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L^2-INDEX THEORY ON HOMOGENEOUS SPACES AND DISCRETE SERIES REPRESENTATIONS

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INTRODUCTION. These notes are an exposition, with occasional brief indications of proofs, of our results concerning the existence of global L^2-solutions for invariant elliptic systems of differential equations on (non-compact) homogeneous spaces of Lie groups. In a somehow less precise form, they were announced in [10,11]. Full details will be published elsewhere [12].

Let G be a unimodular Lie group, H a compact subgroup, and D a G-invariant elliptic differential operator on the homogeneous space $M = G/H$. Let $\operatorname{Ker} D$ denote the space of all L^2-solutions of the equation $Du = 0$. Usually, it is an infinite-dimensional Hilbert space, and thus the ordinary index of D is meaningless. However, because $\operatorname{Ker} D$ is G-invariant, it makes perfect sense to measure its "size" with respect to the Plancherel measure, and the elliptic nature of D implies that $\operatorname{Ker} D$ has finite "G-dimension". Thus one can define the "G-index" of D as the difference

$$\operatorname{ind}_G D = \dim_G \operatorname{Ker} D - \dim_G \operatorname{Ker} D^* ,$$

where D^* denotes the formal adjoint of D. Our main result is a formula for this index in terms of the principal symbol $\sigma(D)$ of D; it is formally similar to the standard index formula, except that the characteristic forms it involves are viewed as defining elements in $H_G^*(M, \mathbb{R})$ (the cohomology ring of G-invariant differential forms on M) rather than just ordinary De Rham cohomology classes. As in the compact case, this formula enables one to check, in specific instances, that $\operatorname{ind}_G D > 0$ and thus to prove existence of non-zero global L^2-solutions. On the other hand, we will also show that (with some natural restrictions on the structure of G) the representation of G on $\operatorname{Ker} D$ is a finite direct sum of irreducible discrete series representations. In particular, this implies that there are no non-zero global L^2-solutions for the equation $Du = 0$ whenever G has no discrete series, which in turn can be easily determined by knowing the structure of G.

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When G has a discrete, torsion-free, cocompact subgroup, as happens for example if G is linear and semisimple, our L^2-index theorem is a consequence of the L^2-index theorem for covering spaces of Atiyah and Singer ([2],[17]), combined with the Atiyah-Singer index theorem for compact manifolds [5]. However, this is far from being the case in general. Indeed, it is known (see [16]) that if a unimodular Lie group G with solvradical R, nilradical N and Levi factor S admits a discrete subgroup Γ of finite covolume then, assuming for simplicity that S acts effectively on R, both $R/R\cap\Gamma$ and $N/N\cap\Gamma$ are compact. In particular, this cannot happen unless the Lie algebra of N has a rational form. Even when G does admit discrete subgroups of finite covolume it still can fail to possess discrete subgroups which are cocompact, as illustrated by the following simple example. Let G be the semidirect product of $S = SL(n,\mathbb{R})$ with $N = \mathbb{R}^n$, the action of S on N being the standard one. If Γ is a discrete subgroup in G of finite covolume, then $\Gamma\cap N$ is conjugated under an inner automorphism to $\mathbb{Z}^n$, therefore $\Gamma\cap S$ is conjugated to a subgroup of $SL(n,\mathbb{Z})$, and this clearly prevents Γ of being cocompact.

In view of the Atiyah-Schmid approach to the discrete series representations of semisimple Lie groups [4], based on the use of the L^2-index theorem for covering spaces of compact manifolds, it is reasonable to expect that our index theorem can provide valuable information of representation-theoretic nature. Although we defer a closer look into this matter for a future paper, it is already transparent that our results, in conjunction with the recent results of Ash [1], can be used to work out geometric realizations for the discrete series. To illustrate this, we will outline in §6 a new geometric way to construct the discrete series representations for unimodular solvable Lie groups, involving, as in the semisimple case, L^2-harmonic spinors. This construction together with the fact that the symbol class of the "Dirac complex" provides a canonical generator for the equivariant K-theory of the co-isotropy representation (see §3) seem to indicate that the right ingredient to be used in the "geometric quantization" process is the Dirac operator rather than the $\bar{\partial}$-operator.

§1. G-INVARIANT PSEUDO-DIFFERENTIAL OPERATORS. As in the standard theory, in order to develop the analytical index machinery, one has to enlarge the class of differential operators. We shall introduce in this section the appropriate class of pseudo-differential operators.

