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The L^2-index theorem for homogeneous spaces of Lie groups

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Introduction

The L^2-index theorem for covering spaces of Atiyah [3] and Singer [20] asserts that, given a discrete group Γ acting smoothly and freely on a manifold $\tilde{M}$ with compact quotient $M = \Gamma \backslash \tilde{M}$, and an elliptic differential operator $\tilde{D}$ on $\tilde{M}$ which is Γ-invariant and thus descends to an elliptic differential operator D on the compact manifold M, then the Γ-index $\operatorname{ind}_\Gamma \tilde{D} = \dim_\Gamma \operatorname{Ker} \tilde{D} - \dim_\Gamma \operatorname{Ker} \tilde{D}^*$ of $\tilde{D}$ coincides with the ordinary index $\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^*$ of D. Here $\operatorname{Ker} \tilde{D}$ is the space of L^2-solutions of $\tilde{D}u = 0$, and $\dim_\Gamma$ denotes the dimension function corresponding to the trace (on the commutant of Γ acting on the Hilbert space of L^2-sections over $\tilde{M}$) naturally associated to Γ. The importance of this theorem lies in the fact that $\operatorname{ind} D > 0$ implies the existence of nontrivial L^2-solutions for the equation $\tilde{D}u = 0$, and as such it was used in a crucial way by Atiyah and Schmid [5] to construct explicit realizations of the discrete series representations for semisimple Lie groups. Indeed, if G is a Lie group which possesses a discrete, torsion-free, cocompact subgroup Γ, and H is a compact subgroup of G, then the L^2-index theorem applied to the covering space $\tilde{M} = G/H$ of $M = \Gamma \backslash G/H$, combined with the index formula of Atiyah-Singer [6], yields existence results for L^2-solutions of G-invariant elliptic equations on the homogeneous space G/H.

It is relevant to note that, with D denoting this time a G-invariant elliptic differential operator on G/H, the ratio between the Γ-index of D and the covolume of Γ gives a number independent of Γ. This real number, $\operatorname{ind}_G D$, can be in fact intrinsically defined as the difference of the two "formal degrees" corresponding to the representations of G on $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$ respectively, and hence makes sense for any unimodular Lie group G, even when it has no discrete, cocompact subgroups (which, outside the semisimple case, is the generic situation). Furthermore, it follows from the abstract theory of traces that the G-index $\operatorname{ind}_G D$ depends only on the principal symbol $\sigma(D)$ of D, and hence should be computable in terms involving only $\sigma(D)$, according to a formula which is easy to guess from the Γ-index case. In the present paper we shall (1) prove this formula in full generality, and (2) show, for a large class of groups G, that the representation of G on $\operatorname{Ker} D$ is a finite direct sum of irreducible discrete series. These two results give criteria for both existence and vanishing (see Corollary 6.3) of L^2-solutions.

Our proof of (1) is based, with no surprise, on the heat equation method (cf. [4]) and extensive use of the theory of von Neumann algebras, with the McKean-Singer identity applied to the G-trace on the commutant of the left quasi-regular representation of G. An expected difficulty arises at the K-theory level, since the equivariant K-theory of the isotropy representation is not simply generated as a module over the representation ring $R(H)$ (unless there is a G-invariant spin-structure, which is an undesirable restriction). We overcome it by showing that, for a priori reasons, the formula for the G-index of D depends on the principal symbol $\sigma(D)$ in a continuous way with respect to the I_H-adic topology on $R(H)$ (I_H being the augmentation ideal), and then using the signature operator. The proof we outlined in [10] avoided this difficulty from the start, but the use of the index theorem for foliations (cf. [8]), on which it essentially relied, was not very enlightening.

As already mentioned in [11], the main step towards (2) is to handle the case when G is semisimple. The main ingredients we use in order to show that the support of the measure class on $\hat{G}$ corresponding to $\operatorname{Ker} D$ is located in the discrete series are the very precise results of Harish-Chandra on the construction of the Plancherel measure, and the existence of a G-invariant parametrix for D (cf. Proposition 1.3 below).

The plan of the paper is the following. In Section 1 we develop the basic calculus with G-invariant pseudo-differential operators on a homogeneous space, with special emphasis on the properties of order zero operators. The main device we use in the process is the "averaging" operation, which, as shown in Section 2, also allows us to control the G-trace in terms of the local, ordinary trace. Section 3 is devoted to the definition and elementary properties of the analytical index for G-invariant elliptic pseudo-differential operators, which are then used to prove that it factors through the symbol map. The topological index is introduced in Section 4, and the index theorem is proved in Section 5. In Section 6 we establish the fact that the space of L^2-solutions of the equation $Pu = 0$, where P is a G-invariant pseudo-differential operator and G satisfies certain conditions, decomposes as a finite direct sum of irreducible square-integrable representations. Finally, in Section 7, we collect a few applications of the main results, together with some comments on the behaviour and the nature of the G-index map.

1. Invariant pseudo-differential calculus

To begin with, we shall discuss some basic facts concerning the pseudo-differential calculus with G-invariant operators on a homogeneous space. Throughout, G will denote a unimodular Lie group with at most countably many connected components, H a compact subgroup, and $M = G/H$ the corresponding homogeneous space of left cosets gH, $g \in G$.

Given a representation ε of H on a finite-dimensional complex vector space E, we denote by $\mathcal{E}$ the associated homogeneous vector bundle over M. The space $C^\infty(M, \mathcal{E})$ (resp. $C_c^\infty(M, \mathcal{E})$) of all C^∞-sections (resp. C^∞-sections with compact support) of $\mathcal{E}$ will be identified with the space $(C^\infty(G) \otimes E)^H$ (resp. $(C_c^\infty(G) \otimes E)^H$) of all elements in $C^\infty(G) \otimes E$ (resp. $C_c^\infty(G) \otimes E$) which are invariant under the representation $h \mapsto R(h) \otimes \varepsilon(h)$ of H, where R stands for the right regular representation of G. Suppose now that the representation ε is unitary. Then, the H-invariant inner product on E determines a G-invariant Hermitian structure on $\mathcal{E}$. Further, after choosing a Haar measure dg on G and giving M the G-invariant measure dg/dh, where dh denotes the normalized Haar measure on H, one can define, in an obvious way, a global inner product on $C_c^\infty(M, \mathcal{E})$, invariant under the left action $L(g)$ of any $g \in G$. The corresponding completion, denoted $L^2(M, \mathcal{E})$, will be also regarded as being the (closed) subspace $(L^2(G) \otimes E)^H$ of all H-invariant elements in the Hilbert tensor product $L^2(G) \otimes E$.

If $\mathcal{E}_1, \mathcal{E}_2$ are homogeneous vector bundles over M, we will write $\Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ for the space of all pseudo-differential operators P: $C_c^\infty(M, \mathcal{E}_1) \to C^\infty(M, \mathcal{E}_2)$, of order n. Here, the term "pseudo-differential" refers to the class of operators used in [6], which corresponds to the case $\rho = 1$, $\delta = 0$ in Hörmander's notation (see [15]). Given $P \in \Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$, we write $\sigma^n(P)$, or simply $\sigma(P)$, for its principal symbol. Recall that, if $T'M$ denotes the cotangent bundle T^*M with the zero-section removed and π: $T'M \to M$ stands for the corresponding projection, then $\sigma(P)$: $\pi^*\mathcal{E}_1 \to \pi^*\mathcal{E}_2$ is a smooth bundle homomorphism and, on each fibre of $T'X$, $\sigma(P)$ is positively homogeneous of degree n.

For $P \in \Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ we shall denote by K_P its distributional kernel, which will be regarded as an element in the following space of distributions on $G \times G$:

$$(C^{-\infty}(G \times G) \otimes \operatorname{Hom}(E_1, E_2))^{H \times H} = \{K \in C^{-\infty}(G \times G) \otimes \operatorname{Hom}(E_1, E_2);$$
$$K(x, y) = \varepsilon_2(a) K(xa, yb) \varepsilon_1(b)^{-1}, (x, y) \in G \times G, (a, b) \in H \times H\}.$$

Thus, if $u \in C_c^\infty(M, \mathcal{E}_1) = (C_c^\infty(G) \otimes E_1)^H$, one has

$$(Pu)(x) = \int K_P(x, y) u(y)\, dy, \qquad x \in G.$$

We will say that $P \in \Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ is compactly supported when K_P has this property, and we will denote by $\Psi_{cc}^n(M, \mathcal{E}_1, \mathcal{E}_2)$ the subspace of $\Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ consisting of all such operators. Further, we say that $P \in \Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ is G-compactly supported if $K_P(x, y) = 0$ for $x^{-1}y$ outside a compact set in G; these operators form a subspace of $\Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ which will be denoted by $\Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)$. Note that if $P \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)$ and $Q \in \Psi_c^m(M, \mathcal{E}_2, \mathcal{E}_3)$ their composition makes sense and $QP \in \Psi_c^{n+m}(M, \mathcal{E}_1, \mathcal{E}_3)$.

A pseudo-differential operator $P \in \Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ is called G-invariant if $L(g)PL(g)^{-1} = P$ for any $g \in G$. For such an operator

$$K_P(x, y) = k_P(x^{-1}y), \qquad (x, y) \in G,$$

where k_P is an element in the space of distributions on G,

$$\begin{aligned}\big(C^{-\infty}(G) \otimes \operatorname{Hom}(E_1, E_2)\big)^{H \times H} = \{&k \in C^{-\infty}(G) \otimes \operatorname{Hom}(E_1, E_2);\\ &k(x) = \varepsilon_2(a) k(a^{-1}xb) \varepsilon_1(b)^{-1}, x \in G, (a, b) \in H \times H\big\}.\end{aligned}$$

The subspace of $\Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)$ formed by G-invariant operators will be denoted $\Psi^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$. Note that $P \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$ if and only if k_P has compact support.

G-invariant pseudo-differential operators can be constructed by averaging over G ordinary pseudo-differential operators, which are compactly supported. Indeed, if $P \in \Psi_{cc}^n(M, \mathcal{E}_1, \mathcal{E}_2)$, then for each $u \in C_c^\infty(M, \mathcal{E}_1)$ the integral

$$\operatorname{Av}(P)u = \int_G L(g) P L(g)^{-1} u\, dg$$

makes sense, since it only involves integration over a compact subset of G, and it defines a C^∞-section of $\mathcal{E}_2$. Moreover, using the formula which describes the transformation of the asymptotic symbol under a diffeomorphism (see, for instance, [15], Theorem 2.16), one can show that:

1.1 Lemma. *If $P \in \Psi_{cc}^n(M, \mathcal{E}_1, \mathcal{E}_2)$ then $\operatorname{Av}(P) \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$ and one has*

$$\sigma(\operatorname{Av}(P)) = \int_G L(g) \sigma(P) L(g)^{-1} dg.$$

Conversely, every element $P \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$ arises in this way. Before giving the precise statement, let us introduce the following definition: a function

$f \in C_c^\infty(G)$ is said to be a cut-off function if

(a) f is positive;
(b) $f(gh) = f(g)$ for any $g \in G$ and $h \in H$;
(c) $\int_G f(x)\,dx = 1$.

1.2. LEMMA. *Let $P \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$. Then $P = \mathrm{Av}(fP)$, for any cut-off function f on G.*

Proof. Clearly, $fP \in \Psi_{cc}^n(M, \mathcal{E}_1, \mathcal{E}_2)$; therefore $\mathrm{Av}(fP)$ makes sense. For $u \in C_c^\infty(M, \mathcal{E}_1)$ one has

$$\begin{aligned}(\mathrm{Av}(fP)u)(x) &= \int_G \big(L(g)fPL(g)^{-1}u\big)(x)\,dg = \int_G \big(L(g)fL(g)^{-1}Pu\big)(x)\\ &= \int_G f(g^{-1}x)(Pu)(x)\,dg = (Pu)(x).\end{aligned}$$

Q.e.d.

Using this averaging procedure one can construct G-invariant parametrices out of ordinary parametrices.

1.3. PROPOSITION. *Assume that $P \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$ is elliptic. Then there exists $Q \in \Psi_c^{-n}(M, \mathcal{E}_2, \mathcal{E}_1)^G$ such that $I - QP \in \Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_1)^G$ and $I - PQ \in \Psi_c^{-\infty}(M, \mathcal{E}_2, \mathcal{E}_2)^G$.*

Proof. Let $Q' \in \Psi_c^{-n}(M, \mathcal{E}_2, \mathcal{E}_1)$ be an ordinary parametrix for P; in particular $S' = I - Q'P \in \Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_1)$. Choose a cut-off function f and define $Q = \mathrm{Av}(fQ') \in \Psi_c^{-n}(M, \mathcal{E}_2, \mathcal{E}_2)^G$. Then, for any $u \in C_c^\infty(M, \mathcal{E}_1)$,

$$\begin{aligned}(QPu)(x) &= \int_G \big(L(g)fQ'L(g)^{-1}Pu\big)(x)\,dg = \int_G \big(L(g)fQ'PL(g)^{-1}u\big)(x)\,dg\\ &= \int_G f(g^{-1}x)\big(Q'PL(g)^{-1}u\big)(g^{-1}x)\,dg\\ &= \int_G f(g^{-1}x)\big(u(x) - \big(S'L(g)^{-1}u\big)(g^{-1}x)\big)\,dg\\ &= u(x) - (Av(fS')u)(x).\end{aligned}$$

Since $S = \mathrm{Av}(fS') \in \Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_1)^G$, Q is a left parametrix for P. Similarly, $\mathrm{Av}(Q'f) \in \Psi_c^{-n}(M, \mathcal{E}_2, \mathcal{E}_1)^G$ is a right parametrix for P. Therefore, Q is also a right parametrix for P. Q.e.d.

The remainder of this section will be devoted to the study of G-invariant pseudo-differential operators of order zero. Recall that any $P \in \Psi_{cc}^0(M, \mathcal{E}_1, \mathcal{E}_2)$ satisfies an estimate of the form

$$\|Pu\|_{L^2} \leq c\|u\|_{L^2}; \qquad u \in C_c^\infty(M, \mathcal{E}_1)$$

and thus extends to a bounded operator $P\colon L^2(M, \mathcal{E}_1) \to L^2(M, \mathcal{E}_2)$. This is false for arbitrary $P \in \Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)$. However, as follows from the next proposition, it is true for operators in $\Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$.

1.4. PROPOSITION. *Let $P \in \Psi_{cc}^0(M, \mathcal{E}_1, \mathcal{E}_2)$ with support $(K_P) \subset C \times C$ for some compact C in G. Then $\mathrm{Av}(fP) \in \Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$ extends to a bounded operator from $L^2(M, \mathcal{E}_1)$ to $L^2(M, \mathcal{E}_2)$. In addition*

$$\|\mathrm{Av}(P)\| \leq c\|P\|,$$

where the constant c depends only on C.

The proof will be based on the following useful lemma, which we will also need later.

1.5. LEMMA. *Let $\mathcal{H}$ be a Hilbert space and assume that $F \in L^2(G, \mathcal{H})$ has the property: there exists a compact K in G such that $\langle F(x), F(y)\rangle = 0$ for $x^{-1}y \notin K$. Then*

(i) *$\int F(x)\,dx$ exists, as an element in $\mathcal{H}$;*

(ii) *$\|\int F(x)\,dx\|^2 \leq |K| \int \|F(x)\|^2\,dx$,*

where $|K|$ denotes the Haar measure of K.

