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Ergodic Theory and Dynamical Systems / Volume 5 / Issue 02 / June 1985, pp 203 - 236
DOI: 10.1017/S0143385700002868, Published online: 19 September 2008

Ergod. Th. & Dynam. Sys. (1985), **5**, 203–236
Printed in Great Britain

Approximately transitive flows and ITPFI factors

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(*Received* 27 *October* 1983)

This paper is dedicated to Richard V. Kadison on the occasion of his sixtieth birthday.

Abstract. We define a new property of a Borel group action on a Lebesgue measure space, which we call approximate transitivity. Our main results are (i) a type III_0 hyperfinite factor is ITPFI if and only if its flow of weights is approximately transitive, and (ii) for ergodic transformations preserving a finite measure, approximate transitivity implies zero entropy.

0. *Introduction*

von Neumann algebras are the non-commutative analogue of measure theory spaces. The product measures of measures on finite sets give rise to a class of factors called ITPFI factors (see terminology). However, in the classification problem the most natural class turned out to be the approximately type I factors [**4**] (those factors which are well-approximated by finite-dimensional ones – see terminology). It is trivial that ITPFI implies approximately type I, but the converse is false and non-trivial [**12**], [**2**], [**6**]. (Since all non-atomic standard Borel measures are Borel isomorphic, the corresponding problem does not arise in measure theory.) The ITPFI factors are certainly the most natural subclass of the approximately type I factors. Their exact position among the approximately type I factors (up to isomorphism of factors) has remained an interesting mystery for some time. In particular, there is still no direct spatial construction of a non-ITPFI approximately type I factor. The crucial existential step is always carried out in the flow of weights (an ergodic flow which is naturally defined as an invariant of the factor). The problem only arises for factors of type III_0 or III_1. The known examples of non-ITPFI approximately type I factors are all of type III_0. In this paper we completely characterize the ITPFI factors among the type III_0 approximately type I factors by a new ergodic property of their complete invariant, the flow of weights, which we call *approximate transitivity*. Of course this transfers the original problem to understanding approximate transitivity for ergodic flows. Our second major result is that for finite measure preserving flows, approximate transitivity implies zero entropy.

Approximate transitivity is a new and apparently interesting notion in ergodic theory. Our proof that ITPFI implies approximate transitivity is rather straightforward (lemma 8.1) and is, in fact, how this property was discovered. Our proof of the converse is of particular interest because it is obtained by attempting directly to 'invert the flow of weights arrow'. The argument is quite similar to the Murray–von Neumann proof of the uniqueness of the hyperfinite II_1 factor **[15]**. They embed a finite-dimensional algebra in a finite type I factor in a very precise manner relative to the trace. For type III factors no trace exists and, instead of only comparing minimal projections inside the finite-dimensional algebra, we compare such minimal projections together with the restriction of a state to these projections. This comparison directly yields measure theoretical objects on the flow of weights, and (except in the III_1 case) is non-trivial even though the comparison of projections is trivial. Our paper is intended to illustrate this technique. In fact an alternate proof of Krieger's theorem, not using the cohomological technique of Krieger, can be based on the original Murray–von Neumann proof of the uniqueness of the hyperfinite II_1 together with the above refined comparison of projections. (Krieger's theorem **[13]** states, in part, that the flow of weights considered as a mapping from type III_0 Krieger factors with algebraic isomorphism as the equivalence relation, to strictly ergodic flows with conjugacy as the equivalence relation, is one-to-one and onto between equivalence classes.)

It is easy to translate our proof to the purely ergodic setting of non-singular transformations. Our result would then follow from Krieger's theorem (our proof does not use Krieger's theorem). However, as mentioned above, part of our goal was to exhibit the flow of weights as a useful technique. Indeed we present more of the comparison theory of finite weights than is needed for our proof.

§ 1 contains some terminology. In § 2 we define approximate transitivity (hereafter referred to as AT) for Borel group actions and give some elementary properties. In § 3 we prove that for finite measure preserving transformations, AT implies zero entropy. In § 4 we give three different constructions of the flow of weights which will be used later. § 5 contains a comparison theory for finite periodic weights, and § 6 gives the comparison theory for finite (not necessarily periodic) weights. In § 7 we introduce a 'product property' which is equivalent to being ITPFI. In § 8 we prove the equivalence of the ITPFI and AT properties.