From now on G will denote a unimodular Lie group with countably many connected components, H a compact subgroup and $M = G/H$ the corresponding homogeneous space. Given a representation $\varkappa$ of H on a finite-dimensional complex vector space E, we denote by $\mathcal{E}$ the associated homogeneous vector bundle over M and by $C^\infty(M,\mathcal{E})$ (resp. $C_c^\infty(M,\mathcal{E})$) its space of C^∞ (resp. C^∞ and compactly supported) sections. When ϵ is unitary, $\mathcal{E}$ carries a Hermitian structure which together with a G-invariant measure on M, gotten from a fixed Haar measure dg on G and the normalized Haar measure dh on H, determine a G-invariant inner product on $C_c^\infty(M,\mathcal{E})$. By completion, one obtains a Hilbert space $L^2(M,\mathcal{E})$; for $g \in G$, the corresponding unitary operator on $L^2(M,\mathcal{E})$ will be denoted $L(g)$. It is convenient to identify $L^2(M,\mathcal{E})$ (resp. $C^\infty(M,\mathcal{E})$, $C_c^\infty(M,\mathcal{E})$) with the subspace $(L^2(G)\otimes E)^H$ (resp. $(C^\infty(G)\otimes E)^H$) consisting of all elements in $L^2(G)\otimes E$ (resp. $C^\infty(G)\otimes E$, $C_c^\infty(G)\otimes E$) which are invariant under the representation $h \to R(h)\otimes \epsilon(h)$ of H; here R stands for the right regular representation of G.

Given two finite-dimensional unitary representations ϵ_1, ϵ_2 of H, let $\Psi^n(\mathcal{E}_1,\mathcal{E}_2)$ (resp. $\Psi^n(\mathcal{E})$, if $\epsilon_1 = \epsilon_2 = \epsilon$) denote the space of all order n pseudo-differential operators $P: C_c^\infty(M,\mathcal{E}_1) \to C^\infty(M,\mathcal{E}_2)$ which are G-invariant, i.e. $L(g)PL(g)^{-1} = P$ for any $g \in G$. In view of the above identification of $C_c^\infty(M,\mathcal{E}_1)$ (resp. $C^\infty(M,\mathcal{E}_2)$) with $(C_c^\infty(G)\otimes E_1)^H$ (resp. $(C^\infty(G)\otimes E_2)^H$), the Schwartz kernel K_P of P can be regarded as a $\mathrm{Hom}(E_1,E_2)$-valued distribution on $G \times G$ satisfying the covariance condition

$$K_P(x,y) = \epsilon_2(a)K(xa,yb)\epsilon_1(b)^{-1}, \quad x,y \in G, \quad a,b \in H.$$

Furthermore, since P is G-invariant, K_P is necessarily of the form $K_P(x,y) = k_P(x^{-1}y)$, where $k_P \in (C^{-\infty}(G)\otimes \mathrm{Hom}(E_1,E_2))^{H\times H}$, i.e. k_P is a $\mathrm{Hom}(E_1,E_2)$-valued distribution on G such that

$$k_P(x) = \epsilon_2(a)k_P(a^{-1}xb)\epsilon_1(b)^{-1}, \quad x \in G, \quad a,b \in H.$$

We will denote by $\Psi_c^n(\mathcal{E}_1,\mathcal{E}_2)$ (resp. $\Psi_c^n(\mathcal{E})$, if $\epsilon_1 = \epsilon_2 = \epsilon$) the subspace of $\Psi^n(\mathcal{E}_1,\mathcal{E}_2)$ consisting of those operators P for which the distribution k_P has compact support. Note that any differential operator $D \in \Psi^n(\mathcal{E}_1,\mathcal{E}_2)$ is automatically contained in $\Psi_c^n(\mathcal{E}_1,\mathcal{E}_2)$.

Some of the basic properties of pseudo-differential operators on compact manifolds remain valid for our class of G-invariant pseudo-differential operators, and we list below, for later use, the most important ones.

(1.1) Any $P \in \Psi_c^0(\mathcal{E}_1,\mathcal{E}_2)$ defines a bounded G-invariant operator $P: L^2(M,\mathcal{E}_1) \to L^2(M,\mathcal{E}_2)$.

(1.2) (Existence of a parametrix) Let $P \in \Psi_c^n(\mathcal{E}_1,\mathcal{E}_2)$ be elliptic. Then there exists $Q \in \Psi_c^{-n}(\mathcal{E}_2,\mathcal{E}_1)$ such that $I-QP \in \Psi_c^{-\infty}(\mathcal{E}_1)$ and $I-PQ \in \Psi_c^{-\infty}(\mathcal{E}_2)$.