Proof. Let C be an arbitrary compact in G. Then

$$\begin{aligned}\left\|\int_C F(x)\,dx\right\|^2 &\leq \int_C\int_C |\langle F(x), F(y)\rangle|\,dx\,dy\\ &= \int_C\int_C \chi_K(x^{-1}y)\,|\langle F(x), F(y)\rangle|\,dx\,dy\\ &\leq \int_C \|F(x)\|\left(\int_C \chi_K(x^{-1}y)\|F(y)\|\,dy\right)dx,\end{aligned}$$

where χ_K is the characteristic function of K. If we put

$$f(x) = \int \chi_K(x^{-1}y)\|F(y)\|\,dy,$$

it follows that

$$\left\|\int_C F(x)\,dx\right\|^2 \leq \|F\|_{L^2(G,\mathcal{H})}\|f\|_{L^2(G)} \leq |K|\,\|F\|^2_{L^2(G,\mathcal{H})},$$

since

$$\|f\|_{L^2(G)} \leq \|\chi_K\|_{L^1(G)}\left(\int \|F(y)\|^2\,dy\right)^{1/2} = |K|\,\|F\|_{L^2(G,\mathcal{H})}. \quad \text{Q.e.d.}$$

Proof of Proposition 1.4. With P as in the statement and $u \in C_c^\infty(M, \mathcal{E}_1)$, define

$$F(x) = L(x)PL(x)^{-1}u.$$

It is easy to check that $\langle F(x), F(y)\rangle = 0$ for $x^{-1}y \notin CC^{-1}$. Choose now $f \in C_c^\infty(G)$ such that $f \equiv 1$ on C. Then $Pf = P$, hence

$$\int \|F(x)\|^2\,dx = \int \|PfL(x)^{-1}u\|^2\,dx \leq \|P\|^2 \int \|L(x)fL(x)^{-1}u\|^2\,dx$$

$$= \|P\|^2 \int\int |f(x^{-1}y)|^2 \|u(y)\|^2\,dx\,dy \leq \|P\|^2 \|f\|^2_{L^2(G)} \|u\|^2.$$

Applying now the previous lemma one gets

$$\|\mathrm{Av}(P)u\|^2 = \left\|\int F(x)\,dx\right\|^2 \leq |C\cdot C^{-1}|\,\|P\|^2\|f\|^2_{L^2(G)}\|u\|^2. \qquad \text{Q.e.d.}$$

From Proposition 1.4 and Lemma 1.2 it follows:

1.6. Corollary. *Any $P \in \Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$ defines a bounded G-invariant operator P: $L^2(M, \mathcal{E}_1) \to L^2(M, \mathcal{E}_2)$.*

Due to this result, we can form a larger class of G-invariant bounded operators, which turns out to be very useful. Precisely, we shall define $\Psi_G^*(M, \mathcal{E}_1, \mathcal{E}_2)$ as being the norm-closure in $\mathfrak{B}(L^2(M, \mathcal{E}_1), L^2(M, \mathcal{E}_2))$ of $\Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$. Also, we define $C_G^*(M, \mathcal{E}_1, \mathcal{E}_2)$ as being the norm closure of $\Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_2)^G$. The reason behind this notation is the following. Remark first that

$$(1.1)\quad \Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_2)^G$$
$$= \left\{R(\varphi)\colon L^2(M, \mathcal{E}_1) \to L^2(M, \mathcal{E}_2);\ \varphi \in \left(C_c^\infty(G) \otimes \mathrm{Hom}(E_1, E_2)\right)^{H\times H}\right\},$$

where

$$(R(\varphi)u)(x) = \int \varphi(x^{-1}y)u(y)\,dy, \qquad u \in C_c^\infty(M, \mathcal{E}_1).$$

When $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, both $\Psi_G^*(M, \mathcal{E}, \mathcal{E})$ and $C_G^*(M, \mathcal{E}, \mathcal{E})$ are C^*-algebras of G-invariant operators on $L^2(M, \mathcal{E})$, and $\Psi_G^*(M, \mathcal{E}, \mathcal{E})$ is an extension of $C_G^*(M, \mathcal{E}, \mathcal{E})$. Moreover, as we shall see in Section 7, when $\mathcal{E}$ corresponds to the left regular representation of H, then $C_G^*(G, \mathcal{E}, \mathcal{E})$ is canonically isomorphic to the reduced C^*-algebra $C_r^*(G)$ of the group G.

To simplify the notation a little bit, we shall drop the letter M in most of the symbols defined up to now. Also, whenever $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ we shall write $\mathcal{E}$ only once.

The symbol map, initially defined on $\Psi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$, can be extended to the whole $\Psi_G^*(\mathcal{E}_1, \mathcal{E}_2)$. Indeed recall that, for $P \in \Psi^0(\mathcal{E}_1, \mathcal{E}_2)$, its principal symbol is given by the formula

$$\sigma(P)(x, d\varphi(x))u(x) = \lim_{t\to\infty} e^{-it\varphi(x)}P(e^{it\varphi}u)(x), \tag{1.2}$$

where $\varphi \in C^\infty(M)$, $d\varphi(x) \neq 0$, and $u \in C_c^\infty(M, \mathcal{E}_1)$. It is easy to check that

$$\sigma\big(L(g)PL(g)^{-1}\big) = L(g)\sigma(P)L(g)^{-1}, \quad \text{for any } g \in G.$$

In particular, when P is G-invariant, $\sigma(P)$ is completely determined by its value at $o = 1 \cdot H \in M$, which we will denote $\sigma_0(P)$; precisely,

$$\sigma_0(P)(\xi) = \sigma(P)(o, \xi) \in \operatorname{Hom}(E_1, E_2), \qquad 0 \neq \xi \in T_o^*M.$$

Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H. We identify T_0^*M with $V = \mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^*; \xi \mid \mathfrak{h} = 0\}$. Since $\sigma_0(P)(t\xi) = \sigma_0(P)(\xi)$, $t > 0$, for any P of order zero, $\sigma_0(P)$ can be viewed as a smooth H-equivariant map of the unit sphere (with respect to a fixed $\mathrm{Ad}^*(H)$-invariant metric on V) $S = S(V)$ to $\operatorname{Hom}(E_1, E_2)$, i.e. $\sigma_0(P) \in (C^\infty(S, \operatorname{Hom}(E_1, E_2))^H$. The completion of the latter space with respect to the sup-norm is, obviously, the space of all continuous H-equivariant functions from S to $\operatorname{Hom}(E_1, E_2)$, $(C(S, \operatorname{Hom}(E_1, E_2))^H$, for which we shall also use the simpler notation $C_H(S, E_1, E_2)$. By use of (1.2) it can be seen that

$$\|\sigma_0(P)\| \leq \|P\|, \qquad P \in \Psi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G.$$

Therefore σ_0 extends by continuity to a map

$$\sigma_0\colon \Psi_G^*(\mathcal{E}_1, \mathcal{E}_2) \to C_H(S, E_1, E_2).$$

1.7. Proposition. *The following sequence of separable C^*-algebras*

$$0 \to C_G^*(\mathcal{E}) \to \Psi_G^*(\mathcal{E}) \xrightarrow{\sigma_0} C_H(S, E) \to 0$$

is exact.

Proof. Clearly, $C_G^*(\mathcal{E})$ is contained in $\operatorname{Ker}\sigma_0$, since σ_0 vanishes on $\Psi_c^{-\infty}(\mathcal{E})^G$. Conversely, let us prove that $\operatorname{Ker}\sigma_0$ is contained in $C_G^*(\mathcal{E})$. Clearly, it is enough to check that, if $P \in \Psi_c^0(\mathcal{E})^G$ and $\|\sigma_0(P)\| < \delta$, then the spectrum of the class of P in the quotient C^*-algebra $\Psi_G^*(\mathcal{E})/C_G^*(\mathcal{E})$ is contained in $\{z \in \mathbf{C};\ |z| \leq \delta\}$. To see this, assume that $\lambda \in \mathbf{C}$ and $|\lambda| > \delta$; then $P - \lambda I \in \Psi_c^0(\mathcal{E})^G$ is elliptic and, by Proposition 1.3, invertible modulo $\Psi_c^{-\infty}(\mathcal{E})^G$.

Finally, let us prove that any $\alpha \in (C^\infty(S, \operatorname{End}(E))^H$ is the symbol of an operator $P \in \Psi_c^0(\mathcal{E})^G$. Since α is H-equivariant, there exists a globally defined G-invariant symbol $\tilde{\alpha}\colon \pi^*\mathcal{E} \to \pi^*\mathcal{E}$ whose value at the base point $o \in M$ coincides with α. Let f be a cut-off function. Using the usual construction of a pseudo-differential operation out of its symbol, we can produce $Q \in \Psi^0_{cc}(\mathcal{E})$ such that $\sigma(Q) = f\tilde{\alpha}$. Applying now Lemma 1.1 to Q, we get $P = \mathrm{Av}(Q) \in \Psi^0_c(\mathcal{E})^G$ with $\sigma(P) = \mathrm{Av}(f\tilde{\alpha}) = \tilde{\alpha}$; thus $\sigma_0(P) = \alpha$. Q.e.d.

1.8. Corollary. *The sequence*

$$0 \to C^*_G(\mathcal{E}_1, \mathcal{E}_2) \to \Psi^*_G(\mathcal{E}_1, \mathcal{E}_2) \xrightarrow{\sigma_0} C_H(S, E_1, E_2) \to 0$$

is exact, and σ_0 induces an isometry of $\Psi^*_G(\mathcal{E}_1, \mathcal{E}_2)/C^*_G(\mathcal{E}_1, \mathcal{E}_2)$ onto $C_H(S, E_1, E_2)$.

Proof. When $\mathcal{E}_1 = \mathcal{E}_2$ this is just the preceding proposition. The present statement follows by the standard trick of replacing $P \in \Psi^*_G(\mathcal{E}_1, \mathcal{E}_2)$ by $\tilde{P} \in \Psi^*_G(\mathcal{E}_1 \oplus \mathcal{E}_2)$, with

$$\tilde{P} = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}$$

and observing that $\|\tilde{P}\| = \|P\|$ and $\|\sigma_0(\tilde{P})\| = \|\sigma_0(P)\|$.

2. The natural trace

Let us first recall the definition of the natural trace on the commutant $\mathfrak{R}_G \subset \mathfrak{B}(L^2(G))$ of the left regular representation L of the unimodular Lie group G. The von Neumann algebra $\mathfrak{R}_G$ is canonically equipped with a faithful, normal, semi-finite trace tr_G, which, once a choice of a (two-sided) Haar measure dg on G has been made, is determined uniquely by the property

$$\mathrm{tr}_G(R(f)^*R(f)) = \int |f(g)|^2\, dg,$$

for $f \in L^2(G)$ with $R(f)$ bounded on $L^2(G)$. Here R denotes, as before, the right regular representation of G.

Let now $\mathcal{E}$ be a homogeneous bundle over M, associated to a unitary representation ε of H on E. We continue to denote by L the restriction to the closed, G-invariant subspace $L^2(M, \mathcal{E}) = (L^2(G) \otimes E)^H$ of the representation $L \otimes I$ of G on $L^2(G) \otimes E$. Its commutant $\mathfrak{R}_G(\mathcal{E})$ is the reduced algebra of $\mathfrak{R}_G \otimes \mathrm{End}(E)$ corresponding to $(L^2(G) \otimes E)^H$ and we let tr_G be its natural trace, obtained from the natural trace on $\mathfrak{R}_G$ and the ordinary trace on $\mathrm{End}(E)$.

The crucial fact for us, pointed out in [8, § 6] in a more general context, is that the "G-trace" tr_G on $\mathfrak{R}_G(\mathcal{E})$ can be related, by means of an averaging operation, to the ordinary trace tr on $\mathfrak{B}_G(\mathcal{E}) := \mathfrak{B}(L^2(M, \mathcal{E}))$, as we shall now explain. A general reference for the notions and results used in the following few lines is [13]. First of all tr_G, initially defined on the positive part $\mathfrak{R}_G^+(\mathcal{E})$ has a unique extension to the "extended" positive part $\hat{\mathfrak{R}}_G^+(\mathcal{E})$. Next, there exists a normal, faithful, semi-finite operator valued weight Av_G: $\mathfrak{B}_G^+(\mathcal{E}) \to \hat{\mathfrak{R}}_G^+(\mathcal{E})$, given by

$$\mathrm{Av}_G(T) = \int_G L(g)TL(g)^{-1}\,dg, \qquad T \in \mathfrak{B}_G^+(\mathcal{E}),$$

such that

$$\mathrm{tr}_G \mathrm{Av}_G(T) = \mathrm{tr}(T) \quad \text{for any } T \in \mathfrak{B}_G^+(\mathcal{E}).$$

The various domains of Av_G are defined, as usual, by

$$\mathrm{Dom}_{1/2}(\mathrm{Av}_G) = \{T \in \mathfrak{B}_G(\mathcal{E});\ \|\mathrm{Av}_G(T^*T)\| < \infty\},$$

$$\mathrm{Dom}(\mathrm{Av}_G) = \left\{\sum_{i=1}^{n} S_i^* T_i;\ S_i,\ T_i \in \mathrm{Dom}_{1/2}(\mathrm{Av}_G),\ n \in \mathbf{N}\right\}$$

and one has:

(a) $\mathrm{Dom}(\mathrm{Av}_G) = \{\sum_{i=1}^n \lambda_i T_i;\ T_i \in \mathfrak{B}_G^+(\mathcal{E}),\ \|\mathrm{Av}_G(T_i)\| < \infty,\ \lambda_i \in \mathbf{C},\ n \in \mathbf{N}\}$;

(b) $\mathrm{Dom}(\mathrm{Av}_G)$ and $\mathrm{Dom}_{1/2}(\mathrm{Av}_G)$ are two-sided modules over $\mathfrak{R}_G(\mathcal{E})$;

(c) Av_G has a unique linear extension Av_G: $\mathrm{Dom}(\mathrm{Av}_G) \to \mathfrak{R}_G(\mathcal{E})$, which satisfies

$$\mathrm{Av}_G(PTQ) = P\,\mathrm{Av}_G(T)Q, \qquad T \in \mathrm{Dom}(\mathrm{Av}_G),\ P, Q \in \mathfrak{R}_G(\mathcal{E}).$$

2.1. LEMMA. *Let* $T = T^* \in \mathrm{Dom}(\mathrm{Av}_G) \cap \mathrm{Dom}(\mathrm{tr})$. *Then*

(i) $\mathrm{Av}_G(T) \in \mathrm{Dom}(\mathrm{tr}_G)$ *and* $\|\mathrm{Av}(T)\|_{\mathrm{tr}_G} \le \|T\|_{\mathrm{tr}}$;

(ii) $\mathrm{tr}_G \mathrm{Av}_G(T) = \mathrm{tr}(T)$.

Proof. Let $T = T^+ - T^-$ be the Jordan decomposition of T. Since by hypothesis $\mathrm{tr}(T^{\pm}) < \infty$, it follows that $\mathrm{Av}_G(T^{\pm}) \in L^1(\mathfrak{R}_G(\mathcal{E}), \mathrm{tr}_G)$ and $\|\mathrm{Av}_G(T^{\pm})\|_{\mathrm{tr}_G} = \|T^{\pm}\|_{\mathrm{tr}}$. Next, the equality $\mathrm{Av}_G(T^+) = \mathrm{Av}_G(T^-) + \mathrm{Av}_G(T)$ holds in $L^1(\mathfrak{R}_G(\mathcal{E}), \mathrm{tr}_G)$, because $T \in \mathrm{Dom}(\mathrm{Av}_G)$, so that for any $u \in L^2(M, \mathcal{E})$,

$$\int_G \langle L(g)T^+L(g)^{-1}u, u\rangle\,dg = \int_G \langle L(g)T^-L(g)^{-1}u, u\rangle\,dg + \langle \mathrm{Av}_G(T)u, u\rangle.$$

Applying the triangle inequality in $L^1(\mathfrak{R}_G(\mathcal{E}), \mathrm{tr}_G)$, we get

$$\|\mathrm{Av}_G(T)\|_{\mathrm{tr}_G} \le \|T^+\|_{\mathrm{tr}} + \|T^-\|_{\mathrm{tr}} = \|T\|_{\mathrm{tr}}.$$

Finally, the equality (ii) follows from the linearity of tr_G as a functional on $L^1(\mathfrak{R}_G(\mathcal{E}), \mathrm{tr}_G)$. Q.e.d.

The link between the two averaging operations Av (see Section 1) and Av_G is given by:

2.2. LEMMA. *Let* $P \in \Psi_{cc}^0(\mathcal{E})$. *Then* $P \in \mathrm{Dom}(\mathrm{Av}_G)$ *and* $\mathrm{Av}_G(P) = \mathrm{Av}(P)$.

Proof. Let C be a compact in G, such that support $(K_p) \subset C \times C$. Choose a function $f \in C_c^\infty(G)$ with the following properties: $f(xh) = f(x)$, for $x \in G$, $h \in H$; $f(x) \geq 0$, $x \in G$; $f(x) = 1$ for $x \in C$. One has $P = f^{1/2}Pf^{1/2}$, and because $f^{1/2}I \in \mathrm{Dom}_{1/2}(\mathrm{Av}_G)$, it follows that $P \in \mathrm{Dom}(\mathrm{Av}_G)$. The equality in the statement is then obvious. Q.e.d.

2.3. COROLLARY. *One has* $\Psi_c^{-\infty}(M, \mathcal{E})^G \subset \mathrm{Dom}(\mathrm{tr}_G)$. *Moreover, for any* $\varphi \in (C_c^\infty(G) \otimes \mathrm{End}(E))^{H \times H}$, $\mathrm{tr}_G R(\varphi) = \mathrm{tr}\, \varphi(1)$.

Proof. Let $P \in \Psi_c^{-\infty}(M, \mathcal{E})^G$ and let f be a cut-off function. Then $fP \in \Psi_{cc}^{-\infty}(M, \mathcal{E}) \subset \mathrm{Dom}(\mathrm{Av}_G) \cap \mathrm{Dom}(\mathrm{tr})$, so that, by Lemma 2.1, $P = \mathrm{Av}(fP) \in \mathrm{Dom}(\mathrm{tr}_G)$ and

$$\mathrm{tr}_G(P) = \mathrm{tr}(fP) = \int_G f(x) K_P(x, x)\, dx = k_P(1). \qquad \text{Q.e.d.}$$

3. The analytical index

In this section we introduce the analytical index for G-invariant elliptic pseudo-differential operators and establish its elementary properties. We then use these properties to prove that it factors through the symbol map, and thus defines a real-valued map on the equivariant K-theory group of the isotropy representation.

