PROPERTY T FOR VON NEUMANN ALGEBRAS

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0. *Introduction*

Kazhdan's property T for groups **[11]** was first used in von Neumann algebras in [3] where it was shown that if Γ is a countable discrete group with property T (with infinite conjugacy classes), the fundamental group of the von Neumann algebra of Γ is countable. In [4] the first author defines a property T for type II_1 factors and claims that a discrete group Γ has property T if and only if its von Neumann algebra has this property. The rigidity problem is posed and some results are mentioned.

In this paper we define property T in a way that makes sense for any von Neumann algebra and mades clear the analogy with Kazhdan's property. The key concept is that of a correspondence which plays the role of a representation of a group. Whereas the representation theory of a II_1 factor is simple (just the coupling constant), the structure of its correspondences is very rich. There are notions of trivial correspondence and coefficients, the latter allowing one to topologize the space of correspondences. Property T means that the trivial correspondence is isolated from those that do not contain it. For II_1 factors the property is the same as that of [4], but any type I factor has property T.

We show that II_1 factors with property T fail to be amenable in a strong sense: the identity is never the limit of a sequence of completely positive compact trace decreasing maps. By U. Haagerup's result in [7], this shows that such a factor can never be embedded in the von Neumann algebra of a free group.

As an application of this result we give a solution to a problem posed by C. Sutherland in **[16]**. If Q is a (countable discrete) group and M is a factor, a Q-kernel is a homomorphism $\theta : Q \to \operatorname{Out} M$ $(= \operatorname{Aut} M / \operatorname{Int} M)$. In **[13]** M. Nakamura and Z. Takeda show how to associate a cohomology class $\operatorname{Ob}(\theta) \in H^3(Q, \pi)$ which is an obstruction to the existence of a lifting $\hat{\theta} : Q \to \operatorname{Aut} M$. In **[16]** Sutherland shows that if Q is finite or if M is infinite then $\operatorname{Ob}(\theta)$ is the only obstruction. In **[15]** A. Ocneanu shows that if Q and M are amenable then $\operatorname{Ob}(\theta)$ is the only obstruction. We show that if Γ is any infinite discrete group with property T and M is the von Neumann algebra of the free group with infinitely many generators, there is a $\theta : \Gamma \to \operatorname{Out} M$ with $\operatorname{Ob}(\theta) = 0$ but no lifting $\hat{\theta} : \Gamma \to \operatorname{Aut} M$. This also solves some related problems which we mention. The problems remain open for more general Γ, for example $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$.

1. *Property* T

Let M and N be von Neumann algebras. By a *correspondence* from M to N we shall mean a Hilbert space H which is a left M module and a right N module with commuting normal actions. Thus $a\xi b$ makes sense for $a \in M$, $b \in N$ and $\xi \in H$.

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We define a basis of neighbourhoods of a given correspondence H as follows: given $\varepsilon > 0$, $\xi_1, \ldots, \xi_n \in H$, $x_1, \ldots, x_p \in M$ and $y_1, \ldots, y_q \in N$ let $U(\varepsilon, \xi_i, x_j, y_k)$ be the set of correspondences H' from M to N such that there exist $\eta_1, \ldots, \eta_n \in H'$ with $|\langle x_j \eta_i y_k, \eta_{i'}\rangle - \langle x_j \xi_i y_k, \xi_{i'}\rangle| < \varepsilon$ for all i, i', j, k. The sets U form the basis of a topology on any set of correspondences. The bilinear functionals of the form $x \otimes y \to \langle x\xi y, \eta\rangle$ are called the *coefficients* of the correspondence. Coefficients of the form $\langle x\xi y, \xi\rangle$ define positive linear functionals on the algebraic tensor product $M \otimes N^{\mathrm{opp}}$.

For any von Neumann algebra M, the standard representation **[6]** is by construction a Hilbert space which is an M-bimodule. We shall call it the identity correspondence from M to M.

We shall say that M *has property* T if there is a neighbourhood U of the identity correspondence id_M such that any correspondence in U contains id_M as a direct summand.