(1.3) Let $P \in \Psi_c^n(\mathcal{E}_1,\mathcal{E}_2)$ $(n \geq 1)$ be elliptic. When regarded as an operator from $L^2(M,\mathcal{E}_1)$ to $L^2(M,\mathcal{E}_2)$ with (dense) domain $C_c^\infty(M,\mathcal{E}_1)$, the domain of its closure coincides with the subspace of all $u \in L^2(M,\mathcal{E}_1)$ such that $Pu \in L^2(M,\mathcal{E}_2)$.

§2. THE ANALYTICAL INDEX. Let $\mathcal{E}$ be a homogeneous vector bundle over M, associated to a unitary representation ϵ of H on E. The commutant $\mathcal{R}_G(\mathcal{E})$ of the representation L of G on $L^2(M,\mathcal{E}) \cong (L^2(G) \otimes E)^H$ carries a natural (faithful, normal, semi-finite) trace which will be denoted tr_G. This "G-trace" is uniquely determined, once a choice of a (two-sided) Haar measure dg on G has been made, and can be characterized by the following property:

$$tr_G(R(f)^*R(f)) = \int tr(f(g)^*f(g))dg ,$$

whenever $f \in (L^2(G) \otimes End(E))^{H \times H}$ is such that the convolution operator it defines

$$R(f) = \int f(g)R(g)dg$$

is bounded on $L^2(M,\mathcal{E})$. One can prove that $\Psi_c^{-\infty}(\mathcal{E})$ is contained in the domain of tr_G and that, for any smoothing operator $S \in \Psi_c^{-\infty}(\mathcal{E})$,

$$tr_G S = tr\ k_S(1) ,$$

where tr stands for the usual trace on $End(E)$ and 1 for the unit element of G. Given a closed, G-invariant subspace $\mathcal{K}$ of $L^2(M,\mathcal{E})$, we define its "G-dimension" by

$$\dim_G \mathcal{K} = tr_G K ,$$

where K is the orthogonal projection onto $\mathcal{K}$.

Now let $P \in \Psi_c^n(\mathcal{E}_1,\mathcal{E}_2)$ be elliptic, and let $Q \in \Psi_c^{-n}(\mathcal{E}_2,\mathcal{E}_1)$ be a parametrix for P, so that $S = I-QP \in \Psi_c^{-\infty}(\mathcal{E}_1)$. For any $u \in Ker\ P$ in $L^2(M,\mathcal{E}_1)$ one has $Su = u$. Since, as noted above, S is of G-trace class, it follows immediately that $\dim_G Ker\ P < \infty$. Therefore, we can define the analytical index of P as being the real number

$$ind_G P = \dim_G Ker\ P - \dim_G Ker\ P^*$$

As in the classical case, the G-index of P can be computed in terms of an arbitrary parametrix $Q \in \Psi_c^{-n}(\delta_2,\delta_1)$:

(2.1) $\operatorname{ind}_G P = \operatorname{tr}_G(I-QP) - \operatorname{tr}_G(I-PQ)$.

Using this, it is not difficult to derive the standard properties of the index map:

(2.2) if $P_1 \in \Psi_c^p(\delta_1,\delta_2)$ and $P_2 \in \Psi_c^q(\delta_2,\delta_3)$ are elliptic, then P_2P_1 is elliptic and $\operatorname{ind}_G P_2P_1 = \operatorname{ind}_G P_1 + \operatorname{ind}_G P_2$;

(2.3) ind_G is locally constant on the set of all elliptic operators in $\Psi_c^0(\delta_1,\delta_2)$, with respect to the norm topology;

(2.4) if $P \in \Psi_c^0(\delta_1,\delta_2)$ is elliptic and $S \in \Psi_c^{-1}(\delta_1,\delta_2)$, then $\operatorname{ind}_G(P+S) = \operatorname{ind}_G P$.