1. *Terminology*

A von Neumann algebra M is said to be *approximately type* I if it is of the form

$$M = \left(\bigcup_{n=1}^{\infty} M_n \right)'',$$

where $M_n \subset M_{n+1}$ for each n, and each M_n is a finite-dimensional matrix algebra (the names approximately finite, hyperfinite, approximately finite dimensional, and matricial have all been used in the literature for this concept). A factor M is said to be ITPFI if it is of the form $M=\bigotimes_{n=1}^{\infty}(M_n, \phi_n)$ where each M_n is a finite type I factor ([1], [21]). A factor M is called a *Krieger factor* if it can be obtained from an ergodic action of $\mathbb{Z}$ by the Murray-von Neumann group measure space construction. (It is straightforward that ITPFI⇒Krieger⇒approximately type I.)

For a detailed explanation of the following standard terminology see, for example [20]. The term weight always means a normal semi-finite weight. If ϕ is a weight on the von Neumann algebra M then σ_t^ϕ is the modular automorphism group. The invariant $T(M)$ is the set of all t such that σ_t^ϕ is inner. M_* denotes the predual of M, and M_*^+ is then the set of all finite weights. If $\psi\in M_*^+$ then $s(\psi)$ denotes the support of ψ. The flow of weights of M is an ergodic action of $\mathbb{R}_+^*$ on some measure space (X_M, μ_M). The construction of [5] gives not that measure space, but the measure algebra whose elements are equivalence classes $[\Phi]$ of integrable weights Φ of infinite multiplicity. The flow is then defined by $\mathcal{F}_t^M[\Phi]=[t\Phi]$. It is sometimes convenient to consider the flow as an action of $\mathbb{R}$, in which case it is written $F_t^M=\mathcal{F}_{e^t}^M$ (if M is understood, it is usually omitted).

If f is a function on a measure space (X,μ) then $\|f\|$ denotes the L^1-norm of f. If μ, ν are finite measures on X then $\|\mu-\nu\|$ is the L^1-norm defined by $\|d\mu/d\sigma-d\nu/d\sigma\|$ where $\mu, \nu<\sigma$. If x is a finite weight or operator then $\|x\|$ denotes the usual norm.

2. AT *actions-elementary properties*

We define approximate transitivity of a Borel group action on a Lebesgue measure space, and establish some elementary properties.

Definition 2.1. Let G be a Borel group, (X,ν) a Lebesgue measure space, and $\alpha: G\to \mathrm{Aut}(X,\nu)$ a Borel homomorphism. We say that the action is *approximately transitive* (AT) if given $n<\infty$, finite measures $\mu_1,\dots,\mu_n<\nu$, and $\varepsilon>0$, there exists a finite measure $\mu<\nu$, $g_1,\dots,g_m\in G$ for some $m<\infty$, and $\lambda_{jk}\geq 0$, $k=1,\dots,m$ such that

$$\left\|\mu_j-\sum_{k=1}^{m}\lambda_{jk}\alpha_{g_k}\mu\right\|\leq\varepsilon, \qquad j=1,\dots,n. \tag{2.1}$$

If $G=\mathbb{Z}$ and α is AT, then we say that $T=\alpha(1)$ is AT.

Remark 2.2. There are a number of elementary variations on this definition.

(i) The index k need not be restricted to a finite set. Typically we will take $k\in\mathbb{Z}$ and consider λ_{jk} as a function $\lambda_j\in\ell_+^1(\mathbb{Z})$.

(ii) One can demand that $\|\mu\|=1$ and $\|\lambda_j\|=\sum_k\lambda_{jk}=\|\mu_j\|$.

(iii) By taking $\|\mu\|$ sufficiently small, one can take the λ_{jk} to be integers.

(iv) It is sufficient to ask that eq. (2.1) hold for $n=2$. (If μ' approximates $\mu_1,\dots,\mu_{n-1}$ in the sense of eq. (2.1), choose μ to approximate μ' and μ_n).

(v) For continuous actions of a locally compact group, the λ_{jk} can be replaced by functions $\lambda_j\in L_+^1(G, dg)$ such that

$$\left\|\mu_j-\int_G dg\,\lambda_j(g)\alpha_g\mu\right\|\leq\varepsilon, \qquad j=1,\dots,n. \tag{2.2}$$

To show that eq. (2.2) implies eq. (2.1), approximate the $\lambda_j(g)$ by simple functions. To prove the converse, write the sum as an integral over delta functions $\delta(gg_k^{-1})\lambda_{jk}$ and then approximate by functions in $L^1_+(G, dg)$.