The set of all elliptic operators in $\Psi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$ will be denoted $\Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$. Since we need to let them act on $L^2(M, \mathcal{E})$, we shall first prove:

3.1. LEMMA. *Let* $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$, *with* $n \geq 1$, *be regarded as an operator from* $L^2(M, \mathcal{E}_1)$ *to* $L^2(M, \mathcal{E}_2)$ *with (dense) domain* $C_c^\infty(M, \mathcal{E}_1)$. *Then, the domain of its closure coincides with the subspace of all* $u \in L^2(M, \mathcal{E}_1)$ *for which* $Pu \in L^2(M, \mathcal{E}_2)$ *in the distributional sense.*

Proof. One inclusion being obvious, we have only to prove that if $u \in L^2(M, \mathcal{E}_1)$ is such that $Pu \in L^2(M, \mathcal{E}_2)$, then (u, Pu) is in the closure of the graph of P. Using Proposition 1.3, we can assume, by the same argument as in [3, Proof of Proposition 3.1], that $u \in C^\infty(M, \mathcal{E}_1)$.

Let $\{K_j\}$ be an increasing sequence of compacta with $\cup K_j = G$, and let f be a cut-off function. Then

$$u_j = \int_{K_j} L(x) f L(x)^{-1} u\, dx \in C_c^\infty(M, \mathcal{E}_1)$$

and, because $\mathrm{Av}(fI)u = u$, $u_j \to u$ in the L^2-norm. To prove the convergence in the graph norm of P, we will show that $\{Pu_j\}$ is a Cauchy sequence. Note that, since P is G-invariant,

$$(3.1.1)\qquad Pu_{j+k} - Pu_j = \int_{K_{j+k}\setminus K_j} L(x)PfL(x)^{-1}u\,dx.$$

With χ_j denoting the characteristic function of K_j, let us put

$$F_{j,k}(x) = \big(\chi_{j+k}(x) - \chi_j(x)\big)L(x)PfL(x)^{-1}u.$$

We want to show that

$$(3.1.2)\qquad \int \|F_{j,k}(x)\|^2\,dx \to 0, \quad \text{when } j \to \infty.$$

Using the usual local estimate for the elliptic operator P, one has

$$\|P(fv)\| \le c(\|f'Pv\| + \|f'v\|), \qquad v \in C_c^\infty(M, \mathcal{E}_1),$$

where $f' \in C_c^\infty(G)$ and $f' \equiv 1$ on a neighbourhood of support (f), and c is a constant depending on P and f. So, (3.1.2) will follow from

$$(3.1.3)\qquad \int_{G\setminus K_j} \|f'(x)L(x)^{-1}v\|^2\,dx \to 0, \quad \text{when } j \to \infty.$$

Let us prove (3.1.3). Given $\delta > 0$, choose $w \in C_c^\infty(M, \mathcal{E}_1)$ such that $\|v - w\| < \delta$. Then

$$\int_{G\setminus K_j} \|f'(x)L(x)^{-1}w\|^2\,dx = \int_{G\setminus K_j} \langle L(x)f'L(x)^{-1}w, w\rangle\,dx = 0$$

for j large enough (precisely, when $C \cdot C^{-1} \subset K_j$, C being the support of w), and on the other hand

$$\int_G \|f'L(x)^{-1}(v - w)\|^2\,dx = \mathrm{Av}_G\big(|f'|^2 I\big)\|v - w\|^2 \le \delta^2 \|f'\|^2_{L^2(G)}.$$

This proves (3.1.3), and hence (3.1.2).

Lastly, by applying Lemma 1.5 to the function $F_{j,k}$, we get, in view of (3.1.1),

$$\|Pu_{j+k} - Pu_j\| \to 0 \quad \text{when } j \to \infty. \qquad \text{Q.e.d.}$$

With this result in mind, we can afford to keep the same notation P and P^* for the closure of these operators.

Given a closed, G-invariant subspace $\mathcal{K}$ of $L^2(M, \mathcal{E})$, we define its G-dimension by

$$\dim_G \mathcal{K} = \mathrm{tr}_G K,$$

where K is the orthogonal projection onto $\mathcal{K}$.

3.2. LEMMA. *Let* $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$. *Then* $\dim_G \operatorname{Ker} P < \infty$.

Proof. Let Q be a G-invariant parametrix for P, so that $S_1 = I - QP \in \Psi_c^{-\infty}(\mathcal{E}_1)^G$. For any $u \in \operatorname{Ker} P$ in $L^2(M, \mathcal{E}_1)$ one has $S_1 u = u$. As, by Corollary 2.3, S_1 is of G-trace class, it follows immediately that $\dim_G \operatorname{Ker} P < \infty$. Q.e.d.

In view of this lemma, we can define the analytical index of $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$ as being the real number

$$\operatorname{ind}_G P = \dim_G \operatorname{Ker} P - \dim_G \operatorname{Ker} P^*.$$

3.3. LEMMA. *Let* $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$ *and let* $Q \in \Phi_c^{-n}(\mathcal{E}_2, \mathcal{E}_1)^G$ *be a parametrix for* P. *Then*

$$\operatorname{ind}_G P = \operatorname{tr}_G S_1 - \operatorname{tr}_G S_2,$$

where $S_1 = I - QP \in \Psi_c^{-\infty}(\mathcal{E}_1)$, $S_2 = I - PQ \in \Psi_c^{-\infty}(\mathcal{E}_2)$.

The proof is formally the same as in [3, § 5], so we shall omit it.

The basic properties of the G-index are summarized in the following statement.

3.4. PROPOSITION. (i) *If* $P_1 \in \Phi_c^p(\mathcal{E}_1, \mathcal{E}_2)^G$, $P_2 \in \Phi_c^q(\mathcal{E}_2, \mathcal{E}_3)^G$ *then* $\operatorname{ind}_G P_2 P_1 = \operatorname{ind}_G P_1 + \operatorname{ind}_G P_2$.

(ii) ind_G: $\Phi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G \to \mathbf{R}$ *is locally constant with respect to the norm topology.*

(iii) *If* $P \in \Phi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$ *and* $S \in \Psi_c^{-1}(\mathcal{E}_1, \mathcal{E}_2)^G$ *then* $\operatorname{ind}_G(P + S) = \operatorname{ind}_G P$.

Proof. (i) Let $Q_1 \in \Phi_c^p(\mathcal{E}_2, \mathcal{E}_1)^G$, $Q_2 \in \Phi_c^q(\mathcal{E}_3, \mathcal{E}_2)$ be parametrices for P_1, P_2 respectively. Precisely,

$$I - Q_1P_1 = S_1^{(1)} \in \Psi_c^{-\infty}(\mathcal{E}_1)^G, \qquad I - P_1Q_1 = S_1^{(2)} \in \Psi_c^{-\infty}(\mathcal{E}_2)^G,$$
$$I - Q_2P_2 = S_2^{(1)} \in \Psi_c^{-\infty}(\mathcal{E}_2)^G, \qquad I - P_2Q_2 = S_2^{(2)} \in \Psi_c^{-\infty}(\mathcal{E}_3)^G.$$

Then

$$I - Q_1Q_2P_2P_1 = S^{(1)}, \quad \text{with } S^{(1)} = S_1^{(1)} + Q_1S_2^{(1)}P_1$$

and

$$I - P_2P_1Q_1Q_2 = S^{(2)}, \quad \text{with } S^{(2)} = S_2^{(2)} + P_2S_1^{(2)}Q_2.$$

Since

$$\operatorname{tr}_G Q_1S_2^{(1)}P_1 = \operatorname{tr}_G S_2^{(1)}P_1Q_1 = \operatorname{tr}_G S_2^{(1)} - \operatorname{tr}_G S_2^{(1)}S_1^{(2)}$$

and

$$\operatorname{tr}_G P_2S_1^{(2)}Q_2 = \operatorname{tr}_G S_1^{(2)}Q_2P_2 = \operatorname{tr}_G S_1^{(2)} - \operatorname{tr}_G S_1^{(2)}S_2^{(1)},$$

one has:

$$\begin{aligned}\operatorname{ind}_G P_2P_1 &= \operatorname{tr}_G S^{(1)} - \operatorname{tr}_G S^{(2)} = \operatorname{tr}_G S_1^{(1)} + \operatorname{tr}_G S_2^{(1)} - \operatorname{tr}_G S_2^{(2)} - \operatorname{tr}_G S_1^{(2)} \\ &= \operatorname{ind}_G P_1 + \operatorname{ind}_G P_2.\end{aligned}$$

(ii) Fix $P \in \Psi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$ and then a parametrix $Q \in \Phi_c^0(\mathcal{E}_2, \mathcal{E}_1)^G$. Let us first prove a particular case of (iii):

$$\operatorname{ind}_G(P + S) = \operatorname{ind}_G P, \quad \text{for any } S \in \Psi_c^{-\infty}(\mathcal{E}_1, \mathcal{E}_2)^G. \tag{3.4.1}$$

Indeed, since Q is a parametrix for $P + S$ too, one has

$$\begin{aligned}\operatorname{ind}_G(P+S) &= \operatorname{tr}_G(I - QP - QS) - \operatorname{tr}_G(I - PQ - SQ) \\ &= \operatorname{tr}_G(I - QP) - \operatorname{tr}_G(I - PQ) - \operatorname{tr}_G QS + \operatorname{tr}_G SQ = \operatorname{ind}_G P.\end{aligned}$$

We will now show that any $P' \in \Phi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$ such that

$$\|P' - P\| < \|Q\|^{-1}$$

has the same G-index as P. Indeed, let $Q' \in \Phi_c^0(\mathcal{E}_2, \mathcal{E}_1)^G$ be a parametrix for P' and put:

$$S_1 = I - QP, \qquad S_2 = I - PQ, \qquad S_1' = I - Q'P', \qquad S'_2 = I - P'Q'.$$

Notice that, since $\|QP - QP'\| < 1$,

$$S_1 + QP' = I - (QP - QP')$$

is invertible and thus, by the definition of the G-index, $\operatorname{ind}_G(S_1 + QP') = 0$. In view of (i) and (3.4.1),

$$\operatorname{ind}_G Q' = \operatorname{ind}_G(S_1 + QP')Q' = \operatorname{ind}_G P'Q' = \operatorname{ind}_G Q(I - S_2') = \operatorname{ind}_G Q.$$

But, clearly,

$$\operatorname{ind}_G P' = -\operatorname{ind}_G Q' \quad \text{and} \quad \operatorname{ind}_G P = -\operatorname{ind}_G Q,$$

which concludes the proof of (ii).

Finally, (iii) follows from (ii) and the fact that the map $t \mapsto P + tS$ from $[0,1]$ to $\Phi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$ is norm continuous. Q.e.d.

In analogy with the classical case, we proceed now to define the index map at the K-theory level. Recall that $V = \{\xi \in \mathfrak{g}^*;\ \xi \mid \mathfrak{h} = 0\}$. By $K_H(V)$ we denote, as usual, the abelian group associated to the locally compact H-space V in the equivariant K-theory (with compact supports). Since V happens to be a vector space on which H acts linearly, one way to describe $K_H(V)$, convenient for our purposes, is the following. The basic objects to start with are smooth H-equivariant maps α: $S \to \operatorname{Iso}(E_1, E_2)$, where $S = S(V)$ is the unit sphere in V with respect to an $\mathrm{Ad}^*(H)$-invariant metric, and E_1, E_2 are finite-dimensional unitary

H-modules. Two such maps

$$\alpha_0 \in \left(C^\infty(S, \mathrm{Iso}(E_0, F_0))\right)^H, \qquad \alpha_1 \in \left(C^\infty(S, \mathrm{Iso}(E_1, F_1))\right)^H$$

are called isomorphic, and we shall write $\alpha_0 \cong \alpha_1$, if there exist $\varphi \in \mathrm{Iso}_H(E_0, E_1)$, $\psi \in \mathrm{Iso}_H(F_0, F_1)$, where Iso_H denotes the H-equivariant isomorphisms, such that

$$\psi\alpha_0(\xi) = \alpha_1(\xi)\varphi, \quad \text{for any } \xi \in S.$$

Further, α_0 and α_1 are said to be homotopic if there is an $\alpha \in (C^\infty(S \times I, \mathrm{Iso}(E, F))^H$, where $I = [0,1]$ with the trivial action of H, such that $\alpha \mid S \times \{0\} \cong \alpha_0$ and $\alpha \mid S \times \{1\} \cong \alpha_1$. The set of all homotopy classes of such maps will be denoted $\mathcal{C}$. This is an abelian semigroup, under the obvious direct sum operation. Let $\mathcal{C}_0$ denote the subsemigroup of all classes which can be represented by a constant map α, $\alpha(\xi) = \varphi$, $\xi \in S$, with $\varphi \in \mathrm{Iso}_H(E_1, E_2)$. Then $\mathcal{C}/\mathcal{C}_0$ is not only a semigroup but actually a group, which is isomorphic to $K_H(V)$.

We are now in a position to formulate the main result of this section.

3.5. Proposition. *There exists a unique homomorphism of abelian groups* $\mathrm{ind}_a\colon K_H(V) \to \mathbf{R}$ *such that, for any* $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$,

$$\mathrm{ind}_a\left[\sigma_0^n(P)\right] = \mathrm{ind}_G P,$$

where $[\sigma_0^n(P)]$ *denotes the class in* $K_H(V)$ *defined by* $\sigma_0^n(P)$.

Proof. Let $\alpha \in (C^\infty(S, \mathrm{Iso}(E_1, E_2))^H$. As shown in Section 1 (during the proof of Proposition 1.7), there always exists $P_\alpha \in \Phi_c^0(\mathcal{E}_1, \mathcal{E}_2)^G$ such that $\alpha = \sigma_0(P)$. The problem is to prove that $\mathrm{ind}_G P_\alpha$ depends only on the K-theory class $[\alpha]$ of α.

1°. Assume that $\alpha_i \in (C^\infty(S, \mathrm{Iso}(E_i, F_i))^H$, $i = 0, 1$, are isomorphic, via the isomorphisms $\varphi \in \mathrm{Iso}_H(E_0, E_1)$, $\psi \in \mathrm{Iso}_H(F_0, F_1)$, and choose $P_i = P_{\alpha_i} \in \Phi_c^0(\mathcal{E}_i, \mathcal{F}_i)^G$ such that $\sigma_0(P_i) = \alpha_i$. Let $\tilde{\varphi}\colon \mathcal{E}_0 \to \mathcal{E}_1$, $\tilde{\psi}\colon \mathcal{F}_0 \to \mathcal{F}_1$ be the isomorphisms of homogeneous bundles induced by φ, ψ respectively. Then $\tilde{P}_1 = \tilde{\psi}^{-1}P_1\tilde{\varphi}$ has the same principal symbol as P_0, hence $\tilde{P}_1 - P_0 \in \Psi_c^{-1}(\mathcal{E}_0, \mathcal{F}_0)^G$. By Proposition 3.4, $\mathrm{ind}_G P_0 = \mathrm{ind}_G \tilde{P}_1$. On the other hand, it is clear from the definition of the G-index that $\mathrm{ind}_G \tilde{P}_1 = \mathrm{ind}_G P_1$.

2°. In order to check that $\mathrm{ind}_G P_\alpha$ depends only on the homotopy class of α in $\mathcal{C}$, we have to show that if $\alpha_0, \alpha_1 \in (C^\infty(S, \mathrm{Iso}(E, F))^H$ are sufficiently close in the sup-norm topology, then $\mathrm{ind}_G P_0 = \mathrm{ind}_G P_1$. So, let us suppose that $\|\alpha_0 - \alpha_1\| < \delta$. By Corollary 1.8, there exists $S \in \Psi_c^{-\infty}(\mathcal{E}, \mathcal{F})^G$ such that $\|P_0 - P_1 + S\| < 2\delta$. In view of Proposition 3.4, it follows then that, when δ is small, $\mathrm{ind}_G P_0 = \mathrm{ind}_G P_1$.

3°. Finally, we must check that $\mathrm{ind}_G P_\alpha = 0$ when α is a constant map, i.e. $\alpha(\xi) = \varphi \in \mathrm{Iso}_H(E_1, E_2)$, $\xi \in S$. But this is obvious, since we can take P_α:

$C_c^\infty(M, \mathcal{E}_1) \to C_c^\infty(M, \mathcal{E}_2)$ as being given by the formula

$$(P_\alpha u)(x) = \varphi(u(x)).$$

Summing up, we get a well-defined map ind_a: $K_H(V) \to \mathbf{R}$, which is clearly a group homomorphism. The last fact we have to prove is that, for any $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$ with $n \in \mathbf{Z}$,

$$\operatorname{ind}_G P = \operatorname{ind}_a[\sigma_0^n(P)]. \tag{3.5.1}$$

Choose $R \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_1)^G$ such that $\sigma_0^n(R) = I$; by replacing R with $\frac{1}{2}(R + R^*)$ we can assume that R is self-adjoint. With $Q \in \Phi_c^{-n}(\mathcal{E}_2, \mathcal{E}_1)^G$ being a parametrix for P, set

$$P' = RQ \in \Phi_c^0(\mathcal{E}_2, \mathcal{E}_1)^G.$$

Then, by Proposition 3.4(i), as $R = R^*$ and thus $\operatorname{ind}_G R = 0$,

$$\operatorname{ind}_G P' = \operatorname{ind}_G Q = -\operatorname{ind}_G P; \tag{3.5.2}$$

on the other hand

$$\sigma_0^0(P')\sigma_0^n(P) \equiv I,$$

so that, again by Proposition 3.4(i), and by the very construction of ind_a: $K_H(V) \to \mathbf{R}$, one has

$$\operatorname{ind}_a[\sigma_0^0(P')] = -\operatorname{ind}_a[\sigma_0^n(P)]. \tag{3.5.3}$$

Clearly (3.5.1) follows now from (3.5.2), (3.5.3) and our definition of the index for a symbol of class.