We can now proceed to define the index map at the K-theory level. Let $\mathfrak{G}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). We identify the tangent space T_oM , where $o = 1\cdot H \in M$, with $\mathfrak{G}/\mathfrak{h}$, and the cotangent space T_o^*M with $V = \mathfrak{h}^\perp$; explicitly

$$V = \{\xi \in \mathfrak{G}^*;\ \xi|\mathfrak{h} = 0\} .$$

The compact group H acts on V by the co-isotropy representation $\mathrm{Ad}^*: H \to GL(V)$. We denote by $K_H(V)$ the abelian group associated to the H-space V in the equivariant K-theory with compact supports. Since V happens to be a vector space on which H acts linearly, the group $K_H(V)$ can be described as follows. Consider all smooth H-equivariant maps $\alpha: S^V \to \mathrm{Iso}(E_1,E_2)$, where S^V denotes the unit sphere in V with respect to a fixed H-invariant metric, and E_1,E_2 are finite-dimensional H-modules. There is an obvious notion of an isomorphism between two such objects, and further an obvious notion of a homotopy. The set $\mathfrak{C}$ of all homotopy classes is an abelian semigroup under the direct sum operation. Let $\mathfrak{C}_o$ denote the sub-semigroup of all classes in $\mathfrak{C}$ which can be represented by a constant map. The quotient semigroup $\mathfrak{C}/\mathfrak{C}_o$ is actually a group, isomorphic to $K_H(V)$.

Observe now that for any $P \in \Psi^n(\delta_1,\delta_2)$ its principal symbol $\sigma(P)$ is G-invariant, and hence completely determined by $\sigma(P)_o$, the principal symbol at $o \in M$. Further, when P is elliptic $\sigma(P)_o$ maps S^V , H-equivariantly, to $\mathrm{Iso}(E_1,E_2)$ and thus defines an element in $K_H(V)$, which will be denoted $[\sigma(P)]$. The properties (2.2)-(2.4) imply:

(2.5) There exists a unique homomorphism $\operatorname{ind}_a: K_H(V) \to \mathbb{R}$ such that, for any elliptic $P \in \Psi_c^n(\delta_1,\delta_2)$, $\operatorname{ind}_a[\sigma(P)] = \operatorname{ind}_G P$.

§3. THE TOPOLOGICAL INDEX. In this section and the next one it will be assumed that M has even dimension $m = 2n$, and also that M is G-invariantly orientable. If, in addition, M admits a G-invariant spin-structure (i.e. the co-isotropy representation $Ad^*: H \to SO(V)$ lifts to $Spin(V)$), one can construct a G-invariant Dirac operator D^+ on M and it is known that its symbol class $\delta = [\sigma(D^+)]$ generates freely $K_H(V)$ as a module over the representation ring $R(H)$, giving thus rise to an isomorphism $\tau: R(H) \to K_H(V)$. By inverting this "Thom isomorphism" it is then easy to define the "topological" index of an arbitrary element in $K_H(V)$. In general, $K_H(V)$ need not be simply generated over $R(H)$. However, as we shall see in a moment, it is not difficult to get around this difficulty.

Consider the subgroup $\tilde{H}$ of $H \times Spin(V)$ consisting of all elements (h,s) such that $Ad^*(h)$ and the orthogonal transformation of V determined by $s \in Spin(V)$ coincide. The group $\tilde{H}$ is a double covering of H and it comes up with two natural representations: $(h,s) \to s \in Spin(V)$ which lifts the co-isotropy representation, and $(h,s) \to Ad^*(h) \in SO(V)$ which gives V the structure of an $\tilde{H}$-space. The standard half-spin representations $S^{\pm}(V)$, when regarded as $\tilde{H}$-modules, define a "Dirac" complex over V whose class $\tilde{\delta} \in K_{\tilde{H}}(V)$ generates freely $K_{\tilde{H}}(V)$ over $R(\tilde{H})$. Let $u \in Spin(V)$ denote the generator of the kernel of the covering homomorphism $Spin(V) \to SO(V)$. Since u is central in $\tilde{H}$, for any irreducible representation ϵ of $\tilde{H}$, $\epsilon(u) = \pm I$. Clearly, $R(\tilde{H}) = R(\tilde{H})^0 \oplus R(\tilde{H})^1$, where $R(\tilde{H})^1$ is generated by the equivalence classes of irreducible representations ϵ of $\tilde{H}$ such that $\epsilon(u) = (-1)^i I$; in particular $R(\tilde{H})^0$ can be identified with $R(H)$. Similarly let us define $K_{\tilde{H}}(V)^i$ to be the subgroup of all classes in $K_{\tilde{H}}(V)$ representable by $\tilde{H}$-complexes with compact support on which u acts by $(-1)^i I$, $i = 0,1$. Then $K_{\tilde{H}}(V) = K_{\tilde{H}}(V)^0 \oplus K_{\tilde{H}}(V)^1$ and with respect to this decomposition $K_{\tilde{H}}(V)$ is a $\mathbb{Z}_2$-graded module over the $\mathbb{Z}_2$-graded ring $R(\tilde{H}) = R(\tilde{H})^0 \oplus R(\tilde{H})^1$; in addition, $K_{\tilde{H}}(V)^0$ is canonically isomorphic to $K_H(V)$. In conclusion, one has:

(3.1) Given $k \in K_H(V)$ there exists a unique $c \in R(\tilde{H})^1$ such that $k = c\tilde{\delta}$ in $K_{\tilde{H}}(V)$.

