of type III is of the form $N \otimes_{\theta} R^*_+$ where N is a von Neumann algebra of type II_{∞} (i.e. in its central decomposition $N = \{(x(u))_{u \in A}, x(u) \in N(u) \forall u \in A\}$ every N(u) is a factor of type II_{∞}) and where for some trace τ on N one has $\tau \circ \theta_{\lambda} = \lambda \tau$. Moreover this decomposition is unique as above, and:

The restriction of θ_{λ} to A defines an ergodic flow F(M), which is an invariant of M. This flow has a very natural interpretation as an abstract flow of weights on M.

One has $S(M) = \{\lambda, F_{\lambda} = id\}$; when M is of type III₁ it follows that N is a factor so one gets the analogue of (a), (b) with the group Z replaced by R.

In the III_{λ} case, $N = \{(x(u))_{u \in S^1}, x(u) \in N(u), \forall u \in S^1\}$ so that N "fibers over a circle", and the θ of (a), (b) is θ_{λ} . The above structure theorem for factors of type III_{λ} reduces the problem of classification in this case to

(1) Classify factors of type II_{∞} .

(2) Given a factor of type II_{∞} , N, classify (up to conjugacy) its automorphism with module $\lambda, \lambda \in]0, 1[$.

Every factor of type II_{∞} is the tensor product of a factor of type II_1 by the type I_{∞} factor. In the last of their papers, Murray and von Neumann had shown that, though there exists more than one factor of type II_1 (they exhibited 2, in 1968 D. MacDuff constructed a continuum of them) there is among them, only one having the following approximation property: \forall finite subset F of $N, \forall \varepsilon > 0, \exists$ a finite dimensional * subalgebra K with distance $(x, N) < \varepsilon$, $\forall x \in F$ (where the distance is in the hilbert space L^2 of the trace_N). As any other factor of type II_1 contains this hyperfinite one, it was hence natural to think it is the simplest of all and to consider problem (2) in this case. The answer is the following:

For $\lambda \in]0, 1[$ there is, up to conjugacy, only one automorphism of $R_{0,1} = R \otimes I_{\infty}$ with module λ .

If $1/\lambda$ is an integer *n*, one can construct θ_{λ} as the shift on $R_{0,1}$ built as an infinite tensor product of $n \times n$ matrices. As another example, if *T* is the Anosov diffeomorphism of the 2 torus R^2/Z^2 defined by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ then *T* defines an automorphism of its stable foliation, and hence of the corresponding factor which is $R_{0,1}$, this automorphism has module λ where (λ, λ^{-1}) are the eigenvalues of the above matrix. A crucial motivation in the proof of the above theorem is that, since the study of automorphisms of abelian von Neumann algebras is equivalent to ergodic theory of a single transformation, one would expect many results of this theory to have an analogue in the non abelian situation. This turns out to be the case in particular for the Rokhlin tower theorem.

There is however a striking difference with usual ergodic theory, the existence of a *complex valued* invariant for periodic automorphisms. If N is a factor, it can happen for $\theta \in \operatorname{Aut} N$ that θ^k is inner for some k > 0, but that no automorphism $\theta', \varepsilon(\theta') = \varepsilon(\theta)$ satisfies $\theta'^k = 1$, the resulting obstruction is a kth root of 1 in C, $\gamma(\theta)$

which is invariant by multiplication of θ by an inner automorphism. This happens when N=R, every pair (k, z), k>0, $z \in C$, $z^k=1$ appears from a $\theta \in \operatorname{Aut} R$ and moreover the pair (k, z) is the only invariant of $\varepsilon(\theta) \in \operatorname{Out} R$, in other words the group Out $R=\operatorname{Aut} R/\operatorname{Int} R$ has only countably many conjugacy classes parametrized by (k, z). As a corollary one gets that Int R is the only normal subgroup of Aut R.

Elaborating on the existence of this complex valued invariant, we showed that not all factors (even of type II_1) are antiisomorphic to themselves.

In general if N is a factor of type II_{∞} one has a lot of non conjugate automorphisms with the same module $\lambda \in]0, 1[$; it was thus very natural to decide when, given a factor M of type III_{λ} , the corresponding factor of type II_{∞} is $R_{0,1}$. If one knows that it is $R_{0,1}$ then by the above theorem one knows M is isomorphic to Powers factor R_{λ} .

As seen above R is characterized, among factors of type II₁, by the approximation property of Murray and von Neumann. In the general (non II₁) case, a factor Mis called *approximately finite dimensional*⁵ when:

 $\forall F$ finite subset of $M, \forall *$ strong neighborhood V of 0

 $\exists K$ finite dimensional * subalgebra with K+V.