3.6. Corollary. *Assume that $M = G/H$ is odd-dimensional and H is connected. Then, for any $P \in \Phi_c^n(\mathcal{E}_1, \mathcal{E}_2)^G$,* $\operatorname{ind}_G P = 0$.

Proof. Let i: $T \to H$ be the inclusion of a maximal torus. It is known that the induced map i^*: $K_H(V) \to K_T(V)$ is injective. Since V has odd dimension, $K_T(V) = 0$ by the periodicity theorem. Thus, $K_H(V) = 0$ and the result follows from Proposition 3.5.

4. The topological index

The purpose of this section is to define the "topological" index of an element in $K_H(V)$. Due to Corollary 3.6, only the case when V has even dimension is meaningful, at least when H is connected. So, we shall assume that $M = G/H$ is even dimensional. The connectedness assumption can be slightly relaxed by requiring only that H preserve the orientation of V. We shall do so and fix an H-invariant volume element $\omega \in \Lambda^m V$, which determines the orientation of V.

There is a very natural way to define the topological index when one has a Thom isomorphism τ: $R(H) \to K_H(V)$, where $R(H)$ denotes the representation ring of H. For instance, if the co-isotropy representation Ad^*: $H \to \mathrm{SO}(V)$ lifts to $\mathrm{Spin}(V)$ (in other words if M admits a G-invariant spin-structure), one can exhibit explicitly the Bott class $\beta \in K_H(V)$, which freely generates $K_H(V)$ as an $R(H)$-module; indeed, β is just the symbol class of the Dirac operator. The general case will be dealt with by passing to a suitable double covering of H and then reducing the problem to the previous situation.

So, let us first suppose that Ad^*: $H \to \mathrm{SO}(V)$ lifts to $\mathrm{Spin}(V)$. Then, the half-spin representations $S^{\pm}$ can be regarded as H-modules, and we can form the "Dirac complex" over V. Indeed, let $\mathbf{S}^{\pm} = V \times S^{\pm}$ be the corresponding trivial H-bundles, and consider the homomorphism c: $\mathbf{S}^+ \to \mathbf{S}^-$ given by $c(\xi, \mu) = (\xi, c(\xi)\mu)$, where $c(\xi)$ denotes the Clifford multiplication by $\xi \in V$. Then

$$0 \to \mathbf{S}^+ \xrightarrow{c} \mathbf{S}^- \to 0$$

is a complex with compact support over V, so it defines an element $\alpha \in K_H(V)$.

4.1. LEMMA. *The Bott class $\beta \in K_H(V)$ is given by $\beta = (-1)^n\alpha$, so that the Thom isomorphism τ: $R(H) \to K_H(V)$ has the expression*:

$$\tau(a) = a\beta = (-1)^n a\alpha, \qquad a \in R(H).$$

This is certainly well-known. The case $m \equiv 0 \pmod 8$ is explicitly treated in [2, § 6], and the arguments given there work actually for m even too.

Let us now take up the general case, when M does not necessarily admit a G-invariant spin-structure. We construct a double covering $\tilde{H}$ of H as follows: $\tilde{H}$ is the subgroup of $H \times \mathrm{Spin}(V)$ consisting of all elements (h, s) such that $\mathrm{Ad}^*(h)$ coincides with the orthogonal transformation of V defined by $s \in \mathrm{Spin}(V)$. The group $\tilde{H}$ comes up with two natural representations: $(h, s) \in \tilde{H} \mapsto \mathrm{Ad}^*(h) \in \mathrm{SO}(V)$ which gives V the structure of an $\tilde{H}$-space and $(h, s) \in \tilde{H} \mapsto s \in \mathrm{Spin}(V)$ which shows that the Dirac complex over V is an $\tilde{H}$-complex. If we denote by $\tilde{\alpha}$ the class in $K_{\tilde{H}}(V)$ defined by the Dirac complex, in view of Lemma 4.1, the Thom isomorphism $\tilde{\tau}$: $R(\tilde{H}) \to K_{\tilde{H}}(V)$ is given by

$$\tilde{\tau}(a) = a\tilde{\beta}, \qquad a \in R(\tilde{H}),$$

where $\tilde{\beta} = (-1)^n\tilde{\alpha}$ is the Bott class in $K_{\tilde{H}}(V)$.

Let $u \in \mathrm{Spin}(V)$ denote the generator of the kernel of the covering homomorphism $\mathrm{Spin}(V) \to \mathrm{SO}(V)$. Then $u = (1, u)$ is central in $\tilde{H}$ so that, if ε is an irreducible representation of $\tilde{H}$, $\varepsilon(u) = \pm I$. Clearly

$$R(\tilde{H}) = R(\tilde{H})^0 \oplus R(\tilde{H})^1,$$

where $R(\tilde{H})^i$ is generated by the equivalence classes of irreducible representations ε of $\tilde{H}$ such that $\varepsilon(u) = (-1)^i I$; in particular $R(\tilde{H})^0$ can be identified with $R(H)$.

Similarly, let us define $K_{\tilde{H}}(V)^i$ to be the subgroup of all classes in $K_{\tilde{H}}(V)$ represented by $\tilde{H}$-complexes with compact support on which u acts by $(-1)^i I$, $i = 0, 1$.

4.2. LEMMA. *$K_{\tilde{H}}(V) = K_{\tilde{H}}(V)^0 \oplus K_{\tilde{H}}(V)^1$ and, with respect to this decomposition, $K_{\tilde{H}}(V)$ is a $\mathbf{Z}_2$-graded module over the $\mathbf{Z}_2$-graded ring $R(\tilde{H}) = R(\tilde{H})^0 \oplus R(\tilde{H})^1$. In addition, $K_{\tilde{H}}(V)^0$ is canonically isomorphic to $K_H(V)$.*

Proof. Let W be an $\tilde{H}$-bundle over V. Since u leaves each fibre invariant and since $u^2 = 1$, it is easily seen that $W = W^0 \oplus W^1$, with $W^i = \{w \in W; uw = (-1)^i w\}$, $i = 0, 1$. The same argument shows that any $\tilde{H}$-complex over V splits into two complexes on which u acts by $\pm I$ respectively. With this understood, the proof is just a matter of routine.

Combining the two lemmas we get:

4.3. PROPOSITION. *Every class $k \in K_H(V) \cong K_{\tilde{H}}(V)^0$ can be written in a unique way under the form $k = a\tilde{\beta}$, with $a \in R(\tilde{H})^1$.*

The next ingredient we need, in order to define the topological index, is the Chern character ch: $R(\tilde{H}) \to H^*(\mathfrak{g}, H, \mathbf{R})$. Here $H^*(\mathfrak{g}, H, \mathbf{R})$ denotes the relative Lie algebra cohomology with trivial coefficients, i.e. the cohomology of the complex $(C(\mathfrak{g}, H, \mathbf{R}), d)$, where

$$C^q(\mathfrak{g}, H, \mathbf{R}) = \{\alpha \in \Lambda^q \mathfrak{g}^*; \iota_X \alpha = 0 \text{ for } X \in \mathfrak{h}, \operatorname{Ad}^*(h)\alpha = \alpha \text{ for } h \in H\}$$

and d: $C^q(\mathfrak{g}, H, \mathbf{R}) \to C^{q+1}(\mathfrak{g}, H, \mathbf{R})$ is given by

$$d\alpha(x_1, \ldots, x_{q+1}) = \frac{1}{q+1} \sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \alpha([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}).$$

Let us fix for the moment an $\operatorname{Ad}(H)$-invariant splitting of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. This specifies a G-invariant connection on the principal bundle $H \to G \to M$, whose connection form is given by the projection θ: $\mathfrak{g} \to \mathfrak{h}$ parallel to $\mathfrak{m}$, and whose curvature form is prescribed by

$$\Theta(X, Y) = -\tfrac{1}{2}\theta([X, Y]), \qquad X, Y \in \mathfrak{m}.$$

Now let ε be a unitary representation of $\tilde{H}$ on E. We denote by the same letter its differential, ε: $\mathfrak{h} \to \mathfrak{gl}(E)$, and we then define $\Theta_\varepsilon \in \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{gl}_{\mathbf{C}}(E)$ by the formula

$$\Theta_\varepsilon(X, Y) = \frac{1}{2i\pi}\varepsilon(\Theta(X, Y)), \qquad X, Y \in \mathfrak{m}.$$

Further, let us consider the form $\operatorname{tr}\exp\Theta_\varepsilon \in \Lambda\mathfrak{m}^* \otimes \mathbf{C}$. Actually, because ε is unitary, $\operatorname{tr}\exp\Theta_\varepsilon \in \Lambda\mathfrak{m}^*$. In addition, since for $\tilde{h} \in \tilde{H}$ with image $h \in H$ one has

$$\Theta_\varepsilon(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y) = \varepsilon(\tilde{h})\Theta_\varepsilon(X, Y)\varepsilon(\tilde{h})^{-1}, \qquad X, Y \in \mathfrak{m},$$

one can see that $\operatorname{tr}\exp\Theta_\varepsilon$ is H-invariant. This shows that its pullback to $\Lambda\mathfrak{g}$, via the projection $I - \theta$: $\mathfrak{g} \to \mathfrak{m}$, defines a cochain in $\sum_q^\oplus C^q(\mathfrak{g}, H, \mathbf{R})$, which we will continue to denote by the same symbol. Standard arguments in the Chern-Weil approach to characteristic classes imply first that $\operatorname{tr}\exp\Theta_\varepsilon$ is closed, and next that the cohomology class in $H^*(\mathfrak{g}, H, \mathbf{R})$ it defines, and which will be denoted $\operatorname{ch}\varepsilon$, does not depend on the choice of the $\mathrm{Ad}(H)$-invariant splitting of $\mathfrak{g}$. Finally, it is clear that $\operatorname{ch}\varepsilon_1 = \operatorname{ch}\varepsilon_2$ if ε_1 and ε_2 are equivalent unitary representations, so that we can define unambiguously ch: $R(\tilde{H}) \to H^*(\mathfrak{g}, H, \mathbf{R})$.

The last ingredient we need is the analogue of the $\hat{\mathfrak{A}}$-polynomial of Hirzebruch. To define it, we start from the (real) H-module V (which can be also viewed as an $\tilde{H}$-module) and form, as above, $\Theta_V \in \Lambda^2\mathfrak{m}^* \otimes \mathfrak{gl}_{\mathbf{C}}(V)$. Then we construct the element in $\Lambda\mathfrak{m}^*$

$$\det\frac{\Theta_V}{\exp(\frac{1}{2}\Theta_V) - \exp(-\frac{1}{2}\Theta_V)},$$

pull it back to an element in $\Lambda\mathfrak{g}^*$ and, after noting that it is in fact a cocycle, we define $\hat{\mathfrak{A}}(\mathfrak{g}, H) \in H^*(\mathfrak{g}, H, \mathbf{R})$ as being its cohomology class.

Remark now that $\dim H^m(\mathfrak{g}, H, \mathbf{R}) = 1$, since G and H are unimodular; in fact, $H^m(\mathfrak{g}, H, \mathbf{R}) = C^m(\mathfrak{g}, H, \mathbf{R}) \cong \Lambda^m V$. If $\Omega = \sum\Omega^{(q)} \in H^*(\mathfrak{g}, H, \mathbf{R})$, we define the scalar $\Omega[V]$ by the relation $\Omega^{(m)} = (\Omega[V])\omega$.

With these preparations, we are finally able to define the topological index ind_t: $K_H(V) \to \mathbf{R}$, as follows. Let $k \in K_H(V)$; in view of Proposition 4.3 there is a unique class $a \in R(\tilde{H})^1$ such that $k = a\beta$, and we define

$$\mathrm{ind}_t k = \left(\operatorname{ch}(a)\hat{\mathfrak{A}}(\mathfrak{g}, H)\right)[V],$$

Note that the definition depends, up to a multiplicative constant, on the choice of a G-invariant volume element on M, which gives the orientation and the evaluation map $\Omega \mapsto \Omega[V]$, $\Omega \in H^*(\mathfrak{g}, H, \mathbf{R})$.

5. The index theorem

Before taking up the proof of the index theorem, let us recall our basic assumptions: G is a separable unimodular Lie group, H is a compact subgroup,

$M = G/H$ has even dimension and it is G-invariantly orientable. Further, in defining both the analytical and the topological index, we had to fix a scale: the Haar measure dg on G for the analytical index, and a G-invariant volume element on $M = G/H$ for the topological index; we shall assume that they are chosen in a coherent way, i.e. the quotient measure dg/dh (where dh is the normalized Haar measure on H) is given precisely by the volume element chosen on M.

As a first step, we shall prove the index formula for the signature operator with twisted coefficients. So, let us fix a G-invariant Riemannian metric on M and denote by $\Lambda T^*_{\mathbf{C}}M = \Lambda^+ T^*_{\mathbf{C}}M \oplus \Lambda^- T^*_{\mathbf{C}}M$ the bundle decomposition of the total exterior bundle of the cotangent bundle corresponding to the canonical involution associated to the metric (for details see [4, §5] or [6, p. 575]). Then $D = d + d^*$, where d is the exterior differentiation on forms, splits into two operators, $D = D^+ \oplus D^-$, with

$$D^{\pm} : C_c^{\infty}(M, \Lambda^{\pm} T^*_{\mathbf{C}}M) \to C_c^{\infty}(M, \Lambda^{\mp} T^*_{\mathbf{C}}M),$$

elliptic, G-invariant and formal adjoints of each other. Now if ε is a finite-dimensional unitary representation of H and $\mathcal{E}$ denotes the induced homogeneous vector bundle over M, we set $\mathcal{E}^{\pm} = \mathcal{E} \otimes \Lambda^{\pm} T^*_{\mathbf{C}}M$ and then, using a G-invariant connection on $\mathcal{E}$, we form as in [4, §6] the $(\pm)$-signature operators with coefficients in $\mathcal{E}$, $D_{\varepsilon}^{\pm}$: $C_c^{\infty}(M, \mathcal{E}^{\pm}) \to C_c^{\infty}(M, \mathcal{E}^{\mp})$.

With the same conventions as in Section 4, let us define $\mathfrak{L}(\mathfrak{g}, H) \in H^*(\mathfrak{g}, H, \mathbf{R})$ as being the cohomology class determined by

$$\det \frac{\frac{1}{2}\Theta_V}{\tanh\left(\frac{1}{2}\Theta_V\right)} \in \Lambda \mathfrak{m}^*,$$

This being agreed, we can now state:

5.1. THEOREM. $\operatorname{ind}_G D_{\varepsilon}^+ = 2^n(\operatorname{ch}(\varepsilon)\mathfrak{L}(\mathfrak{g}, H))[V]$.

Proof. Let $\Delta_{\varepsilon}^{\pm} = D_{\varepsilon}^{\mp} D_{\varepsilon}^{\pm}$. It is an (unbounded) self-adjoint operator on $L^2(M, \mathcal{E}^{\pm})$, so we can form the bounded operator $e^{-t\Delta_{\varepsilon}^{\pm}}$, for any $t > 0$. We claim that $e^{-t\Delta_{\varepsilon}^{\pm}}$ is of G-trace class and that its G-trace has an asymptotic expansion of the form

$$\operatorname{tr}_G e^{-t\Delta_{\varepsilon}^{\pm}} \sim \sum_{k \geq -m} c_k\left(\Delta_{\varepsilon}^{\pm}\right) t^{k/2} \quad \text{as } t \to 0, \tag{5.1.1}$$

whose coefficients can be computed from the local expression of $\Delta_{\varepsilon}^{\pm}$ in any coordinate chart.

Let us fix, once and for all, a relatively compact open subset U in M, which is also a coordinate chart. Further, let $\mathcal{H}_s(U, \mathcal{E}^{\pm})$ be the "local" Sobolev spaces, obtained by completing $C_c^\infty(U, \mathcal{E}^{\pm})$ with respect to the usual s-norm ($s \in \mathbf{Z}^+$), which we will denote by $\| \ \|_{s,U}$. The norm of a bounded linear operator B: $L^2(U, \mathcal{E}^{\pm}) \to \mathcal{H}_s(U, \mathcal{E}^{\pm})$ will be denoted $\|B\|_{s,U}$.