(vi) The equation $h = d\sigma/d\nu$ gives a one-to-one correspondence between functions $h \in L^1_+(X, \nu)$ and finite measures $\sigma < \nu$. We have

$$d(\alpha_g\sigma)/d\nu = \rho_g\alpha_g(d\sigma/d\nu),$$

where $\rho_g = d(\alpha_g\nu)/d\nu$. Then

$$(\beta_g f)(x) = f(g^{-1}x)\rho_g(x) \tag{2.3}$$

defines a homomorphism β from G into the invertible isometries on $L^1(X, \nu)$. It is therefore equivalent to ask that for any $f_1, \ldots, f_n \in L^1_+(X, \nu)$ and $\varepsilon > 0$, there exist $f \in L^1_+(X, \nu)$, $g_1, \ldots, g_m \in G$ and $\lambda_{jk} \geq 0$ such that

$$\left\| f_j - \sum_{k=1}^{m} \lambda_{jk}\beta_{g_k} f \right\| \leq \varepsilon, \qquad j = 1, \ldots, n. \tag{2.4}$$

If $\alpha_g\nu = \nu$ for all $g \in G$, then $\beta = \alpha$.

LEMMA 2.3. *An* AT *action is ergodic.*

Proof. Let $B \subset X$, $\nu(B) > 0$, $\nu(X \setminus B) > 0$, and $\alpha_g B = B$ for all $g \in G$. Choose $B_1 \subset B$ and $B_2 \subset X \setminus B$ such that $0 < \nu(B_j) < \infty$, $j = 1, 2$. Let $\mu_j = \nu|_{B_j}$. Then eq. (2.1) for $j = 2$ implies that $\mu(B) < \varepsilon$. If $\varepsilon < \frac{1}{2}\nu(B_1)$ this contradicts eq. (2.1) for $j = 1$. □

Remark 2.4. Let $(X, \mathscr{B}, \nu, G, \alpha)$ be a Borel group action, $\nu(X) < \infty$. Let $\mathscr{B}_0$ be a sub-σ-algebra of the σ-algebra $\mathscr{B}$ of Borel subsets of X, such that $\alpha_g\mathscr{B}_0 = \mathscr{B}_0$ for all $g \in G$. Then the restriction $(X, \mathscr{B}_0, \nu, G, \alpha)$ is called a *factor action* of the given action. If σ is any finite measure on $(X, \mathscr{B})$ we have $\|\sigma\|_{(X,\mathscr{B}_0)} \leq \|\sigma\|_{(X,\mathscr{B})}$. Hence any factor action of an AT action is again AT.

The base and ceiling function construction of a flow is particularly useful when the ceiling function is constant. In this situation one naturally expects that the flow will have a certain property if and only if the base transformation has the corresponding property.

LEMMA 2.5. *Let (X, ν, F_s) be a flow built over the base transformation (B, ν_B, T) with a ceiling function with constant height H. Then F_s is* AT *if and only if T is* AT.

Proof. We can write $X = B \times I$ where the interval $I = [0, H)$ carries Lebesgue measure. We have

$$F_s(b, t) = (T^n b, u), \tag{2.5}$$

where $s + t = nH + u$, $n \in \mathbb{Z}$, $0 \leq u < H$.

Assume that T is AT. Let $\mu_1, \ldots, \mu_n < \nu$, $\mu_j(X) < \infty$, and $\varepsilon > 0$. Since rectangles generate the measure algebra, it follows by a straighforward but tedious argument that one can approximate the μ_j by a sum of product measures. More precisely, there exists an integer L and measures μ_{jk}, $k = 0, \ldots, L-1$ on B such that

$$\left\| \mu_j - \sum_{k=0}^{L-1} \mu_{jk} \times m_k \right\| < \varepsilon, \qquad j = 1, \ldots, n \tag{2.6}$$

where m_k is the restriction of Lebesgue measure to the interval $L_k = [kH/L, (k+1)H/L)$. Since T is AT there exist $\sigma_B < \nu_B$ and $\lambda_{jk} \in \ell^1_+(\mathbb{Z})$ such that

$$\left\| \mu_{jk} - \sum_{q\in\mathbb{Z}} \lambda_{jk}(q) T^q \sigma_B \right\| < \varepsilon L^{-1}. \tag{2.7}$$

Define

$$\sigma = \sigma_B \times m_0 \tag{2.8}$$

and

$$\lambda_j(qL+k) = \lambda_{jk}(q), \qquad q \in \mathbb{Z}, k = 0, \ldots, L-1. \tag{2.9}$$

Then

$$\begin{aligned} \left\| \mu_j - \sum_{r\in\mathbb{Z}} \lambda_j(r) F_{rH/L}\sigma \right\| &< \varepsilon + \sum_{k=0}^{L-1} \left\| \left(\mu_{jk} \times \varepsilon_k - \sum_{r\in\mathbb{Z}} \lambda_j(r) F_{rH/L}\sigma \right) \Big|_{B\times I_k} \right\| \\ &= \varepsilon + \sum_{k=0}^{L-1} \left\| \mu_{jk} - \sum_{q\in\mathbb{Z}} \lambda_{jk}(q) T^q \sigma_B \right\| < 2\varepsilon. \end{aligned} \tag{2.10}$$

Thus F_s is AT.