The first ingredient one needs in order to define the topological index of an element $k \in K_H(V)$ is the "Chern character". Let $H^*(\mathfrak{G},H,\mathbb{R})$ be the relative Lie algebra cohomology of the pair $(\mathfrak{G},H)$, with trivial real coefficients. If we fix an $Ad(H)$-invariant splitting $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{M}$, this specifies a G-invariant connection on the principal bundle $H \to G \to M$, whose connection form is given by the projection $\theta: \mathfrak{G} \to \mathfrak{H}$ parallel to $\mathfrak{M}$, and whose curvature form is determined by

THEOREM. For any G-invariant elliptic pseudo-differential operator $P \in \Psi_c^p(\mathcal{E}_1, \mathcal{E}_2)$, $\mathrm{ind}_G P = \mathrm{ind}_t[\sigma(P)]$.

We shall now outline the main ideas of the proof, which is essentially based, in the same spirit as the proof of the index theorem for foliations [9], on the heat equation method.

4.1. Let $A^+ : C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^-)$ be a G-invariant elliptic first-order differential operator with formal adjoint $A^- : C^\infty(M, \mathcal{E}^-) \to C^\infty(M, \mathcal{E}^+)$. Consider the associated Laplace operators $\Delta^{\pm} = A^{\mp}A^{\pm}$. Each of them is an unbounded self-adjoint operator, so that $H_t^{\pm} = e^{-t\Delta^{\pm}}$ $(t > 0)$ is a well-defined family of bounded operators on $L^2(M, \mathcal{E}^{\pm})$ satisfying the heat equation

$$\frac{d}{dt} H_t^{\pm} + \Delta^{\pm} H_t^{\pm} = 0$$

with initial value $H_0^{\pm} = I$. The invariance under G forces $H_t^{\pm}$ to be of the form

$$H_t^{\pm} = R(h_t^{\pm}) \text{ , with } h_t^{\pm} \in (C^\infty(G) \otimes \mathrm{End}(E^{\pm}))^{H \times H} .$$

Although $h_t^{\pm}$ is not compactly supported one can still prove that $H_t^{\pm}$ are of G-trace class and that

$$\mathrm{tr}_G H_t^{\pm} = \mathrm{tr}\, h_t^{\pm}(1) .$$

It is now a consequence of the classical theory of the asymptotic heat expansion (cf. [3]), and a crucial fact for us, that $\mathrm{tr}\, h_t^{\pm}(1)$ has an asymptotic expansion of the form

$$\mathrm{tr}\, h_t^{\pm}(1) \sim \sum_{k \geq -m} c_k(A^{\pm}) t^{k/2} ,$$

where the coefficients $c_k(A^{\pm})$ are purely local invariants of $\Delta^{\pm}$. On the other hand, for essentially formal reasons, one has the analogue of the McKean-Singer identity

$$\mathrm{ind}_G A^+ = \mathrm{tr}_G H_t^+ - \mathrm{tr}_G H_t^- \text{ , for any } t > 0 .$$

One can conclude that

$$\mathrm{ind}_G A^+ = c_0(A^+) - c_0(A^-) .$$

4.2 Choose a G-invariant Riemannian metric on M, or, which amounts to the same thing, an H-invariant inner product on V. To any unitary representation ϵ of H one can then associate the twisted signature operators $L_\epsilon^{\pm}$, with "coefficients" in the corresponding homogeneous bundle $\mathcal{E}$. The heat equation formula for the G-index explained above in conjunction with the local version of the generalized signature theorem (cf. [3]) give the G-index formula for L_ϵ^{+}:

$$\operatorname{ind}_G L_\epsilon^{+} = 2^n(\operatorname{ch}(\epsilon)\mathcal{L}(\mathfrak{G},H))[V] ,$$

where $\mathcal{L}(\mathfrak{G},H)$ denotes the cohomology class in $H^*(\mathfrak{G},H,\mathbb{R})$ determined by the form

$$\det \frac{\frac{1}{2}\,\Theta_V}{\tanh(\frac{1}{2}\,\Theta_V)} \in \Lambda V^* \quad .$$