To prove (5.1.1), we start as in [12, Ch. 3] by inverting locally the analytic family of elliptic operators $\Delta_\varepsilon^+ - \lambda I$, $\lambda \in \mathbf{C} \setminus \mathbf{R}^+$. So, let f be a cut-off function whose support (modulo H) is contained in U. As in [12, loc. cit.], one can construct an analytic family of pseudo-differential operators $P_\lambda \in \Psi_{cc}^{-2}(U, \mathcal{E}^+)$ such that

$$(5.1.2) \qquad fP_\lambda = P_\lambda \quad \text{and} \quad R_\lambda = P_\lambda(\Delta_\varepsilon^+ - \lambda I) - f \in \Psi_{cc}^{-\infty}(U, \mathcal{E}^+).$$

Moreover, according to [12, Lemma, p. 59], given $s \in \mathbf{Z}^+$, there exists a number $c_k > 0$, independent of λ, such that

$$(5.1.3) \qquad \|R_\lambda\|_{s,U} \leq c_k(1 + |\lambda|)^{-k-1}.$$

Let us now observe that the operators R_λ: $L^2(U, \mathcal{E}^+) \to L^2(U, \mathcal{E}^+)$ are trace-class and, assuming $s > m$,

$$(5.1.4) \qquad \|R_\lambda\|_{\mathrm{tr}} \leq C\|R_\lambda\|_{s,U}$$

for some $C > 0$, independent of λ. Indeed, since $s > m$, the inclusion operator I_s: $L^2(U, \mathcal{E}^+) \to \mathcal{H}_s(U, \mathcal{E}^+)$ is nuclear; on the other hand, $R_\lambda = I_s R_\lambda$ so that

$$\|R_\lambda\|_{\mathrm{tr}} \leq \|I_s\|_{\mathrm{tr}} \|R_\lambda\|_{s,U}.$$

The operator R_λ (resp. P_λ) can be extended, by zero outside U, to a bounded operator on $L^2(M, \mathcal{E}^+)$ which we will continue to denote R_λ (resp. P_λ). Evidently, (5.1.4) remains valid when, in the left hand side of the equality, R_λ is viewed as an operator on the whole $L^2(M, \mathcal{E}^+)$.

From now on we will assume that $\mathrm{dist}(\lambda, \mathbf{R}^+) \geq 1$. This clearly implies that $\|(\Delta_\varepsilon^+ - \lambda I)^{-1}\| \leq 1$, therefore, in view of (5.1.2), (5.1.3) and (5.1.4), one has

$$(5.1.5) \qquad \|P_\lambda - f(\Delta_\varepsilon^+ - \lambda I)^{-1}\|_{\mathrm{tr}} = 0\big((1 + |\lambda|)^{-k-1}\big).$$

Let $\Gamma = \{\lambda \in \mathbf{C};\ \mathrm{dist}(\lambda, \mathbf{R}^+) = 1\}$, clock-wise oriented, and consider the operator

$$E(t) = \frac{1}{2i\pi} \int_\Gamma e^{-t\lambda} P_\lambda \, d\lambda.$$

Assuming $0 < t < 1$ and using the analyticity in λ, one has

$$\begin{aligned} E(t) - fe^{-t\Delta_\varepsilon^+} &= \frac{1}{2i\pi} \int_\Gamma e^{-\lambda t}\big(P_\lambda - f(\Delta_\varepsilon^+ - \lambda I)^{-1}\big)\, d\lambda \\ &= \frac{1}{2i\pi} \int_\Gamma e^{-\lambda}\left(P_{\lambda/t} - f\left(\Delta_\varepsilon^+ - \frac{\lambda}{t} I\right)^{-1}\right) \frac{d\lambda}{t}; \end{aligned}$$

by (5.1.5), we then get

$$(5.1.6)\quad \|E(t) - fe^{-t\Delta_\varepsilon^+}\|_{\mathrm{tr}} \le \frac{1}{2\pi}\int_\Gamma |e^{-\lambda}|\left(1+\frac{|\lambda|}{t}\right)^{-k-1}\frac{|d\lambda|}{t} = 0(t^k).$$

As proved in [12, Ch. 3], $E(t)$ is a smoothing operator, so that $E(t) \in \Psi_{cc}^{-\infty}(M, \mathcal{E}^+)$. In particular it is a trace-class operator. Furthermore, in view of Lemma 2.2, $E(t) \in \mathrm{Dom}(\mathrm{Av}_G)$. Now from (5.1.6) we can see that $fe^{-t\Delta_\varepsilon^+}$ is also of trace-class. On the other hand, because $\mathrm{Av}_G(fI) = I$, it follows that $fe^{-\Delta_\varepsilon^+} \in \mathrm{Dom}(\mathrm{Av}_G)$ and $\mathrm{Av}_G(fe^{-t\Delta_\varepsilon^+}) = e^{-t\Delta_\varepsilon^+}$. Thus, $E(t)$ and $fe^{-t\Delta_\varepsilon^+}$ are both in $\mathrm{Dom}(\mathrm{Av}_G) \cap \mathrm{Dom}(\mathrm{tr})$. By Lemma 2.1, $e^{-t\Delta_\varepsilon^+}$ and $\mathrm{Av}_G(E(t))$ are then in $\mathrm{Dom}(\mathrm{tr}_G)$, and moreover

$$|\mathrm{tr}_G e^{-t\Delta_\varepsilon^+} - \mathrm{tr}\,E(t)| \le \|e^{-t\Delta_\varepsilon^+} - \mathrm{Av}_G E(t)\|_{\mathrm{tr}_G} \le 2\|fe^{-t\Delta_\varepsilon^+} - E(t)\|_{\mathrm{tr}}.$$

From (5.1.6) it follows that

$$(5.1.7)\qquad \mathrm{tr}_G e^{-t\Delta_\varepsilon^+} - \mathrm{tr}\,E(t) = 0(t^k), \quad \text{for any } k \in \mathbf{Z}^+.$$

At this stage we use the remarkable fact, proved in the classical theory (cf. [4, §4] or [12, Theorem, p. 57]), that $\mathrm{tr}\,E(t)$ has an asymptotic expansion of the form

$$\mathrm{tr}\,E(t) \sim \sum_{k \ge -m} t^{k/2}\int f\mu_k^+(U),$$

where the local measures $\mu_k^+(U)$ are local invariants of Δ_ε^+. As a first consequence it follows that, Δ_ε^+ being G-invariant, each $\mu_k^+(U)$ is the restriction to U of an invariant measure μ_k^+ on M. In particular one can see that $\int f\mu_k^+(U) = \int f\mu_k^+$ does not depend on the cut-off function f. (Recall that any cut-off function f satisfies, by definition, the condition $\int f(g)\,dg = 1$.) Together with (5.1.7) this proves the assertion (5.1.1) with the additional piece of information that the coefficients $c_k(\Delta_\varepsilon^\pm)$ are given by

$$(5.1.8)\qquad c_k(\Delta_\varepsilon^\pm) = \int f\mu_k^\pm(U).$$

A further consequence of the fact that $\mu_k^\pm(U)$ are purely local invariants is that one can explicitly compute the difference $\mu_0^+(U) - \mu_0^-(U)$. Indeed, as follows from the local version of the "generalized signature theorem" (cf. [4, §§ 5–6]), $\Omega(U) = \mu_0^+(U) - \mu_0^-(U)$ is an m-form, which can be expressed as

$$\Omega(U) = 2^n\left(\mathrm{tr}\exp\Theta_\varepsilon(U)\cdot\det\frac{\frac{1}{2}\Theta_V(U)}{\tanh\left(\frac{1}{2}\Theta_V(U)\right)}\right)^{(m)}$$

where the superscript (m) indicates the top-degree component of the form under brackets, and $\Theta_\varepsilon(U), \Theta_V(U)$ are the restrictions to U of the global G-invariant differential forms $\Theta_\varepsilon(M), \Theta_V(M)$ determined by the corresponding cocycles in $\Sigma_q^{\oplus} C^q(\mathfrak{g}, H, \mathbf{R})$ defined in Section 4. It follows that $\Omega(U)$ is the restriction to U of a global G-invariant form $\Omega(M)$, and also that

$$\Omega(M) = 2^n(\mathrm{ch}(\varepsilon)\mathfrak{L}(\mathfrak{g}, H))[V] \cdot \omega(M)$$

where $\omega(M)$ is the G-invariant volume form on M determined by $\omega \in \Lambda^m V$.

So, the constant term in the asymptotic expansion of $\mathrm{tr}_G e^{-t\Delta_\varepsilon^+} - \mathrm{tr}_G e^{-t\Delta_\varepsilon^-}$ can be computed as

$$(5.1.9)\quad c_0(\Delta_\varepsilon^+) - c_0(\Delta_\varepsilon^-) = \int_U f\Omega(U) = \int_M f\Omega(M) = 2^n(\mathrm{ch}(\varepsilon)\mathfrak{L}(\mathfrak{g}, H))[V],$$

the last equality being true because

$$\int_M f\omega(M) = \int_M f(dg/dh) = \int_G f dg = 1.$$

The McKean-Singer identity with respect to the G-trace

$$(5.1.10)\quad \mathrm{tr}_G e^{-t\Delta_\varepsilon^+} - \mathrm{tr}_G e^{-t\Delta_\varepsilon^-} = \dim_G \mathrm{Ker}\, D_\varepsilon^+ - \dim_G \mathrm{Ker}\, D_\varepsilon^-, \quad \text{for any } t > 0,$$

together with (5.1.9), concludes the proof.

For completeness, let us include a proof for (5.1.10). Denote by $H^{\pm}$ the orthogonal projection onto $\mathrm{Ker}\, D_\varepsilon^{\pm}$. Then

$$\begin{aligned}\mathrm{tr}_G e^{-t\Delta_\varepsilon^{\pm}} - \dim_G \mathrm{Ker}\, D_\varepsilon^{\pm} &= \mathrm{tr}_G\big(e^{-t\Delta_\varepsilon^{\pm}}(1 - H^{\pm})\big)\\ &= \mathrm{tr}_G\big((I - H^{\pm})e^{-t\Delta_\varepsilon^{\pm}}(I - H^{\pm})\big)\\ &= \mathrm{tr}_G e^{-t(I-H^{\pm})\Delta_\varepsilon^{\pm}(I-H^{\pm})}.\end{aligned}$$

On the other hand, with $D_\varepsilon^{\pm} = U^{\pm} | D_\varepsilon^{+} |$ being the polar decomposition, one has

$$U^{\mp} U^{\pm} = I - H^{\pm},$$

which clearly implies

$$\mathrm{tr}_G e^{-t(1-H^+)\Delta_\varepsilon^+(1-H^+)} = \mathrm{tr}_G e^{-t(I-H^-)\Delta_\varepsilon^-(I-H^-)}. \qquad \text{Q.e.d.}$$

As a next step towards the general index theorem, we shall now treat the case of Dirac-type operators. This time by ε we denote a representation of $\tilde{H}$ with $[\varepsilon] \in R(\tilde{H})^1$. As in Section 4, let $S^{\pm}$ denote the half-spin representation of $\mathrm{Spin}(V)$. Since $u \in \tilde{H}$ acts by $-I$ on E and on $S^{\pm}$, it follows that $E^{\pm} = E \otimes S^{\pm}$ are (unitary) H-modules, so we can form the induced homogeneous vector bundles $\mathcal{E}^{\pm}$ over M. Let

$$\nabla_\varepsilon^{\pm} : C^\infty(M, \mathcal{E}^{\pm}) \to C^\infty\big(M, T_{\mathbf{C}}^* M \otimes \mathcal{E}^{\pm}\big)$$

be G-invariant connections, compatible with the Hermitian structure of $\mathcal{E}^{\pm}$

inherited from $E^{\pm}$. We define the $(\pm)$-Dirac operator with coefficients in ε as being the composition

$$A_{\varepsilon}^{\pm}: C_c^{\infty}(M, \mathcal{E}^{\pm}) \xrightarrow{\nabla_{\varepsilon}^{\pm}} C_c^{\infty}\left(M, T_{\mathbf{C}}^* M \otimes \mathcal{E}^{\pm}\right) \xrightarrow{c} C_c^{\infty}(M, \mathcal{E}^{\mp}),$$

where c is the bundle homomorphism induced by the Clifford multiplication. Clearly, A_{ε}^{+} and A_{ε}^{-} are G-invariant first order elliptic differential operators, formal adjoints of each other.

5.2. THEOREM. $\operatorname{ind}_G A_{\varepsilon}^{+} = (-1)^n(\operatorname{ch}(\varepsilon)\hat{\mathcal{A}}(\mathfrak{g}, H))[V]$.

Proof. 1°. We shall first establish this formula for the special case when ε is of the form $\varphi \otimes \sigma$, where φ is a unitary representation of H on a (finite-dimensional, complex) vector space F and so $[\varphi] \in R(\tilde{H})^0$, and $\sigma = \sigma^{+} \oplus \sigma^{-}$ with σ^{+}, σ^{-} being the representations of $\tilde{H}$ on S^{+}, S^{-} obtained from the half-spin representations of $\operatorname{Spin}(V)$. Indeed, if $\Lambda^{\pm} V_{\mathbf{C}}$ denotes the $\mathrm{SO}(V)$-module whose associated vector bundle over M (equipped with the natural $\mathrm{SO}(2n)$-structure) is $\Lambda^{\pm} T_{\mathbf{C}}^* M$, it is well-known that $S \otimes S^{\pm} \cong \Lambda^{\pm} V_{\mathbf{C}}$, as $\mathrm{SO}(V)$-modules, hence $F \otimes S \otimes S^{\pm} \cong F \otimes \Lambda^{\pm} V_{\mathbf{C}}$ as H-modules. Using this, and the formula relating the Clifford multiplication and the exterior differentiation, we then see easily that the principal symbols of the Dirac operator $A_{\varphi\otimes\sigma}^{+}$ and of the signature operator D_{φ}^{+} are related by the equality

$$\sigma_0^1\left(A_{\varphi\otimes\sigma}^{+}\right) = (-1)^n \sigma_0^1\left(D_{\varphi}^{+}\right),$$

no matter which G-invariant connection we employ to construct each of them. From Proposition 3.5 it follows that

$$\operatorname{ind}_G A_{\varphi\otimes\sigma}^{+} = (-1)^n \operatorname{ind}_G D_{\varphi}^{+}$$

and so, by Theorem 5.1,

$$\operatorname{ind}_G A_{\varphi\otimes\sigma}^{+} = (-1)^n 2^n(\operatorname{ch}(\varphi)\mathcal{L}(\mathfrak{g}, H))[V].$$

Since

$$\operatorname{ch}(\sigma) = 2^n \det\cosh\left(\tfrac{1}{2}\Theta_V\right)$$

and thus

$$2^n \mathcal{L}(\mathfrak{g}, H) = \operatorname{ch}(\sigma)\hat{\mathcal{A}}(\mathfrak{g}, H),$$

we finally get

$$\operatorname{ind}_G A_{\varphi\otimes\sigma}^{+} = (-1)^n\left(\operatorname{ch}(\varphi)\operatorname{ch}(\sigma)\hat{\mathcal{A}}(\mathfrak{g}, H)\right)[V]. \tag{5.2.1}$$

2°. Let $\mathcal{Q}$: $R(\tilde{H})^1 \to \mathbf{R}$ be the map given by

$$\mathcal{Q}[\varepsilon] = \operatorname{ind}_G A_{\varepsilon}^{+} \quad \text{for } [\varepsilon] \in R(\tilde{H})^1.$$

We shall first show that $\mathcal{Q}$ depends in a polynomial way on (the matrix coefficients of) ε, and then, using the central theorem of invariant theory for the unitary group as a key argument, that $\mathcal{Q}$ is continuous with respect to the $I_{\tilde{H}}$-adic topology on $R(\tilde{H})$; here $I_{\tilde{H}}$ denotes the augmentation ideal.

Since we are only interested in the G-index, there is no loss of generality in assuming that A_ε^+ is constructed by means of the G-invariant connection induced by an $\mathrm{Ad}(H)$-invariant splitting $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. In that case, if we denote by $\{X_1,\ldots,X_m\}$ an orthonormal basis on $\mathfrak{m}$ and by $\{\omega^1,\ldots,\omega^m\}$ the dual basis on $V \cong \mathfrak{m}^*$, $A_\varepsilon^+ : (C_c^\infty(G) \otimes E \otimes S^+)^H \to (C_c^\infty(G) \otimes E \otimes S^-)^H$ is given by the formula

$$A_\varepsilon^+ = \sum_{i=1}^{n} R(X_i) \otimes I \otimes c(\omega^i),$$

where $R(X_i)$ is the left invariant vector field on G determined by $X_i \in \mathfrak{m}$.