As an elaboration on Glimm's theorem characterizing C^* -algebra with only type I representations, it follows from the work of O. Marechal [22] and Elliott-Woods [13] that for any approximately finite dimensional factor M (not of type I_n, $n < \infty$, or II₁) and any C^* -algebra A not of type I, there is a representation π of A which generates M as a von Neumann algebra. Thus as soon as one goes beyond type I C^* -algebras one meets this whole class of factors. Moreover if A is the C^* -algebra corresponding to the "non commutative Cantor set" i.e. $A = \bigotimes_{1}^{\infty} M_2(C)$, then for any representation of A, $\pi(A)$ is approximately finite dimensional.

This obviously raises two questions:

(α) Classify the approximately finite dimensional factors.

(β) Characterize the C^{*}-algebras which generate only approximately finite dimensional factors.

In 1968 after trying to characterize $R_{0,1}$ (among factors of type II_{∞}) by the approximation property above, V. Ya. Golodets succeeded in showing that this class is stable under crossed products by abelian groups. It follows in particular that if M is of type III_{λ} , and M is AFD then the associated II_{∞} also is AFD. This indicated the interest of the problem: is $R_{0,1}$ unique among AFD of type II_{∞} . The difficulty is that while any II_{∞} is $II_1 \otimes I_{\infty}$ it is very difficult to see what property inherited by the II_1 would force it to be isomorphic to R.

In fact the characterization of R, of Murray and von Neumann involves * subalgebras, and hence has still some descriptive flavor. The second factor of type II₁ which they discovered was distinguished by "property Γ " which they considered

⁵ In short AFD.

technical; this property had no reason to characterize R since for any N, $N \otimes R$ possesses it, and in fact in 1962 J. T. Schwartz distinguished between N, R and $N \otimes R$. But in doing that, he found another property of R which was the germ of many later developments.

Property P. M in H has property P iff for any bounded $T \in L(H)$, the norm closed convex hull of the uTu^* , u unitary of M, intersects M'.

He proved that among N, R and $N \otimes R$ only R has property P, and moreover that the group algebra $R(\Gamma)$ of a discrete group has property P iff Γ is amenable. Any AFD factor possesses property P, but it is not clear from the definition that if $M = N \otimes Q$ then N has property P if M does. In fact the most important consequence of property P is the existence of a projection of norm one E from L(H) to M', with E(1)=1. By a result of J. Tomiyama any such projection satisfies $E(aTb)=aE(T)b \ \forall a, b \in M', \ \forall T \in L(H)$, and the existence of such a projection of L(H) on M is independent of the choice of representation. The family of von Neumann algebras satisfying it has the following remarkable stability properties:

(1) It is a monotone class (under decreasing intersections and weak closure of ascending unions).

(2) It is stable under commutant.

- (3) Stable under cross products by amenable groups.
- (4) Stable under tensor product.

The name used to qualify this class is *injectivity*, since it characterizes, thanks to a noncommutative version of the Hahn-Banach theorem due to W. Arveson, those von Neumann algebras which are injective objects in the category of C^* algebras, with completely positive maps as morphisms. As shown by Choi and Effros, it is also equivalent to the existence of a solution in M of the equation $y \otimes a \leq b$ (where $a \in M_n(C)$, $a = a^*$ and $b \in M \otimes M_n(C)$ are given) as soon as a solution exists in L(H). This is very useful because it allows us to treat direct integrals:

(5) M={(x(s))_{s∈A}, x(s)∈M(s) ∀s∈A} is injective iff almost all M(s) are injective. So let M be an injective von Neumann algebra, (5) and the reduction theory of von Neumann allow to assume that M is a factor, then the corresponding von Neumann algebra of type II_∞ is injective by (3) and again by (5) one can reduce to analysing injective factors of type II_∞ and finally of type II₁, writing M=N⊗I_∞. Then N is injective of type II₁ and by Tomiyama's theorem any projection of norm one E: L(H)→N, with E(1)=1 satisfies E(aTb)=aE(T)b, ∀a, b∈N, ∀T∈L(H). It follows that φ=Trace_N∘E is a state on L(H) invariant under all unitaries of N. We call such a state an hypertrace. In 1960 M. Takesaki had shown that if A₁, A₂ are simple C* algebras then A₁⊗A₂ is also simple (here A₁⊗A₂ acts in H₁⊗H₂ if A₁ and A₂ act in H₁, H₂), his proof involved a characterization of the norm on the algebraic tensor product A₁⊙A₂ coming from the representation in H₁⊗H₂ as the least of all possible C* norms on A₁⊙A₂. The corresponding completion A₁⊗_{min}A₂ is called the minimal tensor product of A₁ and A₂. He showed moreover that (as in Grothendieck theory for locally convex spaces) for certain C*