4.3. Let us now consider the case of Dirac-type operators. This time by ϵ we denote a representation of $\widetilde{H}$ such that $[\epsilon] \in R(\widetilde{H})^1$. Let $S^{\pm}(V)$ be the standard half-spin modules. Since $u \in \widetilde{H}$ acts as $-I$ on E and also on $S^{\pm}(V)$, $E^{\pm} = E \otimes S^{\pm}(V)$ are well-defined H-modules, so one can form the induced homogeneous bundles $\mathcal{E}^{\pm}$. Fix an Ad(H)-invariant splitting $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{M}$ and let

$$\nabla_\epsilon^{\pm}: C^\infty(M,\mathcal{E}^{\pm}) \to C^\infty(M,T_C^*M \otimes \mathcal{E}^{\pm})$$

be the associated G-invariant connections on $\mathcal{E}^{\pm}$. The $(\pm)$-Dirac operator with coefficents in $\mathcal{E}$ is defined as the composition

$$D_\epsilon^{\pm}: C_c^\infty(M,\mathcal{E}^{\pm}) \xrightarrow{\nabla_\epsilon^{\pm}} C_c^\infty(M,T_C^*M \otimes \mathcal{E}^{\pm}) \xrightarrow{C} C_c^\infty(M,\mathcal{E}^{\mp}) ,$$

where C is the bundle homomorphism induced by the Clifford multiplication.

Assume for the moment that $\epsilon = \varphi \otimes s$, where φ is an arbitrary representation of H and $s = s^+ \oplus s^-$, with $s^{\pm}$ denoting the representation of $\widetilde{H}$ on $S^{\pm}(V)$. It is easy to see that

$$\sigma(D_{\varphi\otimes s}^{+}) = (-1)^n\sigma(L_\varphi^{+}) ,$$

therefore, by (2.5),

$$\operatorname{ind}_G D_\epsilon^{+} = (-1)^n(\operatorname{ch}(\epsilon)\hat{\mathcal{A}}(\mathfrak{G},H))[V] .$$

this case, the support of the measure class on $\hat{G}$ corresponding to Ker P is located on the discrete series we use the very precise results of Harish-Chandra on the construction of the Plancherel measure (cf. [13]) and also the existence of a G-invariant parametrix for P (cf. (1.2)).

Combining the results of Anh and the well-known criterion of Harish-Chandra for the existence of discrete series in the real reductive case one gets, as an immediate consequence of the above theorem, the following vanishing theorem:

COROLLARY. If rank K_Γ < rank Γ then there are no non-zero L^2-solutions for the equation $Pu = 0$.

§6. DISCRETE SERIES FOR UNIMODULAR SOLVABLE LIE GROUPS. As an application, we shall describe a *new* geometric construction of the discrete series for a simply connected, unimodular, solvable Lie group G. Let us first recall a few basic facts concerning the parametrizations of the discrete series $\hat{G}_d$ and the structure of G when $\hat{G}_d \neq \emptyset$, due to Charbonnel [8] and Anh [1].

First of all, G has discrete series (modulo its center Z) if and only if the set

$$\Lambda_c = \{\lambda \in \mathfrak{G}^*;\ G_\lambda/Z \text{ is compact}\},$$

where G_λ denotes the isotropy subgroup of G at λ with respect to the co-adjoint action, is non-empty. If $\lambda \in \Lambda_c$, then G_λ is connected (in fact, it is a vector group) and the orbit $M_\lambda = G\cdot\lambda$ is closed in $\mathfrak{G}^*$, so that one can apply the Auslander-Kostant-Kirillov construction (cf. [6]) to produce an irreducible unitary representation π_λ of G. The assignment $\lambda \to \pi_\lambda$ induces a bijection, $M_\lambda \to [\pi_\lambda]$, of the orbit space Λ_c/G onto $\hat{G}_d$.

Assume now that $\hat{G}_d \neq \emptyset$. Define

$$\mathfrak{Z}_c^* = \{\lambda|\mathfrak{Z};\ \lambda \in \Lambda_c\}$$

and fix $\zeta \in \mathfrak{Z}_c^*$. Let χ be the character of the identity component Z_0 of Z whose differential is $2\pi i\zeta$, and denote

$$\Lambda_c^\zeta = \{\lambda \in \Lambda_c; \lambda|\mathfrak{Z} = \zeta\},\ \hat{G}_d^\chi = \{[\pi] \in \hat{G}_d;\ \pi|Z_0 \text{ is a multiple of } \chi\}.$$

Clearly, $\lambda \in \Lambda_c^\zeta$ if and only if $[\pi_\lambda] \in \hat{G}_d^\chi$.