Let us now compute the local expression of A_ε^+. As a local chart we take a neighbourhood U of $o \in M$ such that

$$\Psi(t_1,\ldots,t_m) = (\exp(t_1 X_1)\ldots\exp(t_m X_m))H$$

is a diffeomorphism of a neighbourhood of $0 \in \mathbf{R}^m$ onto U. Also, after fixing a basis $\{Y_1,\ldots,Y_r\}$ of $\mathfrak{h}$, we have

$$\Phi(t_1,\ldots,t_m, s_1,\ldots,s_r) = \exp(t_1X_1)\ldots\exp(t_mX_m)\exp(s_1Y_1)\ldots\exp(s_rY_r)$$

defines a system of local coordinates in a neighbourhood V of $1 \in G$. There exist then functions $\alpha_j^i, \beta_k^i \in C^\infty(\Phi^{-1}(V))$ such that

$$(R(X_i)f)\circ\Phi = \sum_{j=1}^{m} \alpha_j^i \frac{d}{dt_j}(f\circ\Phi) + \sum_{k=1}^{r} \beta_k^i \frac{d}{ds_k}(f\circ\Phi)$$

for any $f \in C^\infty(V)$.

If $f \in C_c^\infty(M, E^+) = (C_c^\infty(G) \otimes E \otimes S^+)^H$ and support $(f) \subset V$, then

$$\begin{aligned}(R(X_i)f)(\exp t_1X_1\ldots\exp t_mX_m) &= \sum_{j=1}^{m} \alpha_j^i(t,0)\frac{d}{dt_j}(f\circ\Phi)(t,0)\\ &\quad+ \sum_{k=1}^{r} \beta_k^i(t,0)\frac{d}{ds_k}(f\circ\Phi)(t,0)\end{aligned}$$

where $t = (t_1,\ldots,t_m) \in \mathbf{R}^m$. Further

$$\begin{aligned}\frac{d}{ds_k}(f\circ\Phi)(t,0) &= \left.\frac{d}{ds_k}\right|_{s=0} f(\exp(t_1X_1)\ldots\exp(t_mX_m)\exp(s_1Y_1)\ldots\exp(s_rY_r))\\ &= \left.\frac{d}{ds_k}\right|_{s=0} \varepsilon^+(\exp(-s_rY_r))\ldots\varepsilon^+(\exp(-s_1Y_1))f(\exp(t_1X_1)\ldots\exp(t_mX_m))\\ &= -(I\otimes\sigma^+(Y_k) + \varepsilon(Y_k)\otimes I)f(\exp(t_1X_1)\ldots\exp(t_mX_m)).\end{aligned}$$

This shows that the local expressions of A_{ε}^{+}, with respect to the system of coordinates $(t_1,\ldots,t_m)$ on U, $A_{\varepsilon,U}^{+}$: $C^{\infty}(U)\otimes E\otimes S^{+}\to C^{\infty}(U)\otimes E\otimes S^{-}$, is

$$(5.2.2)\qquad A_{\varepsilon,U}^{+}=\sum_{i,j}\alpha_j^i(t,0)\frac{d}{dt_j}I\otimes c(\omega^i)$$
$$-\sum_{i,k}\beta_k^i(t,0)\big(I\otimes c(\omega^i)\sigma^{+}(Y_k)+\varepsilon(Y_k)\otimes c(\omega^i)\big).$$

In fact our concern with this formula is only qualitative. As follows from the classical theory (cf. [4], p. 303) the local measures $\mu_0^{\pm}(U)$ associated to $\square_{\varepsilon}^{\pm}=A_{\varepsilon}^{\mp}A_{\varepsilon}^{\pm}$ depend polynomially on the local coefficients of $\square_{\varepsilon}^{\pm}$ and their derivatives. Formula (5.2.1) shows then that $\mu_0^{\pm}(U)$ are polynomial functions in $\varepsilon(Y_1),\ldots,\varepsilon(Y_r)$. By (5.1.9), the same is true for the constant terms $c_0(\square_{\varepsilon}^{\pm})$, and further, by the Singer-McKean identity, for $\mathrm{ind}_G A_{\varepsilon}^{+}$. That is, there exists a polynomial $P\in\mathbf{C}[Z_1,\ldots,Z_N]$, with $N=rp^2$, $p=\dim E$, such that

$$(5.2.3)\qquad \mathrm{ind}_G A_{\varepsilon}^{+}=P\big(c_{11}(\varepsilon(Y_1)),\ldots,c_{ij}(\varepsilon(Y_k)),\ldots,c_{pp}(\varepsilon(Y_r))\big);$$

here $c_{ij}(T)=\langle Te_i,e_j\rangle$ are the coefficients with respect to an orthonormal basis $\{e_1,\ldots,e_p\}$ of E. Each monomial $P_{\alpha}=Z_1^{\alpha_1}\ldots Z_N^{\alpha_N}$ defines, by complete polarization, a multilinear functional on $\otimes^{|\alpha|}\mathrm{End}\,E$, where $|\alpha|=\alpha_1+\cdots+\alpha_N$. By averaging it over the unitary group $U(E)$, acting by the $|\alpha|^{\text{th}}$ tensor power of its adjoint representation on $\mathrm{End}(E)$, we get an invariant element $\tilde{P}_{\alpha}\in\mathrm{Hom}_{U(E)}(\otimes^{|\alpha|}(E\otimes E^*),\mathbf{C})$. Since $\mathrm{ind}_G A_{\varepsilon}^{+}$ does not depend on the orthonormal basis chosen on E, the equality (5.2.2) remains valid when P is replaced by $\tilde{P}=\Sigma^{\oplus}\tilde{P}_{\alpha}$. Now, according to the fundamental basis theorem for the invariants of the unitary group (see [4, p. 325] for a two-line argument translating Weyl's original proof for $\mathrm{GL}(E)$), $\mathrm{Hom}_{U(E)}(\otimes^{q}\mathrm{End}(E),\mathbf{C})$ is spanned by elementary invariants, i.e. of the form

$$\mathrm{Inv}_{\tau}(T_1\otimes\cdots\otimes T_q)=\mathrm{tr}\big(T_1T_{\tau(1)}\ldots T_{\tau^{n_1}(1)}\big)\ldots\mathrm{tr}\big(T_{i_r}T_{\tau(i_r)}\ldots T_{\tau^{n_r}(i_r)}\big),$$

where τ is a permutation of $\{1,\ldots,q\}$ and each factor in the above product of traces corresponds to one of its cycles. We can conclude then that $\mathrm{ind}_G A_{\varepsilon}^{+}$ is a linear combination of products of the form $\mathrm{tr}(\varepsilon(Y_{i_1})\ldots\varepsilon(Y_{i_s}))$ with $i_1,\ldots,i_s$ ranging over the set $\{1,\ldots,r\}$ and $1\le s\le\deg(P)$.

Let us now prove that $\mathcal{Q}$: $R(\tilde{H})^1\to\mathbf{R}$ vanishes on $(I_{\tilde{H}})^p$ for $p>\deg(P)$. It is enough to check that $\mathcal{Q}(a)=0$ if $a=([\alpha_1]\text{-}\dim\alpha_1)\ldots([\alpha_p]\text{-}\dim\alpha_p)$, with $\alpha_1,\ldots,\alpha_p$ representations of $\tilde{H}$ such that $a\in R(\tilde{H})^1$. Of course, we can write $a=[\varepsilon_1]-[\varepsilon_2]$, with ε_1, ε_2 acting on the same space E and $[\varepsilon_1],[\varepsilon_2]\in R(\tilde{H})^1$, and we must show that

$$(5.2.4)\qquad \mathrm{ind}_G A_{\varepsilon_1}^{+}=\mathrm{ind}_G A_{\varepsilon_2}^{+}.$$

Let $\chi(\alpha)$ denote the character of a representation α of $\tilde{H}$. Since

$$\chi(\varepsilon_1) - \chi(\varepsilon_2) = (\chi(\alpha_1) - \dim\alpha_1)\dots(\chi(\alpha_p) - \dim\alpha_p),$$

it is clear that $\chi(\varepsilon_1)$ and $\chi(\varepsilon_2)$ have the same p-jet at $1 \in \tilde{H}$. But

$$\big(R(Y_{i_1}\dots Y_{i_s})\chi(\varepsilon_j)\big)(1) = \operatorname{tr}\big(\varepsilon_j(Y_{i_1})\dots\varepsilon_j(Y_{i_s})\big), \qquad j = 1,2.$$

Combined with (5.2.3), this proves the equality (5.2.4).

3°. We are finally able to prove the index formula for the operators of Dirac type. As we already mentioned in 1°, $\sigma \otimes \sigma$ is isomorphic to the representation λ of H on $\Lambda V_{\mathbf{C}} \cong S \otimes S$. Since $\dim\lambda = 2^m$, $1 - 2^{-m}[\lambda] \in I_{\tilde{H}}$ and the sequence

$$\ell_k = 1 + (1 - 2^{-m}[\lambda]) + \cdots + (1 - 2^{-m}[\lambda])^k$$

converges to $2^m[\lambda]^{-1}$ in the completion $(R(\tilde{H}) \otimes \mathbf{Q})^{\wedge}$; also, note that $\ell_k \in R(\tilde{H})^0 \otimes \mathbf{Q} = R(H) \otimes \mathbf{Q}$. By 2°, for any $\varepsilon \in R(\tilde{H})^1$,

$$2^m\mathcal{Q}[\varepsilon] = \mathcal{Q}([\varepsilon][\sigma]\ell_k[\sigma]), \quad \text{if } k \text{ is large enough.} \tag{5.2.5}$$

On the other hand, as follows from (5.2.1) in 1°, applied to $[\varphi] = [\varepsilon][\sigma]\ell_k \in R(\tilde{H})^0 \otimes \mathbf{Q}$,

$$\mathcal{Q}([\varepsilon][\sigma]\ell_k[\sigma]) = (-1)^n\big(\operatorname{ch}(\varepsilon)\operatorname{ch}(\sigma)^2\operatorname{ch}(\ell_k)\hat{\mathcal{Q}}(\mathfrak{g}, H)\big)[V].$$

Since, for k sufficiently large,

$$\operatorname{ch}(\ell_k) = 2^m\operatorname{ch}(\lambda)^{-1} = 2^m\operatorname{ch}(\sigma)^{-2},$$

it follows that

$$\mathcal{Q}([\varepsilon][\sigma][\ell_k][\sigma]) = (-1)^n 2^m\big(\operatorname{ch}(\varepsilon)\hat{\mathcal{Q}}(\mathfrak{g}, H)\big)[V],$$

which, together with (5.2.5), completes the proof of the theorem.

With all these preparations, it is now an easy matter to get the general index theorem.

5.3. THEOREM. *For any G-invariant pseudo-differential elliptic operator* $P \in \Phi_c^p(M, \mathcal{E}, \mathcal{F})^G$, $\operatorname{ind}_G P = \operatorname{ind}_t[\sigma_0^p(P)]$.

Otherwise formulated, the theorem says that the two index maps ind_a, ind_t: $K_H(V) \to \mathbf{R}$ coincide.

Indeed, by Proposition 4.3, it is enough to check that they coincide on elements of the form $[\varepsilon]\tilde{\alpha}$, where ε is a representation of $\tilde{H}$ such that $[\varepsilon] \in R(\tilde{H})^1$, and $\tilde{\alpha}$ is the K-theory class of the Dirac $\tilde{H}$-complex. But clearly $[\varepsilon]\tilde{\alpha} = [\sigma_0^1(A_\varepsilon^+)]$, so that the statement follows from Theorem 5.2.

6. L^2-solutions and discrete series

Elliptic differential operators of Dirac or Laplace-Beltrami type were used to realize geometrically the discrete series representations, for certain classes of unimodular Lie groups, as spaces of L^2-solutions. The most familiar instance is the geometric realization of the discrete series in the semisimple case (see [5], [19] and the references given there); other examples, occurring for a larger class of Lie groups, are discussed in [18]. It is our purpose here to put in evidence the converse phenomenon: subject to some restrictions on the structure of the group, the space of L^2-solutions of any invariant elliptic system of equations, defined over a homogeneous space whose isotropy subgroups are compact, decomposes as a finite direct sum of irreducible discrete series.

To begin with, we shall treat the semisimple case, which already displays the main difficulties in working out the proof.

6.1. THEOREM. *Let G be a connected semisimple Lie group with finite center, H a compact subgroup, ε, ε' two finite-dimensional unitary representations of H. Then, for any G-invariant pseudo-differential elliptic operator $P \in \Phi_c^p(M, \mathcal{E}, \mathcal{E}')^G$, the unitary representation of G on the space of L^2-solutions of the equation $Pu = 0$ is a finite direct sum of discrete series representations.*

Proof. Due to the existence of a G-invariant parametrix, $\operatorname{Ker} P \subset \operatorname{Ker}(I - S)$ for some $S \in \Psi_c^{-\infty}(M, \mathcal{E})^G$. Thus, it is enough to prove that $\operatorname{Ker}(I - S)$ has the stated property. Recall now that S is of the form

$$(R(\varphi)u)(x) = \int \varphi(x^{-1}y)u(y)\,dy, \qquad u \in \left(L^2(G) \otimes E\right)^H,$$

with $\varphi \in (C_c^\infty(G) \otimes \operatorname{End}(E))^{H \times H}$. In general, if π is a unitary representation of G on the Hilbert space $\mathcal{H}(\pi)$, we define $\pi(\varphi)$: $(\mathcal{H}(\pi) \otimes E)^H \to (\mathcal{H}(\pi) \otimes E)^H$ by the formula

$$\pi(\varphi)\xi = \int \varphi(y)(\pi(y) \otimes I)\xi\,dy, \qquad \xi \in (\mathcal{H}(\pi) \otimes E)^H.$$

The Plancherel theorem gives the following direct integral decomposition for S: $(L^2(G) \otimes E)^H \to (L^2(G) \otimes E)^H$,

$$S = \int_{\hat{G}}^{\oplus} I_\pi \otimes \check{\pi}(\varphi)\,d\mu(\pi);$$

here μ is the Plancherel measure on the unitary dual $\hat{G}$, I_π is the identity operator on $\mathcal{H}(\pi)$ and $\check{\pi}$ denotes the contragredient representation of π. It follows that

$$\text{(6.1.1)} \qquad \operatorname{Ker}(I - S) = \int_{\hat{G}}^{\oplus} \mathcal{H}(\pi) \otimes \operatorname{Ker}(I - \check{\pi}(\varphi))\,d\mu(\pi).$$

We will show that the continuous part of the Plancherel measure gives no contribution to the direct integral decomposition (6.1.1). More precisely, denoting by $\hat{G}_d$ the discrete series of G, we claim that:

(6.1.2) The set of all $\pi \in \hat{G} \setminus \hat{G}_d$ such that $\operatorname{Ker}(I - \check{\pi}(\varphi)) \neq 0$ has zero Plancherel measure.

To prove this we shall rely on Harish-Chandra's work on the explicit description of the Plancherel measure, and we shall freely use some of his results in [14]. The first observation is that it is enough to concentrate our attention to the contribution given to the Plancherel measure by a single conjugacy class of non-compact Cartan subgroups. Accordingly, after fixing a maximal compact subgroup K in G, we shall look at the series attached to a Cartan subgroup $C = T \times A$ with T the anisotropic part and A the vector part (by hypothesis, nontrivial). The component of the Plancherel measure supported on this series can be described as

$$d\mu_C = \mu(\lambda, \nu)\, d\lambda \otimes d\nu,$$

where $(\lambda, \nu) \in \hat{C} = \hat{T} \times A^*$, $\mu(\lambda, \nu)$ is an explicitly known function (invariant under the Weyl group), $d\lambda$ is the counting measure on the lattice $\hat{T}$ and $d\nu$ is the Lebesgue measure on the vector group A^*. Thus, with $\lambda \in \hat{T}$ fixed, π_ν denoting the irreducible representation of G corresponding to $(\lambda, \nu) \in \hat{T} \times A_0^*$ (A_0^* is the open dense subset of A^* consisting of regular elements), and $\psi \in (C_c^\infty(G) \otimes \operatorname{End}(E))^{H \times H}$ denoting the function $\psi(x) = \varphi(x^{-1})^*$, we are reduced to prove:

(6.1.3) $\quad N = \{\nu \in A_0^*; \operatorname{Ker}(I - \pi_\nu(\psi)) \neq 0\}$ has zero Lebesgue measure.