To see the meaning of this notion, let us assume that M is a factor of type II_1. Then in the standard representation there is a vector $\xi_0 \in H$ with $x\xi_0 = \xi_0 x$ for all $x \in M$, $\|\xi_0\| = 1$. Conversely, if H is a correspondence from M to M and $\xi_0 \in H$, $\|\xi_0\| = 1$, is a central vector (that is, $\xi_0 x = x\xi_0$ for all $x \in M$), one has $\langle xy\xi_0, \xi_0\rangle = \langle x\xi_0 y, \xi_0\rangle = \langle x\xi_0, \xi_0 y^*\rangle = \langle yx\xi_0, \xi_0\rangle$ so that $\langle x\xi_0, \xi_0\rangle$ is the normalized trace $\mathrm{Tr}_M(x)$. The closed subspace $\overline{M\xi_0}$ is a direct summand of H which is isomorphic to id_M. Thus H contains a central vector if and only if it contains id_M. Moreover if $\xi \in H$, $\|x\xi - \xi x\|$ can be controlled by coefficients, and since the kernel of Tr_M is the linear span of commutators **[5]**, the "distance" of the linear functional $\langle x\xi, \xi\rangle$ from the trace may be controlled by expressions of the form $\|x\xi - \xi x\|$. With these comments it is a straightforward exercise to show that M has property T if and only if one can find $\varepsilon > 0$, $x_1, \ldots, x_n \in M$ such that "For any correspondence H from M to M and vector $\xi \in H$, $\|\xi\| = 1$ with $\|x_i\xi - \xi x_i\| < \varepsilon$ there exists $\eta \in H$, $\eta \neq 0$ which is central: $x\eta = \eta x$ for all $x \in M$".

The next result allows us to control the distance of an almost central vector from central vectors.

PROPOSITION 1. *If M is a II_1 factor with property* T *then* $\mathrm{Int}\, M$ *is open and there exist $\varepsilon > 0$, $y_1, \ldots, y_m \in M$, $K > 0$ such that for any $\delta \leqslant \varepsilon$ and any correspondence H from M to M with vector $\xi \in H$, $\|\xi\| = 1$, $\|y_i\xi - \xi y_i\| < \delta$ there exists a central vector η with $\|\eta - \xi\| < K\delta$.*

Proof. The proof that $\mathrm{Int}\, M$ is open is exactly as in **[3]**. Thus $\mathrm{Int}\, M$ is closed and by the characterization of full II_1 factors in **[2]**, there is a finite set $u_1, u_2 \ldots u_p$ of unitary elements in M and $c > 0$ such that $\|\xi\|^2 < c\sum \|u_i\xi - \xi u_i\|^2$ for $\xi \in L^2(M)$, $\mathrm{tr}(\xi) = 0$. The same inequality holds if M acts on a direct sum of copies of $L^2(M)$ (and tr is replaced by the projection onto central vectors).

Let $\varepsilon > 0$ and $x_1, \ldots, x_n$ be as in the reformulation of property T above. Let $\{y_i\} = \{x_j\} \cup \{u_k\}$. Let H be an arbitrary correspondence and let $\xi \in H$, $\|\xi\| \leqslant 1$ satisfy $\|y_i\xi - \xi y_i\| \leqslant \delta < \varepsilon$. Decompose H as $H = H_1 \oplus H_2$ with H_1 a multiple of id_M and H_2 containing no central vectors. If $\xi = \xi_1 \oplus \xi_2$, $\|\xi_2\| \leqslant \delta/\varepsilon$. Write $\xi_1 = \xi_1' + \xi_1''$ with ξ_1'' orthogonal to central vectors and ξ_1' central. Then $\|\xi_1''\|^2 \leqslant pc\delta^2$. Altogether

$$\|\xi - \xi_1'\|^2 = \|\xi_1'' + \xi_2\|^2 < \left((1/\varepsilon)^2 + pc\right)\delta^2 .$$

Now let Γ be a discrete group and $\mu : \Gamma \times \Gamma \to T$ be a (normalized) 2-cocycle. The left μ-regular representation of Γ on $\ell^2(\Gamma)$ is defined by

$$(\lambda_\mu(g)f)(h) = \mu(h^{-1}, g)f(g^{-1}h).$$

Then $\lambda_\mu(g)\lambda_\mu(h) = \mu(g,h)\lambda_\mu(gh)$. This projective representation is determined up to unitary equivalence by the cohomology class of μ and it has the property that if $g \mapsto u_g$ is any unitary representation of Γ then $g \to \lambda_\mu(g) \otimes u_g$ is a projective representation equivalent to a direct sum of copies of λ_μ. One may suppose $\mu(g, g^{-1}) = 1$ for all $g \in G$. Then if $Jf(g) = \overline{f(g^{-1})}$, $\ell^2(\Gamma)$ is a correspondence with $x\xi y = xJy^*J\xi$.