Now assume that F_s is AT. Let $\mu_1, \ldots, \mu_n < \nu_B$, $\mu_j(B) = 1$, and $0 < \varepsilon < \frac{1}{2}$. Let $J_k = [kH/6, (k+1)H/6)$, $k = 0, 1, \ldots, 5$, and let m^k denote the restriction of Lebesgue measure to J_k. Let

$$\tilde{\mu}_j = \mu_j \times m^3, \qquad j = 1, \ldots, n. \tag{2.11}$$

Then there exist $\mu < \nu$, $\|\mu\| = 1$ and $\lambda_j \in \ell^1_+(\mathbb{Z})$, $\|\lambda_j\| = 1$, and $s_k \in \mathbb{R}$, $k \in \mathbb{Z}$ such that

$$\left\| \tilde{\mu}_j - \sum_{k\in\mathbb{Z}} \lambda_j(k) F_{s_k}\mu \right\| < \varepsilon/6, \qquad j = 1, \ldots, n. \tag{2.12}$$

In particular we have

$$\left\| \left(\sum_{k\in\mathbb{Z}} \lambda_j(k) F_{s_k}\mu \right) \Big|_{B\times J_3^c} \right\| < \varepsilon/6. \tag{2.13}$$

In order to produce the desired measure σ on B and $\Lambda_j \in \ell^1_+(\mathbb{Z})$ it is necessary to restrict the supports of λ_j and μ somewhat. Since $\|\mu\| = 1$ there is some $0 \le K \le 5$ such that

$$\|\mu|_{B\times J_K}\| \ge \tfrac{1}{6}. \tag{2.14}$$

By shifting the s_k and shifting μ under F_s (if necessary) we can assume that $K = 4$. Let

$$Y = \{k \in \mathbb{Z} : 0 \le s_k < \tfrac{1}{3}H \pmod{H}\}. \tag{2.15}$$

Then $k \notin Y$ implies that $(F_{s_k}(\mu|_{B\times J_4}))(B \times J_3) = 0$. It now follows from eqs. (2.13) and (2.14) that

$$\sum_{k\notin Y} \lambda_j(k) < \varepsilon, \tag{2.16}$$

and thus

$$\sum_{k\in Y} \lambda_j(k) > 1 - \varepsilon, \qquad j = 1, \ldots, n. \tag{2.17}$$

Write $\mu=\mu'+\mu''$ where

$$\mu''=\sum_{k=0}^{2}\mu\Big|_{B\times J_k}. \tag{2.18}$$

If $k\in Y$ then $(F_{s_k}\mu'')(B\times J_3)=0$, and eqs. (2.13) and (2.17) imply that

$$\|\mu''\|<(1-\varepsilon)^{-1}\varepsilon/6<\tfrac{1}{3}\varepsilon. \tag{2.19}$$

Eqs. (2.17) and (2.19) imply that

$$\left\|\sum_{k\in\mathbf{Z}}\lambda_j(k)F_{s_k}\mu-\sum_{k\in Y}\lambda_j(k)F_{s_k}\mu'\right\|<\tfrac{4}{3}\varepsilon. \tag{2.20}$$

Let $P_*\sigma$ denote the canonical projection of the finite measure σ on $X=B\times I$ onto B. We have

$$P_*\tilde{\mu}_j=(H/6)\mu_j \tag{2.21}$$

and

$$P_*F_{s_k}\mu'=T^p(P_*\mu') \tag{2.22}$$

if $s_k\in Y_p$ where

$$Y_p=\{k\in Y\colon pH\le s_k<(p+\tfrac{1}{3})H\}. \tag{2.23}$$

(It is eq. (2.22) that depends crucially on the support properties.) Let

$$\Lambda_j(p)=\sum_{k\in Y_p}\lambda_j(k). \tag{2.24}$$

Since P_* is norm decreasing, eqs. (2.12), (2.20) and (2.21)-(2.24) give

$$\left\|\mu_j-\sum_{p\in\mathbf{Z}}\Lambda_j(p)T^p\sigma\right\|<8H^{-1}\varepsilon, \tag{2.25}$$

where $\sigma=6H^{-1}P_*\mu'$. □

The tower construction of a single transformation (see for example [9]) is the analogue of the base and ceiling function construction of a flow.

COROLLARY 2.6. *Let the transformation (X,ν,S) be constructed as a tower over (B,ν_B,T) with constant height H. Then S is* AT *if and only if T is* AT.