At this stage let us recall (cf. [14, §4]) that the family of representations $\{\pi_\nu; \nu \in A_0^*\}$ can be analytically continued in the parameter ν to the complexification $A_{\mathbf{C}}^*$ of A^*, and that all the representations thus obtained, π_ν with $\nu \in A_{\mathbf{C}}^*$, can be realized, as admissible representations, on the same Hilbert space $\mathcal{H}$. Here the analyticity means that, for any $f \in C_c^\infty(G)$, the mapping $\nu \mapsto \pi_\nu(f)$ from $A_{\mathbf{C}}^*$ to $\mathcal{B}(\mathcal{H})$ is weakly (and thus strongly) analytic. Also, for any $\nu \in A_{\mathbf{C}}^*$ and $f \in C_c^\infty(G)$, $\pi_\nu(f)$ is a trace-class operator. Finally, the restriction of each π_ν to the maximal compact K is a fixed unitary representation π_K of K on $\mathcal{H}$, independent of $\nu \in A_{\mathbf{C}}^*$. Since we can assume that H is contained in K, it follows that $(\mathcal{H}(\pi_\nu) \otimes E)^H$ does not depend on $\nu \in A_{\mathbf{C}}^*$; indeed, it coincides with the space $(\mathcal{H} \otimes E)^H$ of all vectors in $\mathcal{H} \otimes E$ which are invariant under the representation $h \mapsto \pi_K(h) \otimes \varepsilon(h)$ of H. It follows further that with ψ as above, $\pi_\nu(\psi)\colon (\mathcal{H} \otimes E)^H \to (\mathcal{H} \otimes E)^H$ is compact, and that the mapping: $\nu \mapsto I - \pi_\nu(\psi)$, defined on $A_{\mathbf{C}}^*$ and taking values in the set of Fredholm operators on $(\mathcal{H} \otimes E)^H$, is analytic. In turn, this implies that the set of singularities

$$\Sigma = \{\nu \in A_{\mathbf{C}}^*; \dim \operatorname{Ker}(I - \pi_\nu(\psi)) > m_0\},$$

where

$$m_0 = \min\{\dim \operatorname{Ker}(I - \pi_\nu(\psi)); \nu \in A_{\mathbf{C}}^*\},$$

is analytic, and therefore of zero Lebesgue measure. We will show that $N \subset \Sigma$ by proving that $m_0 = 0$. Indeed, let us assume that $m_0 \geq 1$. Then, for any ν, $\|\pi_\nu(\psi)\|_{\mathrm{HS}} \geq 1$, where $\| \ \|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. On the other hand, the Plancherel formula gives

$$\operatorname{tr}_G R(\psi)R(\psi)^* = \int_{\hat{G}} \|\pi(\psi)\|_{\mathrm{HS}}^2 \, d\mu(\pi) \geq \int_{A^*} \|\pi_\nu(\psi)\|_{\mathrm{HS}}^2 \mu_\lambda(\nu)\, d\nu,$$

where $\mu_\lambda(\nu) = \mu(\lambda, \nu)$. Since $\int_{A^*} \mu_\lambda(\nu)\, d\nu = \infty$, it follows that $\operatorname{tr}_G R(\psi)R(\psi)^* = \infty$ which is a contradiction.

Summing up, we can conclude that

$$\text{(6.1.4)} \qquad \operatorname{Ker}(I - S) = \sum_{\pi \in \hat{G}_d}^{\oplus} \mathcal{H}(\pi) \otimes \operatorname{Ker}(I - \check{\pi}(\varphi)).$$

Now $\dim_G \operatorname{Ker}(I - S) < \infty$, while $\dim_G \mathcal{H}(\pi)$ coincides with the usual formal degree $\deg(\pi)$ of π (cf. [8], Proposition 20). Since there is a strictly positive lower bound for the formal degrees (actually, with an appropriate choice of the Haar measure, they are all strictly positive integers), it follows that the direct sum (6.1.4) involves only finitely many non-zero components. Q.e.d.

We will now extend the result to a larger class of Lie groups, whose significance is put in evidence by the recent work of Anh (see [1]). First let us recall that, in Anh's terminology, a connected Lie group A is said to be an "$\mathcal{H}$-group" if there exists a linear functional α on the Lie algebra $\mathfrak{a}$ of A such that $\mathrm{Ad}^*(A)\alpha = \{\alpha + \zeta; \zeta \in \mathfrak{a}^*, \zeta \mid \mathfrak{z} = 0\}$, where $\mathfrak{z}$ is the Lie algebra of the center Z of A. Such a group is necessarily solvable and unimodular ([1], Lemma 2.2 and Theorem 2.9) and its representation theory is very similar to that of nilpotent Lie groups with discrete series (cf. [1], Theorem 2.12 and Proposition 2.14). In particular the irreducible representations of A which are square integrable mod Z are parametrized by the set Λ_A of all $\lambda \in \mathfrak{z}^*$ which integrates to a character χ_λ of Z and can be extended to a functional α on $\mathfrak{a}$ satisfying the above property.

The class of groups we are concerned with can be described as follows: G is a semi-direct product of an $\mathcal{H}$-group A with compact center Z, such that Z is central in G, and a connected reductive Lie group S with compact center. The center Z_G of G is then of the form $Z_G = Z \cdot C$, with C central in S. By K_S we denote a maximal compact subgroup of S.

6.2. THEOREM. *Let G be a Lie group as above, H a compact subgroup of G such that $Z_G \subset H \subset ZK_S$, $\varepsilon_1, \varepsilon_2$ two finite-dimensional unitary representations of H whose restriction to Z is a multiple of a character $\chi = \chi_\lambda$ with $\lambda \in \Lambda_A$, and let $P \in \Phi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$. Then* $\operatorname{Ker} P$ *is a finite direct sum of irreducible discrete series.*

Proof. Clearly $(L^2(G) \otimes E_i)^H$ is a closed subspace of $(L^2(G) \otimes E_i)^Z = L^2(G, \chi) \otimes E_i$, where $L^2(G, \chi) = (L^2(G) \otimes \mathbf{C})^Z$. By Theorem 2.12 in [1],

$$L^2(A, \chi) \cong \mathcal{H}(\pi_\chi) \otimes \mathcal{H}(\check{\pi}_\chi),$$

where π_χ denotes the irreducible representation of A with central character χ. On the other hand, in view of [1, Theorem 3.6] (after passing to an at most double covering of S, which does not affect the substance of our problem), π_χ can be extended to an irreducible representation $\tilde{\pi}_\chi$ of G, and every irreducible representation of G whose restriction to Z is a multiple of χ is of the form

$$\pi_{\chi, \sigma}(as) = \tilde{\pi}_\chi(as) \otimes \sigma(s), \qquad a \in A, s \in S$$

with σ an irreducible representation of S.

It follows that

$$L^2(G, \chi) = \int_{\hat{S}}^{\oplus} \mathcal{H}(\pi_{\chi, \sigma}) \otimes \mathcal{H}(\check{\pi}_{\chi, \sigma})\, d\mu(\sigma),$$

μ being the Plancherel measure on $\hat{S}$, and further that

$$\left(L^2(G) \otimes E_i\right)^H = \int_{\hat{S}}^{\oplus} \mathcal{H}(\pi_{\chi, \sigma}) \otimes \left(\mathcal{H}(\check{\pi}_{\chi, \sigma}) \otimes E_i\right)^H d\mu(\sigma).$$

This being established, the proof of Theorem 6.1 goes through with virtually no change in our present situation, once we notice that, due to the special form of $\pi_{\chi, \sigma}$, the "piece-wise" analyticity in the parameter σ is preserved.

Combining Theorem 3.6 in [1] and the well-known criterion of Harish-Chandra for the existence of discrete series for real reductive groups, one gets, as an immediate consequence of the above theorem, the following vanishing criterion for L^2-solutions of invariant elliptic pseudo-differential operators:

6.3. COROLLARY. *With the same notation as above and under the hypotheses of Theorem* 6.2, *one has* $\operatorname{Ker} P = \{0\}$ *if* $\operatorname{rank} K_S < \operatorname{rank} S$.

It should be mentioned that, when G is semisimple and K (resp. H) is the maximal compact (resp. the compact Cartan), the space of L^2-solutions for Dirac (resp. Laplace-Beltrami) type equations on G/K (resp. G/H) is known to be either irreducible or trivial (for the precise statement see [5, Theorem 9.3]). However, for arbitrary invariant elliptic equations, neither the Decomposition

Theorem 6.1 nor the vanishing result 6.3 were known, even in the semisimple case.

7. Concluding remarks

We collect here a few applications of the main results in the paper, together with a number of further comments on the behavior of the G-index.

7.1. In this subsection G will denote a connected and simply-connected nilpotent Lie group with center Z. Assuming that G has square-integrable (mod Z) irreducible representations (and when this is the case they are sufficiently many to support the Plancherel measure), we will show that they can all be realized as L^2-solutions of invariant elliptic equations, more precisely as L^2-harmonic spinors. The other possibility, of realizing them in terms of L^2-cohomology (see [17], [18]), works only in the presence of a totally complex polarization which, besides being not canonical at all, might even fail to exist.

Given $\lambda \in \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, $G(\lambda)$ will denote the corresponding isotropy subgroup of G (acting on $\mathfrak{g}^*$ by the coadjoint representation). The Lie algebra $\mathfrak{g}(\lambda)$ of $G(\lambda)$ coincides with the radical of the two-form $B_\lambda \in \Lambda^2 \mathfrak{g}$ defined by:

$$B_\lambda(x, y) = \lambda([x, y]); \qquad x, y \in \mathfrak{g}.$$

The assumption on the existence of square-integrable irreducible representations amounts to the existence of a functional $\lambda \in \mathfrak{g}^*$ such that $\mathfrak{g}(\lambda)$ is precisely the center $\mathfrak{z}$ of $\mathfrak{g}$ (cf. [16]). Moreover, the set of regular elements in $\mathfrak{g}^*$

$$\mathfrak{g}^*_{\text{reg}} = \{\lambda \in \mathfrak{g}^*; \dim \mathfrak{g}(\lambda) \leq \dim \mathfrak{g}(\nu) \quad \text{for any } \nu \in \mathfrak{g}^*\}$$

consists entirely of such functionals. Further, if π_λ denotes the irreducible representation of G associated to $\lambda \in \mathfrak{g}^*$ via the Kirillov correspondence, then π_λ is square-integrable mod Z if and only if $\lambda \in \mathfrak{g}^*_{\text{reg}}$, i.e. $\mathfrak{g}(\lambda) = \mathfrak{z}$ (cf. [16], Theorem 1).

Let us fix $\lambda \in \mathfrak{g}^*_{\text{reg}}$ and denote by ε_λ the associated character on $Z = G(\lambda)$: $\varepsilon_\lambda(\exp z) = e^{2\pi i \lambda(z)}$, $z \in \mathfrak{z}$. The corresponding homogeneous line bundle over $M = G/Z$ will be denoted $\mathcal{E}_\lambda$. Since the isotropy representation is trivial, any inner product on $T_o M = \mathfrak{g}/\mathfrak{z}$ gives rise to a G-invariant metric on M, and we will fix one, once and for all. It also follows that, on the Riemannian manifold M, there is a unique G-invariant spin-structure, namely that given by the trivial homomorphism of Z to $\mathrm{Spin}(V)$; here, $V = (\mathfrak{g}/\mathfrak{z})^*$ is endowed with the dual inner product. We can thus form the twisted spin-bundles, with coefficients in $\mathcal{E}_\lambda$, $\mathcal{E}_\lambda^\pm = \mathcal{E}_\lambda \otimes \mathcal{S}^\pm$. Further, by choosing G-invariant connections on $\mathcal{E}_\lambda^+$, $\mathcal{E}_\lambda^-$

we can construct the corresponding Dirac operators $A_\lambda^\pm$. The center Z acts by the character ε_λ on $\operatorname{Ker} A_\lambda^\pm$ and thus, as follows from [16], the representation of G on $\operatorname{Ker} A_\lambda^\pm$ is a multiple of the irreducible representation π_λ.

7.1.A. PROPOSITION. *The multiplicity $m_\lambda^\pm$ of π_λ in* $\operatorname{Ker} A_\lambda^\pm$ *is finite, and one has*

$$\operatorname{Ker} A_\lambda^+ - \operatorname{Ker} A_\lambda^- = \pi_\lambda,$$

in the sense that $m_\lambda^+ - m_\lambda^- = 1$.

Proof. The subgroup $Z_\lambda = \operatorname{Ker}(\varepsilon_\lambda)$ acts trivially on all the entities defined above. Thus we can assume $Z_\lambda = \{1\}$, and therefore that Z is the circle group. Since $A_\lambda^\pm$ is elliptic, $\dim_G \operatorname{Ker} A_\lambda^\pm < \infty$, which proves the first assertion. One has $\dim_G \operatorname{Ker} A_\lambda^\pm = m_\lambda^\pm \deg(\pi_\lambda)$, so that

$$\operatorname{ind}_G A_\lambda^\pm = (m_\lambda^+ - m_\lambda^-)\deg \pi_\lambda. \tag{7.1.1}$$

On the other hand, the index theorem gives

$$\operatorname{ind}_G A_\lambda^+ = \frac{(-1)^n}{n!}(\Lambda^n \Theta_\lambda)[V], \qquad n = \tfrac{1}{2}\dim M$$

since $\Theta_V = 0$, and thus $\hat{\mathfrak{A}}(\mathfrak{g}, Z) = 1$. Also

$$\Theta_\lambda = \frac{(-1)^n}{2^n}\beta_\lambda,$$

where β_λ is the (non-degenerate) 2-form on $\mathfrak{g}/\mathfrak{z}$ induced by B_λ; therefore

$$\operatorname{ind}_G A_\lambda^+ = \frac{1}{n!2^n}(\Lambda^n \beta_\lambda)[V]. \tag{7.1.2}$$

Let us now specify the choice of orientation on $\mathfrak{g}/\mathfrak{z}$ as being given by the condition

$$(\Lambda^n \beta_\lambda)[V] > 0.$$

Then, according to [16, Theorem 4],

$$\deg(\pi_\lambda) = \frac{1}{n!2^n}(\Lambda^n \beta_\lambda)[V]. \tag{7.1.3}$$

From (7.1.1), (7.1.2) and (7.1.3) it follows that $m_\lambda^+ - m_\lambda^- = 1$. Q.e.d.

It is likely that $\operatorname{Ker} A_\lambda^-$ always vanishes and thus that $\operatorname{Ker} A_\lambda^+$ is the representation space of π_λ. Here, we shall only prove that this is "almost always" the case. Precisely, let us assume, as above, that Z is the circle group and let us fix a non-zero element $z \in \mathfrak{z}$. Then, the characters of Z are in a one-to-one correspondence with the lattice of functionals $\{\lambda_p \in \mathfrak{z}^*;\ \lambda_p(z) = p,\ p \in \mathbf{Z}\}$. Further, let us fix a splitting $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{m}$ and then consider the corresponding

G-invariant connection on the principal bundle $Z \to G \to M$; this, in turn, gives rise to G-invariant connections on the twisted spin-bundles $\mathcal{E}_p^{\pm}$ associated to λ_p. We will denote by $A_p^{\pm}$ the Dirac operators constructed by using these connections. Under these assumptions, one has:

7.1.B. PROPOSITION. *There exists* $p_0 \in \mathbf{Z}^+$ *such that, for any* $p \geq p_0$, $\operatorname{Ker} A_p^- = 0$.

Proof. We extend λ_p to functionals on $\mathfrak{g}$, by setting $\lambda_p \mid \mathfrak{m} = 0$, and we denote by β_p the corresponding non-degenerate 2-forms on $\mathfrak{m}$. We next fix a symplectic basis $\{x_1, y_1, \ldots, x_n, y_n\}$ on $\mathfrak{m}$, with respect to β_1, and then give $\mathfrak{m}$ the inner product for which this is an orthonormal basis. (This is not a restriction, since the vanishing of the space of L^2-solutions does not depend on the choice of metric.) Thus, one has

$$\text{(7.1.4)} \quad \beta_1(x_j, y_k) = \delta_{jk}, \quad \beta_1(x_j, x_k) = 0 = \beta_1(y_j, y_k), \qquad 1 \leq j, k \leq n.$$

Note that $C^\infty(M, \mathcal{E}_p^{\pm})$ is canonically isomorphic to $C^\infty(G, \lambda_p) \otimes S^{\pm}(\mathfrak{m})$, where

$$C^\infty(G, \lambda_p) = \{ f \in C^\infty(G); f(g \exp(-tz)) = e^{2i\pi pt} f(g), g \in G, t \in \mathbf{R} \},$$

z is a fixed element in $\mathfrak{z}$ such that $\lambda_1(z) = 1$, and $S^{\pm}(\mathfrak{m})$ denotes the basic $(\pm)$-spin representation of $\operatorname{Spin}(\mathfrak{m})$. With our choice of connection, the Dirac operators $A_p^{\pm}: C^\infty(G, \lambda_p) \otimes S^{\pm}(\mathfrak{m}) \to C^\infty(G, \lambda_p) \otimes S^{\mp}(\mathfrak{m})$ are given by:

$$\text{(7.1.5)} \qquad A_p^{\pm} = \sum_{j=1}^{n} \big(R(x_j) \otimes c(x_j) + R(y_j) \otimes c(y_j) \big),$$

where $R(x)$ is the left invariant vector field on G associated to $x \in \mathfrak{g}$.