THEOREM 2. *Let Γ be a countable discrete group. Suppose $\lambda_\mu(\Gamma)''$ is a factor. Then $\lambda_\mu(\Gamma)''$ has property* T *if and only if Γ has property* T *of Kazhdan.*

Proof. Suppose first that Γ has property T and that Γ is a correspondence. Then if $\pi(g)\xi = \lambda_\mu(g)\xi\lambda_\mu(g)^{-1}$, π is a unitary representation. If ξ' is an almost central vector in H, it is almost fixed by π, so by the Kazhdan property there is a fixed vector ξ for π. Since the $\lambda_\mu(g)$ generate $\lambda_\mu(\Gamma)''$, $x\xi = \xi x$ for all $x \in \lambda(\Gamma)''$.

Now suppose $\lambda_\mu(\Gamma)''$ has property T. We will use the dual action of Rep(Γ) to associate correspondences with representations of Γ. Let $y_1, \ldots, y_n \in \lambda_\mu(\Gamma)''$ and ε be as in Proposition 1. We may suppose $\|y_i\|_2 = 1$ for each i. Write $y_i = \sum_{g \in \Gamma} c_g^i \lambda_\mu(g)$, and let F be a finite subset of Γ such that $\sum_{g \notin F} |c_g^i|^2 < \delta^2$. Let $g \to \pi(g)$ be a unitary representation of Γ on H and suppose that there is a $\xi \in H, \|\xi\| = 1$ with $\|\pi(f)\xi - \xi\| < \delta$ for all $f \in F$. Then define the $\lambda_\mu(\Gamma)''-\lambda_\mu(\Gamma)''$ correspondence $\ell^2(\Gamma) \otimes H$ by

$$\lambda_\mu(g_1)\xi\lambda_\mu(g_2) = (\lambda_\mu(g_1) \otimes \pi(g_1))\zeta(\lambda_\mu(g_2) \otimes \mathrm{id}).$$

Then if $\hat{\xi} \in \ell^2(\Gamma, H)$ is defined by

$$\hat{\xi}(g) = \begin{cases} \xi & \text{if } g = e \\ 0 & \text{otherwise}; \end{cases}$$

we have

$$\|y_i\hat{\xi} - \hat{\xi}y_i\|^2 = \sum_{g \in \Gamma} |c_g^i|^2 \|\pi(g)\xi - \xi\|^2$$

$$\leqslant 4\delta^2 + \sum_{f \in F} |c_f^i|^2 \|\pi(f)\xi - \xi\|^2 \leqslant 5\delta^2.$$

So for δ small we may suppose there is a central vector $\hat{\eta} \in \ell^2(\Gamma) \otimes H$ with $\|\hat{\eta} - \hat{\xi}\| \leqslant \sqrt{5}K\delta$. Thinking of $\hat{\eta}$ as a function from Γ to H, we conclude that for δ small enough, $\eta = \hat{\eta}(e) \neq 0$. But since $\hat{\eta}$ is central, η is a fixed vector for π. Thus Γ has Kazhdan's property T.

REMARK. Pursuing the analogy between correspondences and group representations one can easily check the following characterization of amenable factors: M is amenable if and only if the identity correspondence is weakly contained in the regular one. (Here the regular correspondence means the one coming from the Hilbert space of Hilbert–Schmidt operators on the identity correspondence.)

We shall now show a strong non-amenability of a II_1 factor with property T. It implies that it is not contained in the von Neumann algebra of any free group.

THEOREM 3. *Let M be a factor of type II_1 with property* T. *Then there is no sequence Ψ_n of completely positive maps with $\Psi_n(1) \leqslant 1, \mathrm{tr} \circ \Psi_n \leqslant \mathrm{tr}, \Psi_n(x) \to x$ (in L^2) for all $x \in M$, such that Ψ_n is a compact operator in $L^2(M)$.*

Proof. By interpolation (see [14]) Ψ_n defines a contraction in any $L^p(M)$ space for $p \in [1, \infty]$. (We only need $p = 2$ and 4 and a simple argument can be made to avoid interpolation.) Let H_n be the Hilbert space associated with $M \otimes M^{\mathrm{opp}}$ (algebraic tensor product) and the positive sesquilinear form $\langle a \otimes b, c \otimes d\rangle = \mathrm{Tr}(\Psi_n(ac^*)bd^*)$. Letting M act on the left and right in the obvious way, H_n becomes a correspondence from M to M and $\xi_n = 1 \otimes 1 \in H_n$ has the property that

$$\langle x\xi_n y, \xi_n\rangle = \mathrm{tr}(\Psi_n(x)y) \quad x, y \in M .$$

Since M has property T, there exists by Proposition 1 a sequence of vectors $\eta_n \in H_n, \|\eta_n - \xi_n\| \to 0$ with $\eta_n x = x\eta_n$ for all $x \in M$. To simplify notation suppose $\Psi_n(1) = 1, \|\xi_n\| = 1$, and $\|\eta_n\| = 1$.