Let us now fix $\lambda \in \Lambda_c^\zeta$. Then G can be written as a semi-direct product of a closed, connected, normal subgroup A, which is an H-group with center Z_0, and a closed subgroup T of G_λ, which is a vector group;

moreover, $G_\lambda = Z_0 T$ and $Z_0 \cap T = \{1\}$. Let $\alpha = \lambda|\mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of A. Then $\pi_\alpha \in \hat{A}_d$ and $\pi_\lambda|A$ is unitarily equivalent to π_α. In addition, any irreducible representation π such that $[\pi] \in \hat{G}_d^\chi$ is of the form:

$$\pi(at) = \tau(t)\pi_\lambda(at) \ , \ a \in A, \ t \in T, \ \text{with} \ \tau \in \hat{T} \ . \tag{6.1}$$

We are now ready to describe the realization of these representations in terms of L^2-harmonic spinors. Denote by ϵ_λ the character of G_λ whose differential is $2\pi i\lambda|\mathfrak{G}_\lambda$, where $\mathfrak{G}_\lambda$ is the Lie algebra of G_λ, and by ϵ_λ the corresponding homogeneous line bundle over $M_\lambda = G/G_\lambda$. Since G_λ/Z is compact, there exists a G-invariant metric on M_λ, and we will fix one such. On the other hand, since G_λ is simply connected, the co-isotropy representation of G_λ in $SO(V)$ lifts to $Spin(V)$; here $V = \{v \in \mathfrak{G}^*; v|\mathfrak{G}_\lambda = 0\}$. We can thus form the corresponding twisted spin-bundles with coefficients in $\mathcal{E}_\lambda^\pm = \mathcal{E}_\lambda \otimes S^\pm(V)$ and, after choosing G-invariant connections in $\mathcal{E}_\lambda^\pm$, we can construct the twisted Dirac operators $D_\lambda^\pm$.

THEOREM. The formal difference $\operatorname{Ker} D_\lambda^+ - \operatorname{Ker} D_\lambda^-$ represents a single irreducible discrete series representation π^λ of G with $[\pi^\lambda] \in \hat{G}_d^\chi$. Any $[\pi] \in \hat{G}_d^\chi$ can be realized in these terms.

Let us now sketch the proof. As follows from §5, $\operatorname{Ker} D_\lambda^\pm$ is a finite sum of discrete series representations, therefore the virtual representation $\operatorname{Ker} D_\lambda^+ - \operatorname{Ker} D_\lambda^-$ is a finite integral linear combination of representations in $\hat{G}_d^\chi$. In view of (6.1), it is easily seen that all the representations π with $[\pi] \in \hat{G}_d^\chi$ have the same formal degree d_χ, depending only on χ. Using the index theorem, we will show that

$$\dim_G \operatorname{Ker} D_\lambda^+ - \dim_G \operatorname{Ker} D_\lambda^- = d_\chi \ , \tag{6.2}$$

which proves the first part of the theorem. Indeed, write $\mathfrak{G} = \mathfrak{z} \oplus \mathfrak{M} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of T and $\mathfrak{M}$ is a Ad(T)-invariant supplementary subspace to $\mathfrak{z}$ in $\mathfrak{a}$. Then $\mathfrak{G} = \mathfrak{G}_\lambda \oplus \mathfrak{M}$ defines a connection from $v \in \mathfrak{G}^* \otimes \mathfrak{G}_\lambda$. Since $[\mathfrak{M},\mathfrak{M}] \subset \mathfrak{a}$, the corresponding curvature form θ, a priori an element in $\Lambda^2\mathfrak{M}^* \otimes \mathfrak{G}_\lambda$, is in fact in $\Lambda^2\mathfrak{M}^* \otimes \mathfrak{z}$. It then follows first that

$$\theta_V = 0 \ , \ \text{hence} \ \mathcal{J}(\mathfrak{G}, G_\lambda) = 1 \ ,$$

and next that

$$ch(\varepsilon_\lambda) = \exp(-\frac{1}{2} B_\alpha)$$

where

$$B_\alpha(X,Y) = \alpha([X,Y]) \ , \ X,Y \in \mathfrak{R} \ .$$

Therefore, with a suitable choice of the orientation on $\mathfrak{R}$,

$$\mathrm{ind}_G D_\lambda^+ = \frac{1}{n!2^n} (\Lambda^n B_\alpha)[V] \quad ,$$

which, according to [1, Prop. 2.13], is precisely the formal degree of π_α . Together with the obvious equality

$$\deg \pi_\alpha = d_\lambda \ ,$$

this proves (6.2).