To get the vanishing statement, we shall prove, by a computation similar to that in [17], the inequality

$$\text{(7.1.6)} \qquad \langle A_p^+ A_p^- \varphi, \varphi \rangle \geq c \|\varphi\|^2, \quad \text{for any } \varphi \in C_c^\infty(M, \mathcal{E}_p^-).$$

Using (7.1.5) to compute $A_p^{\mp} A_p^{\pm}$, one gets:

$$\begin{aligned} A_p^{\mp} A_p^{\pm} = {} & \sum_j R([x_j, y_j]) \otimes c(x_j) c(y_j) - \sum_j \big(R(x_j)^2 + R(y_j)^2 \big) \otimes I \\ & + \sum_{j \neq k} R([x_j, y_k]) \otimes c(x_j) c(y_k) + \sum_{j<k} R([x_j, x_k]) \otimes c(x_j) c(x_k) \\ & + \sum_{j<k} R([y_i, y_k]) \otimes c(y_j) c(y_k). \end{aligned}$$

Note now that

$$[x, y] \equiv \beta_1(x, y) z \pmod{\mathfrak{m}}, \qquad x, y \in \mathfrak{g}.$$

By (7.1.4), it follows that:

$$(7.1.7)\quad A_p^{\mp}A_p^{\pm} = \sum_j R(z)\otimes c(x_jy_j) - \sum_j\left(R(x_j)^2 + R(y_j)^2\right)\otimes I + F_0,$$

where F_0 is a first-order operator of the form:

$$(7.1.8)\qquad F_0 = \sum_j R(x_j)\otimes A_j + \sum_j R(y_j)\otimes B_j,$$

with $A_j, B_j \in \operatorname{End} S^+(\mathfrak{m})$.

Let us now define $z_j \in \mathfrak{m}_{\mathbf{C}}$ by $z_j = x_j + iy_j$, $1 \le j \le n$. Then

$$\left(R(x_j)^2 + R(y_j)^2\right)\otimes I = -iR(z)\otimes I + R(\bar{z}_j)R(z_j)\otimes I - F_j,$$

where F_j is also of the form (7.1.8). Substituting this in (7.1.7) we get

$$A_p^{\mp}A_p^{\pm} = \sum_j R(z)\otimes c(x_jy_j) + inR(z)\otimes I - \sum R(\bar{z}_j)R(z_j)\otimes I + F,$$

with F, again, of the form (7.1.8). Since $R(z)$ acts on $C^\infty(G,\lambda_p)$ as the multiplication by $-2i\pi p$, it follows that

$$(7.1.9)\quad A_p^{\mp}A_p^{\pm} = 2\pi npI\otimes I - 2\pi p\sum_j I\otimes ic(x_jy_j) - \sum_j R(\bar{z}_j)R(z_j)\otimes I + F.$$

At this stage we pause to choose a convenient basis on $S^{\pm}(\mathfrak{m})$. Let J denote the set of all multi-indices $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_j = \pm 1$, and set

$$J^{\pm} = \left\{\alpha\in J;\ \alpha_1\cdots\alpha_n = \pm(-1)^n\right\}.$$

It is not difficult to see that there exists an orthonormal basis $\{u_\alpha;\ \alpha\in J\}$ in $S(\mathfrak{m}) = S^+(\mathfrak{m})\otimes S^-(\mathfrak{m})$, with $\{u_\alpha;\ \alpha\in J^{\pm}\}$ an orthonormal basis for $S^{\pm}(\mathfrak{m})$, and such that

$$c(x_jy_j)u_\alpha = i\alpha_ju_\alpha,\qquad j = 1,\dots,n.$$

Now if $\varphi = \sum_{\alpha\in J^{\pm}} f_\alpha\otimes u_\alpha \in C_c^\infty(G,\lambda_p)\otimes S^{\pm}(\mathfrak{m})$, the formula (7.1.9) gives

$$\langle A_p^{\mp}A_p^{\pm}\varphi,\varphi\rangle = 2\pi p\sum_{\alpha\in J^{\pm}}\left(n + \sum_j\alpha_j\right)\|f_\alpha\|^2 + \sum_j\sum_{\alpha\in J^{\pm}}\|R(z_j)f_\alpha\|^2 + \langle F\varphi,\varphi\rangle.$$

Clearly $n + \sum_j\alpha_j = 0$ only if $\alpha = (-1,\dots,-1)$ and this is a multi-index in J^+. Therefore, when $\varphi\in C_c^\infty(M,\mathcal{E}_p^-)$,

$$|\langle A_p^+A_p^-\varphi,\varphi\rangle| \ge 2\pi p\|\varphi\|^2 + \sum_j\sum_{\alpha\in J^-}\|R(z_j)f_\alpha\|^2 - |\langle F\varphi,\varphi\rangle|.$$

Since F is of the form (7.1.8), one has

$$|\langle F\varphi,\varphi\rangle| \le 2c\sum_j\sum_{\alpha,\beta\in J^-}\|R(z_j)f_\alpha\|\cdot\|f_\beta\|,$$

for some $c > 0$, independent on φ. Consequently

$$|\langle A_p^+ A_p^- \varphi, \varphi\rangle \geq (2\pi p - n 2^{2n-2} c^2)\|\varphi\|^2,$$

which implies the vanishing of $\operatorname{Ker} A_p^-$ for $p \in \mathbf{N}$ sufficiently large. Q.e.d.

7.2. The following question arises naturally: given a unimodular Lie group G and a compact subgroup H, is it possible to normalize the Haar measure in such a way that the G-index map takes only integral values? In general, the answer is negative. Indeed, a counterexample is provided by the 4-dimensional nilpotent Lie group G with compact 2-dimensional center considered in [16, §5]. Namely, $\mathfrak{g}$ is the Lie algebra generated by $\{x, y, z, w\}$ with $[x, y] = z + \theta w$, θ irrational, and all other brackets zero, and G is the quotient of the corresponding simply connected nilpotent Lie group by the central subgroup $\Gamma = \{\exp(pz + qw): p, q \in \mathbf{Z}\}$. The discrete series $\hat{G}_d$ is parametrized by pairs of integers (p, q), with $|p| + |q| \neq 0$, and, as shown in [16, loc. cit.], if $\pi_{p,q}$ denotes the irreducible square-integrable representation corresponding to (p, q), then $\deg \pi_{p,q} = |p + \theta q|$. By Proposition 7.1.A, the corresponding Dirac operators $A_{p,q}^+$ will satisfy $\operatorname{ind}_G A_{p,q}^+ = |p + \theta q|$ which is not bounded from zero when (p, q) runs over the lattice $\mathbf{Z} \times \mathbf{Z} \setminus \{0\}$.

7.3. We now come back to the general case and assume that G is a unimodular Lie group, and H a compact subgroup. Let $\varepsilon_1, \varepsilon_2$ be two unitary finite-dimensional representations of H. Then any G-invariant differential operator D: $C^\infty(M, \mathcal{E}_1) \to C^\infty(M, \mathcal{E}_2)$ can be represented as

$$D = \sum_i R(X_i) \otimes A_i, \tag{7.3.1}$$

where $X_i \in \mathfrak{U}(\mathfrak{g}_{\mathbf{C}})$ (the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$), $A_i \in \operatorname{Hom}(E_1, E_2)$, and $\sum_i X_i \otimes A_i \in (\mathfrak{U}(\mathfrak{g}_{\mathbf{C}}) \otimes \operatorname{Hom}(E_1, E_2))^H$; i.e., for any $h \in H$,

$$\sum_i \operatorname{Ad}(h) X_i \otimes \varepsilon_2(h) A_i \varepsilon_1(h)^{-1} = \sum_i X_i \otimes A_i. \tag{7.3.2}$$

Let π be a unitary representation of G on the Hilbert space $\mathcal{H}(\pi)$. We denote by $\mathcal{H}_\infty(\pi)$ the space of C^∞-vectors for π and, with D given by (7.3.1), we define the operator D_π: $(\mathcal{H}_\infty(\pi) \otimes E_1)^H \to (\mathcal{H}_\infty(\pi) \otimes E_2)^H$ by the formula

$$D_\pi = \sum_i \pi(X_i) \otimes A_i; \tag{7.3.3}$$

as before, the superscript H indicates the subspace of H-invariant elements. The invariance condition (7.3.2) ensures that D_π, indeed, maps H-invariant vectors in $\mathcal{H}_\infty(\pi) \otimes E_1$ to H-invariant vectors in $\mathcal{H}_\infty(\pi) \otimes E_2$. D_π can also be viewed as an (unbounded) operator from $(\mathcal{H}(\pi) \otimes E_1)^H$ to $(\mathcal{H}(\pi) \otimes E_2)^H$, with dense domain $(\mathcal{H}_\infty(\pi) \otimes E_1)^H$ and, as such, it is closable.

Assume now that D is elliptic. Then, it can be proved that $(D_\pi)^*$ is the closure of $(D^*)_\pi$, where D^* stands for the formal adjoint of D, and further that

$$\operatorname{Ker} D_\pi = \text{the orthogonal in } \big(\mathcal{H}_\infty(\pi) \otimes E_1\big)^H \text{ of the image of } \big(\mathcal{H}_\infty(\pi) \otimes E_2\big)^H \text{ under } D_\pi^*$$

consists exactly of all solutions $u \in (\mathcal{H}_\infty(\pi) \otimes E_1)^H$ of the equation $D_\pi u = 0$. However, since we do not really use these facts, we shall omit the proof.

The following result can be regarded as a "reciprocity theorem" and also as a generalization to the noncompact case of Theorem I in [7].

7.3.A. Proposition. *Let $G, H, \varepsilon_1, \varepsilon_2$ be as in Theorem* 6.2, *and let* D: $C^\infty(M, \mathcal{E}_1) \to C^\infty(M, \mathcal{E}_2)$ *be a G-invariant elliptic differential operator. Then*:

(i) $\operatorname{Ker} D_\pi = 0$ *for all* $\pi \in \hat{G} \setminus \hat{G}_d$ *except a set of Plancherel measure zero*;

(ii) $\operatorname{Ker} D_\pi = 0$ *for all* $\pi \in \hat{G}_d$ *except a finite number, and* $\dim \operatorname{Ker} D_\pi < \infty$ *for all* $\pi \in \hat{G}_d$;

(iii) $\operatorname{ind}_G D = \sum_{\pi \in \hat{G}_d} (\dim \operatorname{Ker} D_\pi - \dim \operatorname{Ker} D_\pi^*) \deg(\pi)$.

Proof. The Plancherel decomposition

$$\big(L^2(G) \otimes E_1\big)^H = \int_{\hat{G}}^{\oplus} \mathcal{H}(\pi) \otimes \big(\mathcal{H}(\check{\pi}) \otimes E_1\big)^H d\mu(\pi)$$

together with [19, Lemma 5] give the direct integral decomposition

$$\operatorname{Ker} D = \int_{\hat{G}}^{\oplus} \mathcal{H}(\pi) \otimes \operatorname{Ker} D_{\check{\pi}}\, d\mu(\pi). \tag{7.3.4}$$

As we saw in Section 6, the continuous part of the Plancherel measure gives no contribution, which proves (i). The assertion (ii) follows from the fact that $\dim_G \operatorname{Ker} D < \infty$, and (iii) is now obvious.

7.3.B. Corollary. *Let G be semisimple with finite center, $H = K$ a maximal compact subgroup, and $D: C_c^\infty(G/K, \mathcal{E}_1) \to C_c^\infty(G/K, \mathcal{E}_2)$ a G-invariant differential elliptic operator. Then, with $[\pi:\varepsilon_i]$ denoting the intertwining number of ε_i and the restriction of π to K $(i = 1,2)$, one has*

$$\operatorname{ind}_G D = \sum_{\pi \in \hat{G}_d} \big([\pi:\varepsilon_1] - [\pi:\varepsilon_2]\big)\deg(\pi), \tag{7.3.5}$$

the sum involving, necessarily, a finite number of non-zero terms.

Proof. This follows from Proposition 7.3.A and the fact that, for any $\pi \in \hat{G}$, $\dim(\mathcal{H}(\check{\pi}) \otimes E_i)^K = [\pi: \varepsilon_i] < \infty$.

7.4. The above result displays the remarkable fact that, under the given circumstances, $\operatorname{ind}_G D$ depends only on the initial representations ε_1, ε_2 of H, and not on the elliptic operator D. The following simple example shows that this cannot always be true.

Let G be the 3-dimensional Heisenberg group, x, y, z a basis for $\mathfrak{g}$, with $[x, y] = z$, and let $\lambda \in \mathfrak{g}^*$ be a functional such that $\lambda(z) > 0$. With the same notation as in 7.1, let D_λ: $C_c^\infty(M, \mathcal{E}_\lambda) \to C_c^\infty(M, \mathcal{E}_\lambda)$, $M = G/Z$, be the operator

$$D_\lambda = R(x + iy);$$

then $\operatorname{Ker} D_\lambda$ is the representation space of the irreducible representation π_λ, while $\operatorname{Ker} D_\lambda^* = 0$. Thus, $\operatorname{ind}_G D_\lambda = \deg(\pi_\lambda) \neq 0$.

7.5. The index theorem 5.3 applies only when the Lie group G is unimodular. We shall now indicate how to define, using K-theory for C^*-algebras, a more primitive index map. As shown in Section 1 (Proposition 1.7), any finite-dimensional unitary representation ε of the compact subgroup H gives rise to an exact sequence of separable C^*-algebras

$$0 \to C_G^*(\mathcal{E}) \to \Psi_G^*(\mathcal{E}) \to C_H(S(V), E) \to 0, \tag{7.5.1}$$

where $C_G^*(\mathcal{E})$ (resp. $\Psi_G^*(\mathcal{E})$) is the norm closure of the space of G-invariant, $\operatorname{End}(E)$-valued pseudo-differential operators $\Psi_c^{-\infty}(\mathcal{E})^G$ (resp. $\Psi_c^0(\mathcal{E})^G$) in $\mathcal{B}(L^2(M, \mathcal{E}))$. Now let ε be the left regular representation of H on $E = L^2(H)$. If we replace $\operatorname{End}(E)$ by $\mathcal{K}(E)$, the space of compact operators on $L^2(H)$, then (7.5.1) still holds, and the first term is naturally isomorphic to the reduced C^*-algebra $C_r^*(G)$ of G. Indeed the unitary operator U: $L^2(G) \otimes E \to L^2(G) \otimes E$,

$$(Uf)(h) = R(h)f(h), \qquad f \in L^2(H, L^2(G)) \cong L^2(G) \otimes E,$$

commutes with $L(g) \otimes I$ for any $g \in G$, and it transforms the orthogonal projection onto $(L^2(G) \otimes E)^H$, $e_H = \int R(h) \otimes \varepsilon(h)\, dh$, into $I \otimes e_1$, where e_1: $L^2(H) \to L^2(H)$ denotes the projection associated to the constant function 1 in $L^2(H)$. That is, $U^* e_H U = I \otimes e_1$, which shows that reducing $C_r^*(G) \otimes \mathcal{K}(E)$ by the multipliers e_H and $I \otimes e_1$ respectively yields isomorphic C^*-algebras. (There is no mystery about this isomorphism once one notices that U is actually the natural isometry establishing the unitary equivalence between the representation of G induced by the regular representation of H and the regular representation of G.) Now the six-terms exact sequence of K-groups associated to (7.5.1) gives a connecting map

$$\operatorname{Ind}_G \colon K_1(C_H(S(V), E)) \to K_0(C_r^*(G))$$

which allows us to define the index of an elliptic, G-invariant, pseudo-differential operator on $M = G/H$ as an element in the denumerable group $K_0(C_r^*(G))$.

If G is unimodular, and a choice of the two-sided Haar measure μ has been made, the corresponding trace on $C_r^*(G)$ determines an additive map μ_*: $K_0(C_r^*(G)) \to \mathbf{R}$, and one has $\mu_* \circ \operatorname{Ind}_G = \operatorname{ind}_G$. The refined index map Ind_G still enjoys the usual stability properties, so that it is reasonable to expect that it is also computable in terms of the principal symbol. In fact, this is what Bott did in [7] in the compact case; indeed, for G compact, $K_0(C^*(G))$ is canonically isomorphic to the representation ring $R(G)$. Let us also remark that it may well happen that $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$ are trivial (and so $\operatorname{ind}_G D = 0$) without $\operatorname{Ind}_G D$ being zero. For example, if $G = \mathbf{R}^2$, one has $C^*(G) = C_0(\hat{G})$ $(= C_0(\mathbf{R}^2))$ so that $K_0(C^*(G)) = K^0(\mathbf{R}^2) \cong \mathbf{Z}$, whose generator can be obtained as the index of the $\bar{\partial}$-operator on $\mathbf{R}^2 \cong \mathbf{C}$ while the equations $(\partial/\partial\bar{z})u = 0$, $(\partial/\partial z)u = 0$ have no non-trivial global L^2-solutions.

Of course, to obtain a valuable formula for the index map Ind_G, one first has to compute $K_0(C_r^*(G))$. When G is simply connected and solvable, it follows from the Thom isomorphism in [9] that $K_i(C^*(G)) \cong K^{i+j}$ (point), $i, j \in \mathbf{Z}_2$, where j is the dimension mod 2 of G. The computation of the K-theory of $C^*(G)$ for an arbitrary Lie group G and the search for an "intrinsic" index formula certainly deserve further study.

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