Proof. The flow F_s built over (X,ν,S) with constant height one is obviously a flow built over (B,ν_B,T) with constant height H. The result now follows from lemma 2.5. □

Recall that a finite measure preserving transformation is said to have *rank one* if there is a sequence of Rohlin towers which approximate the measure algebra. More precisely one asks that given $f_1,\dots,f_n\in L^1(x,\nu)$ and $\varepsilon>0$, there exist $B\subset X$, $m<\infty$, and $\lambda_{jk}\in\mathbb{R}$ such that $B, TB,\dots,T^mB$ are disjoint and

$$\left\|f_j-\sum_{k=0}^{m}\lambda_{jk}\chi_{T^kB}\right\|<\varepsilon,\qquad j=1,\dots,n. \tag{2.26}$$

(Note that if $f_j\ge 0$ we can require that $\lambda_{jk}\ge 0$.) A slightly weaker condition, called *funny rank one*, is obtained by replacing the sequence $B, TB,\dots,T^mB$ by the sets $T^{n_0}B, T^{n_1}B,\dots,T^{n_m}B$ where $\{n_j\}$ is an arbitrary sequence (depending on $f_1,\dots,f_n$ and $\varepsilon\}$.

LEMMA 2.7. *Given (X, T, ν) with $T\nu=\nu$, $\nu(X)=1$. Then funny rank one implies* AT.

Proof. It is convenient here to use the AT condition on functions in $L^1_+(X, \nu)$ (see remark 2.2 (iv)). Let $f_1, \ldots, f_n \in L^1_+(X, \nu)$, $\varepsilon>0$. Then there exist $B\subset X$, a sequence $\{n_k\}_{k=0,\ldots,m}$ and λ_{jk} such that

$$\left\| f_j - \sum_{k=1}^{m} \lambda_{jk}\chi_{T^{n_k}B}\right\| > \varepsilon. \tag{2.27}$$

Let

$$\lambda'_{jk}=\begin{cases}\lambda_{jk} & \text{if } \lambda_{jk}\geq 0,\\ 0 & \text{if } \lambda_{jk}<0.\end{cases} \tag{2.28}$$

Since the $T^{n_k}B$ are disjoint, we have

$$\|f_j-\sum \lambda'_{jk}T^{-n_k}f\| \leq \|f_j-\sum \lambda_{jk}T^{-n_k}f\|, \tag{2.29}$$

where $f=\chi_B$. □

COROLLARY 2.8. *If (X, T, ν) has pure point spectrum then T is* AT.

Proof. Pure point spectrum implies rank one [**11**]. □

COROLLARY 2.9. *A pure point spectrum flow is* AT.

Proof. Such a flow can be built over a pure point base transformation with a constant ceiling function. The result now folllows from corollary 2.8 and lemma 2.5. □

It is also known that certain diffeomorphisms of the circle are AT [**10**].

3. AT *transformations and entropy*

In this section we prove that if (X, μ, T) is a finite measure preserving transformation, then AT implies that T has zero entropy (theorem 3.5).

In analyzing the implications of the AT condition one immediately observes that an expression of the form

$$T(\lambda)f=\sum \lambda_j T^j f, \tag{3.1}$$

where $\lambda\in\ell^1_+(\mathbb{Z})$ and $f\in L^1_+(X,\mu)$, is in effect a convolution, and 'convolutions spread functions out'. This spreading can present some difficulties when one tries to satisfy the AT condition for functions $f_1=\chi_{A_1}, f_2=\chi_{A_2}$ where $A_1, A_2\subset X$. In order to give a precise meaning to the idea that $T(\lambda)f$ is 'less concentrated' than f, we define upper and lower truncations of L^1 functions (definition 3.1). The upper truncation is used to measure the concentration of f (see eq. (3.5)), and lemma 3.2 then gives a precise meaning to the statement that convolutions spread. We give a 'spectral analysis' of L^1 functions, which allows one to handle the difficulties that arise in an argument when L^1 functions take on either very large or very small values.

Lemmas 3.3 and 3.4 are technical lemmas required for the proof of theorem 3.5. (Hint: read the proof of theorem 3.5 before reading lemmas 3.3 and 3.4.) The basic idea of the argument is as follows. One applies the AT condition to $f_1=\chi_A$ for some $A\subset X$, a second function f_2, and some $\varepsilon>0$. This gives functions λ_1, λ_2 and f satisfying $\|T(\lambda_i)f-f_i\|\leq\varepsilon$, $i=1,2$. Since convolutions spread, choosing f_2 very concentrated relative to the set A forces f to be very concentrated relative to f_1.