Let us estimate $\|x\xi_n - x\eta_n\|$ for any $x \in M$. We have

$$\langle x\xi_n, x\xi_n\rangle = \mathrm{tr}(\Psi_n(x^*x)) \leqslant \langle x\eta_n, x\eta_n\rangle$$

so that

$$\|x\xi_n - x\eta_n\|^2 \leqslant 2\langle x\eta_n, x\eta_n\rangle - 2\mathrm{Re}\langle x\xi_n, x\eta_n\rangle$$

$$= 2\mathrm{Re}\langle \eta_n - \xi_n, x^*x\eta_n\rangle \leqslant 2\|\eta_n - \xi_n\| \, \|x\|_4^2 ,$$

where $\|-\|_4$ is the norm in $L^4(M)$. Thus $\|x\xi_n - x\eta_n\| \leqslant \sqrt{2}\|\eta_n - \xi_n\|^{\frac{1}{2}}\|x\|_4$. Similarly $\|\xi_n y - \eta_n y\| \leqslant \sqrt{2}\|\eta_n - \xi_n\|^{\frac{1}{2}}\|y\|_4$ and $\|\Psi_n(x)\xi_n\| \leqslant \|x\|_2 \leqslant \|x\|_4$. So

$$|\mathrm{tr}(\Psi_n(x)y) - \mathrm{tr}(xy)| \leqslant |\langle \Psi_n(x)\xi_n, (\xi_n - \eta_n)y^*\rangle| + |\langle \Psi_n(x)(\xi_n - \eta_n), \eta_n y^*\rangle|$$

$$\leqslant 2\sqrt{2}\|\eta_n - \xi_n\|^{\frac{1}{2}}\|x\|_4\|y\|_4 .$$

This shows that for arbitrary $\varepsilon > 0$, $\|\Psi_n(x) - x\|_2 < \varepsilon\|x\|_4$ for large n (remember $\|\Psi_n(x)\|_4 \leqslant \|x\|_4$).

Now let $A = L^\infty((\mathbb{Z}_2)^{\mathbb{N}})$ be a maximal abelian subalgebra of M where $\mathbb{Z}_2{}^{\mathbb{N}}$ has the $(\frac{1}{2}, \frac{1}{2})$ product measure. On the linear span $\mathscr{L}$ of the functions

$$\chi_i(x) = (-1)^{x_i} \quad (x = (x_i) \in (\mathbb{Z}_2)^{\mathbb{N}})$$

the Khintchine inequality [12] shows precisely that there is a $C > 0$ such that $\|\xi\|_4 \leqslant C\|\xi\|_2$ for all $\xi \in \mathscr{L}$ (indeed we may choose $C = 3^{\frac{1}{4}}$ by [8]). Thus for large n and any $\xi \in \mathscr{L} \subseteq L^2(M)$ we have $\|\Psi_n(\xi) - \xi\|_2 \leqslant \frac{1}{2}\|\xi\|_2$ which contradicts the compactness of Ψ_n.

COROLLARY 4. *Let M be a factor of type II_1 with property* T. *Then M is not a subfactor of $\lambda(F_n)''$ where F_n is the free group on n generators.*

Proof. In [7] Haagerup shows that for $\mu > 0$ the function $\phi(w) = e^{-\mu \text{ length }(w)}$ is a positive definite function on F_n. He also shows that the map $M_\phi(\lambda(s)) = \phi(s)\lambda(s)$ extends to a completely positive map $M_\phi : C^*_\lambda(F_n) \to C^*_\lambda(F_n)$. But his proof shows that M_ϕ extends to a completely positive map from $\lambda(F_n)'' \to \lambda(F_n)''$ which is clearly compact on $\ell^2(F_n) = L^2(\lambda(F_n)'')$. As $\mu \to 0$, $M_\phi \to \text{id}$ in $\|-\|_2$, also $\text{tr} \circ M_\phi = \text{tr}$. If M were contained in $\lambda(F_n)''$ then composing M_ϕ with the conditional expectation onto M gives a sequence contradicting Theorem 3. Note that the conclusion of the theorem is also valid for $n = \infty$.