algebras (the *nuclear* ones by definition), only one C^* norm exists on $A \odot B$ for arbitrary C^* algebras B ([34], [31]). In 1972 Effros and Lance discovered that some factors (all the Araki—Woods factors at the time) give very good factorizations of L(H) inasmuch as the natural map η from $M \odot M'$ in L(H) given by $\eta(\sum a_i \otimes b_i) =$ $\sum a_i b_i$ is not only an injective homomorphism but is an *isometry* from $M \otimes_{\min} M'$ to the C^* algebra $C^*(M, M')$ generated by M and M' in L(H). They called this remarkable property *semidiscreteness* and proved semidiscreteness \Rightarrow Injectivity. So we get



In fact, these properties are all equivalent.

Assume first that N is a factor of type II_1 and is injective, the existence of an hypertrace on N implies that it is semidiscrete; then Takesaki's theorem shows that $C^*(N, N')$ is simple and hence that it cannot contain a nonzero compact operator in H. The following dichotomy then shows that N has property Γ . Let N be a factor of type II_1 in H; then N has property Γ or $C^*(N, N')$ contains all compact operators. (This was suggested by fine computations of C. Akemann and P. Ostrand showing that for the group algebra of free groups $C^*(N, N')$ contains all compact operators.)

Now N has property Γ iff the group Int N is not closed in Aut N (where Aut N is gifted with its natural topology: $\theta_{\alpha} \rightarrow \theta$ iff $\theta_{\alpha}(x) \rightarrow \theta(x)$ strongly for any $x \in N$). Moreover in general the closure of Int N is characterized in terms of $C^*(N, N')$ by the existence of an extension $\hat{\theta}$ of θ on N which is identity on N'. As in our case $C^*(N, N')$ is $N \otimes_{\min} N'$ we see that Aut $N = \overline{\operatorname{Int}} N \notin \operatorname{Int} N$.

The next step is to show that $N \otimes R$ is isomorphic to N. A remarkable result of D. MacDuff asserts that this is true as soon as N has a central sequence which is not hypercentral, which once translated in terms of automorphisms implies that

$$\overline{\operatorname{Int}} N \subset \operatorname{ct} N \Rightarrow N \sim N \otimes R$$

where ct N is the normal subgroup of all automorphisms θ of N which are trivial on central sequences. Here one has ct N=Int N because if $\theta \in \text{ct } N$ then $\theta \otimes 1 \in \text{ct } (N \otimes N)$ (this is due to a characterization of ct using $C^*(N, N')$) and as the symmetry $\sigma_N(x \otimes y) = y \otimes x$ in $N \otimes N$ is in Int $N \otimes N$ one has $\theta \otimes \theta^{-1}$ inner (and hence θ inner), because $\varepsilon(\text{ct})$ and $\varepsilon(\text{Int})$ always commute and $\varepsilon(\theta \otimes \theta^{-1}) = [\varepsilon(\theta \otimes 1), \varepsilon(\sigma_N)]$. From the properties $N \sim N \otimes R$ and $\sigma_N \in \text{Int}$ one finally deduces the approximation property of Murray and von Neumann. This can be very simply seen if one assumed N to be a subfactor of R but for the general case one uses the existence of an isomorphism of N with a subfactor of the ultraproduct R^{ω} where ω is a free ultrafilter, which in turn follows from the analogue of the Day-Namioka proof of Følner's characterization of amenable groups. The role of the invariant mean is played by the hypertrace and L(H) replaces $l^{\infty}(\Gamma)$ where Γ is the discrete group. Among those proofs the most technical are those relating properties of automorphisms (like $\theta \in \text{Int } N$) with properties of $C^*(N, N')$ (like the existence of $\hat{\theta}$). They involve an exhaustion method, allowing to pass from some infinitesimal information to a global one, and a probabilistic way of taking the polar decomposition of an operator (the usual way $x \rightarrow u(x)|x|$ being too discontinuous), based on the inequality $\int ||E^a(h^2) - E^a(k^2)||_2^2 da \leq ||h-k||_2 ||h+k||_2$ where E^a is the spectral projection $1_{[a, +\infty]}$. So we have now that all injective factors of type II₁ are isomorphic to R. As an immediate corollary, since all von Neumann subalgebras of R are also injective, one gets their complete classification up to isomorphism. It follows that R is the only factor contained in all others, which fully justifies the original belief of Murray and von Neumann that it is the simplest. Also if Γ is a *discrete amenable group* then its group algebra is isomorphic to R as soon as $\{g \in \Gamma, \text{ class of } g \text{ finite}\} = \{e\}$. If M is injective of type II_m then it is isomorphic to $R_{0,1}$. It follows that if G is an arbitrary connected locally compact group then the non type I part of its group algebra R(G) is of the form $A \otimes R_{0,1}$ where A is an abelian von Neumann algebra. Moreover the type III theory, allows to deduce from that, that the above 4 properties are equivalent in general. We thus have only one class which has, on the one hand, the nice characterization seen after Glimm's theorem, and on the other all the stability properties of the injective.