This in turn forces λ_1 to be very 'spread out' (lemma 3.3). However λ_1 being very spread out makes it difficult to keep the support of $T(\lambda_1)f$ close to A, which it must approximate, and simultaneously to keep $T(\lambda_1)f$ small on A^c. In particular, when the partition $(A, X\setminus A)$ moves independently under T, this becomes impossible (theorem 3.5). In order to make this argument precise, one must replace λ_1 and f by functions $\lambda \cong \lambda_1$ and $g \cong f$ with better support properties. This is done in lemma 3.4 by a technical application of the spectral analysis for L^1 functions. The condition that $T(\lambda)g$ be small on A^c then forces the support of $T(\lambda)g$ to be too small to approximate A. In particular, the proof seems somewhat stronger than the statement of the theorem. It suffices that $k\mu(B)\to 0$ (see eqs. (3.30), (3.31), (3.41), (3.42), (3.52), (3.53)).

Let (X, μ) be a Lebesgue measure space. Then (X, μ) is isomorphic to Lebesgue measure on [0, 1]. Let $f\in L^1_+(X, \mu)$. Then one can choose the isomorphism so that the (transformed) function f is monotone decreasing. The upper (resp. lower) truncation of f is a function whose graph is identical to the graph of f, except that the upper left hand corner (resp. lower right hand corner) has been 'chopped off'. More precisely we have:

Definition 3.1. Let $f\in L^1_+(X, \mu)$, $a>0$. We define *the upper truncation of f at a* by

$$f^{[a]}(x)=\begin{cases} f(x) & \text{if } f(x)\le a,\\ a & \text{if } f(x)>a,\end{cases} \tag{3.2}$$

and the *lower truncation of f at a* by

$$f_{[a]}(x)=\begin{cases} f(x) & \text{if } f(x)\ge a,\\ 0 & \text{if } f(x)<a,\end{cases} \tag{3.3}$$

The continuity of upper truncations is expressed by the condition

$$\|f-g\|\le\varepsilon \Rightarrow \|f^{[a]}-g^{[a]}\|\le\varepsilon, \tag{3.4}$$

which follows immediately from eq. (3.2). It should be noted that eq. (3.4) does not hold for lower truncations.

Consider the inequality

$$\|f-f^{[a]}\|\ge\|f\|-\eta, \tag{3.5}$$

where $a, \eta>0$. If η is small compared to $\|f\|$, this inequality forces most of the contribution to the L^1 norm of f to come from that part of the graph of f lying above a. If in addition a is large compared to $\|f\|$, it then forces most of f (in the sense of L^1 norm) to be supported on a set of small measure. It can therefore be used as a measure of the 'concentration' of a function. The following lemma now gives a precise meaning to the statement that 'convolutions spread'.

LEMMA 3.2. *Given (X, μ, T), $\lambda\in\ell^1_+(\mathbb{Z})$, $\|\lambda\|=1$, and $f\in L^1_+(X, \mu)$, then*

$$\|T(\lambda)f-(T\lambda)f)^{[a]}\|\le\|f-f^{[a]}\|. \tag{3.6}$$

Proof. If f_j, $\sum\lambda_j f_j\in L^1_+(X, \mu)$ then it follows directly from eq. (3.2) that

$$(\sum\lambda_j f_j)^{[a]}(x)\ge\sum\lambda_j(f_j^{[a]})(x). \tag{3.7}$$

Since $\|f-f^{[a]}\|=\|f\|-\|f^{[a]}\|$, and $\|T(\lambda)f\|=\|f\|$, eq. (3.6) follows immediately. □

We now proceed to the spectral analysis of L^1 functions. This analysis is, in effect, a consequence of considering $\int f d\mu$ in terms of horizontal slices of the graph of f. Let $f \in L^1_+(X, \mu)$. For each $a > 0$ we define a function $E_a(f)$ on X by

$$(E_a(f))(x) = \begin{cases} 1 & \text{if } f(x) \geq a. \\ 0 & \text{if } f(x) < a. \end{cases} \tag{3.8}$$

We define a measure ν_f on $\mathbb{R}$, absolutely continuous with respect to Lebsgue measure da, by

$$(d\nu_f / da)(a) = \int E_a(f) \, d\mu. \tag{3.9}$$

It follows that a.e. we have

$$f(x) = \int_0^\infty (E_a(f))(x) \, da = \int_0^\infty E'_a(f) \, d\nu_f(a), \tag{3.10}$$

where

$$E'_a(f) = \begin{cases} 0 & \text{if } \int E_a(f) \, d\mu = 0, \\ E_a(f) \left[\int E_a(f) \, d\mu \right]^{-1} & \text{otherwise.} \end{cases} \tag{3.11}$$

We then have

$$\|f\| = \int f d\mu = \int d\nu_f \int d\mu \, E'_a(f) = \int d\nu_f, \tag{3.12}$$

$$\|f - f^{[a]}\| = \int_a^\infty d\nu_f(a'), \tag{3.13}$$

and

$$\|f - f_{[a]}\| \leq \int_0^a d\nu_f(a'). \tag{3.14}$$

(Equality in eq. (3.14) holds only when $f = 0$.) These equations indicate the significance of the measure ν_f.