2. *Q-kernels*

A Q-kernel is a homomorphism $\theta : Q \to \text{Out}\, M$ (M a factor). The obstruction $\text{Ob}(\theta) \in H^3(Q, T)$ is defined as follows. For each $q \in Q$ choose $\alpha_q \in \text{Aut}\, M$ lifting $\theta(q)$. Then $\alpha_q \alpha_r = \text{Ad}\, u(q,r)\alpha_{qr}$ for unitary elements $u(q,r) \in M$ and associativity implies $u(q,r)u(qr,s) = \chi(q,r,s)\alpha_q(u(r,s))u_{q,rs}$ for some 3-cocycle $\chi(q,r,s)$. The class of χ does not depend on the choices made. The class of χ is precisely the obstruction to the existence of an extension N of M together with a map $v : Q \to U(N)$ with $v(q)Mv(q)^* = M$ and $\text{Ad}\, v(q)|_M = \alpha_q \bmod \text{Int}\, M$. The class of χ is also clearly an obstruction to the existence of a homomorphism $\hat{\theta} : Q \to \text{Aut}\, M$ lifting θ. (For details see [**16**].)

THEOREM 5. *There is a separable* II_1 *factor* M, *a countable discrete group* Q *and an injective* Q*-kernel* $\theta : Q \to \text{Out}\, M$ *with* $\text{Ob}(\theta) = 0$ *such that* θ *does not admit a lifting* $\tilde{\theta} : Q \to \text{Aut}\, M$.

Proof. Let Q be any infinite countable discrete group with property T (for example $\text{SL}(3, \mathbb{Z})$). Choose a presentation $1 \to F_\infty \to F_n \xrightarrow{\pi} Q \to 1$ and a set theoretic section $q \to \hat{q}$ for π. Then $\lambda(F_\infty)''$ sits as a subfactor of $\lambda(F_n)''$ in the obvious way. Moreover, since $\hat{q}\hat{r}\widehat{qr}^{-1} \in F_\infty$ it is clear that the map $q \mapsto \alpha_q$, $\alpha_q(x) = \lambda(\hat{q})x\lambda(\hat{q})^{-1}$ defines a homomorphism $\theta : Q \to \text{Out}(\lambda(F_\infty)'')$ which is injective since $\lambda(F_\infty)' \cap \lambda(F_n)'' = \mathbb{C}$. The obstruction $\text{Ob}(\theta)$ is zero because the extension exists by construction. We claim that $q \mapsto \alpha_q$ cannot be perturbed by inner automorphisms so as to become an action. For if this were possible one could find unitary elements u_q in $\lambda(F_\infty)''$ with $(\text{Ad}\, u_q\alpha_q)(\text{Ad}\, u_r\alpha_r) = \text{Ad}\, u_{qr}\alpha_{qr}$. But then $u_q\alpha_q(u_r) = \mu(q,r)u_{qr}$ for a 2-cocycle $\mu : Q \times Q \to I$, and the mutually orthogonal unitary elements $u_q\lambda(\hat{q})$ would generate a copy of $\lambda_\mu(Q)''$ inside $\lambda(F_n)''$. Together with Theorem 2 this contradicts Corollary 4.

If $N \subseteq M$ are II_1 factors with $N' \cap M = \mathbb{C}$ and the unitary normalizer $\mathcal{N}(N)$ generates M (that is, N is regular), it is shown in [**1**] that M decomposes as a kind of crossed product $N \rtimes G$ where $G = \mathcal{N}(N)/U(N)$ is discrete. The question of whether this twisted crossed product is necessarily an ordinary one is Problem 4 in [**9**]. It is clear from the proof of the above theorem that the subfactor $\lambda(F_\infty)'' \subseteq \lambda(F_n)''$ gives a counter example.

It is well known that automorphisms of factors are almost always spatially implemented. The only exception is when a II_∞ factor M acts with II_1 commutant. Here an automorphism is spatially implemented if and only if it preserves the trace on M (see [**10**]). One may further ask if a group of trace preserving automorphisms can be implemented by a unitary representation of the group. The next result shows that the answer is no in general.

COROLLARY 6. *There is a separable* II_∞ *factor* M *acting on a Hilbert space* $\mathscr{H}$ *with* M' *of type* II_1 *and a trace-preserving action of a countable discrete group* G *on* M *which cannot be implemented by a unitary representation of* G *on* $\mathscr{H}$.

Proof. Let $G = Q$ as in Theorem 5. Since $[F_n : F_\infty] = \infty$, $\lambda(F_\infty)'$ is of type II_∞ and $q \to \mathrm{Ad}\,\lambda(\hat{q})$ defines an action of Q which is trace preserving. A unitary implementation of this action would differ from $q \to \lambda(\hat{q})$ by elements in $\lambda(F_\infty)''$ thus giving an embedding of $\lambda(G)''$ in $\lambda(F_n)''$ as above.

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