Furthermore in their work on C^* tensor products, Effros and Lance had shown that (1) all representations of nuclear C^* algebras generate injective von Neumann algebras (2) that if all representations of a C^* algebra are semidiscrete, then the C^* algebra is nuclear. Hence the C^* algebras satisfying condition (3) are exactly the nuclear ones (as a corollary $C^*(G)$ is nuclear for G locally compact connected). Let us mention also that for foliations the injectivity of the associated von Neumann algebra is equivalent to the *amenability* of the foliation, a remarkable and very useful property developed by Zimmer for ergodic group actions. (For instance the action of the fundamental group Γ of a compact Riemann surface V on the natural Poisson boundary $\partial \tilde{\mathcal{V}}$ of its covering space $\tilde{\mathcal{V}}$ is always amenable ergodic and is often of type III₁.) Let us turn now to injective factors of type III. If M is of type III₁, $\lambda \in]0, 1[$ then M is isomorphic to Powers factor R_{λ} . Wolfgang Krieger has shown in 1973 that for factors associated to a single ergodic transformation of a measure space, the flow of weights is a complete invariant and can be any ergodic flow. It follows from a very powerful cohomological lemma in his proof, and from the discrete decomposition of factors of type III_0 , that any injective factor of type III_0 arises from a single ergodic transformation of a measure space and is thus one of Krieger's factors. Thus in the III₀ case, the classification problem is transferred to ergodic theory: there are as many injective factors of type III₀ as ergodic (nontransitive) flows. There is only one injective factor M with $r_{\infty}(M) = [0, +\infty]$, it is the Araki—Woods factor R_1 , arising as algebra of local observables in the free field, but it is still unknown if it is the only injective of type III₁, (i.e. if $r_{\infty}(M) =$ S(M) for any injective). This factor R_1 is associated to the Anosov foliation of the geodesic flow of a Riemann surface of genus >1. We have used foliations above to illustrate the general theory by examples but von Neumann algebras can be very useful for the study of foliations per se. Ruelle and Sullivan have shown how, for an oriented foliation f of the compact manifold V (i.e. the subbundle F of TV, tangent to f is oriented), the holonomy invariant transverse measures Λ correspond exactly to *closed currents* C, "positive in the leaf direction". For such a measured foliation it is natural to define the Euler characteristic as $\langle e(F), [C] \rangle$, the Euler class of the bundle F evaluated on the cycle [C] created by the current C. Now von Neumann algebras allow to define the Betti numbers

$$eta_i = \int \dim H^i(f) \, d\Lambda(f) < \infty,$$

where H'(f) is the space of square integrable harmonic forms on f (with respect to some Euclidean structure on F, of which β_i turns out to be independent). One has then $\sum (-1)^i \beta_i - \langle e(F), [C] \rangle$. As β_0 is the measure of the set of compact leaves with finite holonomy, one gets that for 2 dimensional foliations without such leaves, the mean curvature of leaves is negative. The above formula is a special case of an index theorem computing for elliptic differential operators on f the scalar

$$\int \operatorname{Dim} \left(\operatorname{Ker} D_f\right) d\Lambda(f) - \int \operatorname{Dim} \left(\operatorname{Ker} D_f^*\right) d\Lambda(f)$$

as Ch $D \cdot \tau(F \otimes C)[C]$, where Ch $D \in H^*(V, Q)$ is the chern character of the symbol of D, $\tau(F \otimes C) \in H^*(V, Q)$ the Todd class of $F \otimes C$ and $[C] \in H_p(V, R)$ the homology class of the Ruelle—Sullivan current.

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⁶ Even though it may happen that dim $H^{\iota}(f) = +\infty$, $\forall f \in \Omega$.

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