LEMMA 3.3. *Given (X, μ, T), $\mu \circ T = \mu$, $f_1 \in L^1_+(X, \mu)$, $\|f_1\| \leq 1$ and $\varepsilon > 0$, then there exists $f_2 \in L^1_+(X, \mu)$, $\|f_2\| = 1$, such that for any $f \in L^1_+(X, \mu)$, $\|f\| = 1$, and $\lambda_1, \lambda_2 \in \ell^1_+(\mathbb{Z})$, $\|\lambda_1\| \leq 1$, $\|\lambda_2\| = 1$ satisfying*

$$\|T(\lambda_i)f - f_i\| \leq \varepsilon, \qquad i = 1, 2 \tag{3.15}$$

we have

$$\sup_{j \in \mathbb{Z}} \lambda_{1j} \leq 6\varepsilon. \tag{3.16}$$

Proof. The proof of this lemma consists of a long sequence of inequalities. Nevertheless it is a completely straightforward and obvious application of the above ideas. Choose $a < \infty$ such that

$$\|f_1 - f_1^{[a]}\| \leq \varepsilon. \tag{3.17}$$

Choose $f_2 \in L^1_+(X, \mu)$, $\|f_2\| = 1$, such that

$$\|f_2 - f_2^{[a\varepsilon^{-1}]}\| \geq 1 - \varepsilon. \tag{3.18}$$

It follows from eqs. (3.4) and (3.15) that

$$\|f_2^{[a\varepsilon^{-1}]} - (T(\lambda_2))f^{[a\varepsilon^{-1}]}\| \leq \varepsilon. \tag{3.19}$$

Eqs. (3.15), (3.18) and (3.19) give

$$\|T(\lambda_2)f - (T(\lambda_2)f)^{[a\varepsilon^{-1}]}\| \geq 1 - 3\varepsilon. \tag{3.20}$$

Eq. (3.20) and lemma 3.2 give

$$\|f - f^{[a\varepsilon^{-1}]}\| \geq 1 - 3\varepsilon. \tag{3.21}$$

Let

$$A = \{x : f(x) \geq a\varepsilon^{-1}\}. \tag{3.22}$$

Since $\|f\| \leq 1$ we have

$$\mu(A) \leq a^{-1}\varepsilon. \tag{3.23}$$

Eq. (3.15) gives for each $j \in \mathbb{Z}$,

$$\begin{aligned} \varepsilon \geq \|f_1 - \sum \lambda_{1j} T^j f\| &\geq \int_{T^{-j}A} (\sum \lambda_{1j} T^j f - f_1)\, d\mu \\ &\geq \int_{T^{-j}A} (\lambda_{1j} T^j f - f_1)\, d\mu. \end{aligned} \tag{3.24}$$

Eqs. (3.21) and (3.22) give

$$\int_{T^{-j}A} \lambda_{1j} T^j f\, d\mu \geq \lambda_{1j}(1 - 3\varepsilon). \tag{3.25}$$

Eqs. (3.17) and (3.23) give

$$\int_{T^{-j}A} f_1\, d\mu \leq 2\varepsilon. \tag{3.26}$$

Eqs. (3.24)-(3.26) give $(1 - 3\varepsilon)\lambda_{1j} \leq 3\varepsilon$ and hence

$$\lambda_{1j} \leq 3\varepsilon(1 + \lambda_{1j}) \leq 6\varepsilon. \qquad \square$$

Since convolutions preserve L^1 norms and $\|f\| = 1$, it follows from eq. (3.15) that $\|\lambda_1\| \geq \|f_1\| - \varepsilon$. Thus if ε is very small, eq. (3.16) forces the support of λ_1 to be very large.

LEMMA 3.4. *Given* (X, μ, T), $A \subset X$, $f \in L^1_+(X, \mu)$, $\lambda_1 \in \ell^1_+(\mathbb{Z})$ *and* $\varepsilon > 0$ *such that*

$$\|T(\lambda_1)f - \chi_A\| \leq \varepsilon, \tag{3.27}$$

then there exist $g \in L^1_+(X, \mu)$, $\lambda \in \ell^1_+(\mathbb{Z})$ *such that*

$$\|f - g\| \leq 4\varepsilon^{\frac{1}{2}}, \tag{3.28}$$

$$\|\lambda_1 - \lambda\| \leq \varepsilon^{\frac{1}{2}}, \tag{3.29}$$

and

$$\mu(S) \leq k\mu(B), \tag{3.30}$$

where S is the support of $T(\lambda)g$, *k is the number of elements in the support K of* λ, *and*

$$B=\left\{x\in X: k^{-1}\sum_{j\in K}\chi_{A^c}(T^jx)<\tfrac{1}{4}\right\}. \tag{3.31}$$