The proof of the second part of the theorem is just a routine matter.

The point of this result is that any discrete series representation of G can be realized in terms of L^2-solutions of an elliptic system of equations, not just a "partially" elliptic one as achieved when using polarizations. Although we do not have in mind a specific application of this fact, we feel it is noteworthy.

It would be desirable to know, and reasonable to expect, that $\mathrm{Ker}\ D_\lambda^-$ always vanishes, and thus that $\mathrm{Ker}\ D_\lambda^+$ is in fact the representation space of π^λ . In this direction we note that it would be enough to prove this vanishing result for N-groups only, and also that it does hold indeed when λ is "sufficiently non-singular"; this can be easily seen by imitating the existing vanishing proof (cf. [12, §7] and [15]) in the nilpotent case.

§7. FURTHER REMARKS. Before concluding, let us add a few comments on the index theorem.

7.1. The first remark concerns the role of the C*-algebras in the index theory, so far left inconspicuous in our exposition. Let ε_H be the left regular representation of H on $E_H = L^2(H)$ and let $\mathcal{E}_H$ be the corresponding homogeneous Hilbert bundle over M . Allowing coefficients in $\mathcal{K}(E_H)$, the space of compact operators on $L^2(H)$, we can form as in §2 the spaces of G-invariant pseudo-differential operators $\Psi_c^{-\infty}(\mathcal{E}_H)$ and $\Psi_c^{O}(\mathcal{E}_H)$. By taking the norm closure of these rings of operators in $\mathcal{B}_G(\mathcal{E}_H)$ one obtains two C*-algebras, $C_G^*(\mathcal{E}_H)$ and $\Psi_G^*(\mathcal{E}_H)$ respectively. The symbol map extends to the whole $Y_G^*(\mathcal{E}_H)$, giving rise to the exact sequence of separable C^*-algebras:

(*) $$0 \to C_G^*(\mathcal{E}_H) \to Y_G^*(\mathcal{E}_H) \overset{\sigma}{\to} (C(S^V) \otimes \mathcal{K}(E_H))^H \to 0 .$$

The first term is canonically isomorphic to the reduced C^*-algebra $C_r^*(G)$ of G (this follows from the fact that the representation of G induced by the regular representation of H is unitarily equivalent to the regular representation of G), so that the six term exact sequence of K-groups associated to (*) gives a connecting map

$$\mathrm{Ind}_G\colon K_1((C(S^V) \otimes \mathcal{K}(E_H))^H) \to K_0(C_r^*(G)) \quad .$$

On the other hand, the natural G-trace on $C_r^*(G)$ determines an additive map $\mathrm{tr}_{G*}\colon K_0(C_r^*(G)) \to \mathbb{R}$, and the analytical G-index can be viewed as the composition

$$\mathrm{ind}_G = \mathrm{tr}_{G*} \circ \mathrm{Ind}_G \quad .$$

This explains the stability properties (2.1)-(2.4) of ind_G .

The index map Ind_G allows to attach to any G-invariant elliptic differential operator on $M = G/H$ a more refined index than the numerical one, namely an element in the denumerable group $K_0(C_r^*(G))$. The computation of this index appears to be a very interesting problem. In fact this is precisely what Bott [7] did in the case when G is compact (and thus $K_0(C_r^*(G))$ coincides with the representation ring $R(G)$). We also note that Ind_G makes sense in the non-unimodular case as well, where no natural trace is available.

7.2. A recent result of Phil Green [J. Functional Analysis, 35(1980), Corollary 3, p. 290] implies that the discrete series $\hat{G}_d$ can be viewed as a subset of $K_0(C_r^*(G))$. Proving that Ind_G is surjective would therefore essentially imply the exhaustion character of the geometric realization. It becomes thus apparent that the theory of discrete series may be entirely established on C^*-algebras K-theoretical foundations.

7.3. As shown by Bott [7], if G is compact then $\mathrm{ind}_G D$ only depends on the H-modules E_1, E_2 and not on the particular elliptic operator $D\colon C^\infty(M_1,\mathcal{E}_1) \to C^\infty(M,\mathcal{E}_2)$. In the non-compact case this is not necessarily true. A counterexample can be easily constructed by using the Heisenberg group. However the property still holds when G is semisimple and H is a maximal compact subgroup. Let us mention that in this case it is also