Proof. Eq. (3.27) implies that

$$\int_{A^c} T(\lambda_1)f\,d\mu \le \varepsilon. \tag{3.32}$$

It follows from eq. (3.10) that

$$T(\lambda_1)=\int d\nu_{\lambda_1}(a)\,T(E'_a(\lambda_1)). \tag{3.33}$$

Let

$$\rho(a)=\int_{A^c} T(E'_a(\lambda))f\,d\mu. \tag{3.34}$$

Eqs. (3.32)-(3.34) give

$$\int d\nu_{\lambda_1}(a)\rho(a)\le\varepsilon, \tag{3.35}$$

and hence

$$\nu_{\lambda_1}\{a: \rho(a)>\varepsilon^{\frac{1}{2}}\}<\varepsilon^{\frac{1}{2}}. \tag{3.36}$$

It follows from eqs. (3.34)-(3.36) that there exists $b>0$ such that

$$\rho(b)\le\varepsilon^{\frac{1}{2}} \tag{3.37}$$

and

$$\int_0^b d\nu_{\lambda_1}(a)\le\varepsilon^{\frac{1}{2}}. \tag{3.38}$$

It follows that $\lambda=(\lambda_1)_{[b]}$ satisfies eq. (3.29). Since K is also the support of $E'_b(\lambda)$, eq. (3.37) becomes

$$\begin{aligned}\varepsilon^{\frac{1}{2}}&\ge\int_{A^c} d\mu\, k^{-1}\sum_{j\in K}T^jf=k^{-1}\sum_{j\in K}\int_{T^{-j}A^c}d\mu\, f\\ &=\int_X d\mu\left[k^{-1}\sum_{j\in K}\chi_{A^c}(T^jx)\right]f(x).\end{aligned} \tag{3.39}$$

Eqs. (3.31) and (3.39) give

$$\int_{B^c} d\mu\, f\le 4\varepsilon^{\frac{1}{2}}. \tag{3.40}$$

It follows that $g(x)=\chi_B(x)f(x)$ satisfies eq. (3.28). Eq. (3.30) is satisfied by construction. □

THEOREM 3.5. *Let* (X, μ, T) *be an* AT *transformation,* $\mu(X)=1$, *and* $\mu\circ T=\mu$. *Then the entropy* $h(T)=0$.

Proof. Assume that $h(T)>0$. By a well-known result of Sinai there is a partition $\{A, A^c\}$ of X that moves independently under T (see for example [**18**, p. 43]). We can assume that $0<\mu(A)\le\frac{1}{2}$. Consider the set B given by lemma 3.4 (where ε, f, and λ_1 will be chosen below). Then $\mu(B)$ depends only on the number k of elements in K, and can be calculated directly from the binomial coefficients. Since the special case $\mu(A)=\frac{1}{2}$ dominates, one easily obtains the inequality

$$\mu(B)<2^{-k}. \tag{3.41}$$

Choose k_0 sufficiently large that

$$k_0 2^{-k_0}\leq \tfrac{1}{2}\mu(A), \tag{3.42}$$

and choose $\varepsilon>0$ sufficiently small that

$$\varepsilon+5\varepsilon^{\frac{1}{2}}\leq \tfrac{1}{2}\mu(A) \tag{3.43}$$

and

$$12\varepsilon k_0\leq \mu(A). \tag{3.44}$$

Now let $f_1(x)=\chi_A(x)$. By lemma 3.3 there exists $f_2\in L^1_+(X,\mu)$, $\|f_2\|=1$, such that eq. (3.16) is satisfied. Since T is AT there exists $f\in L^1_+(X,\mu)$, $\|f\|=1$ and $\lambda_1, \lambda_2\in \ell^1_+(\mathbb{Z})$, $\|\lambda_2\|=1$, such that

$$\|T(\lambda_i)f-f_i\|\leq \varepsilon, \qquad i=1,2. \tag{3.45}$$

By lemma 3.4 there exist λ and $g=\chi_B f$ satisfying eqs. (3.28)-(3.30). Eqs. (3.28)-(3.29) and (3.43) give

$$\|T(\lambda)g-\chi_A\|\leq \tfrac{1}{2}\mu(A), \tag{3.46}$$

and hence

$$\mu(S)\geq \tfrac{1}{2}\mu(A), \tag{3.47}$$

where S is the support of $T(\lambda)g$. Since $\|g\|\leq\|f\|=1$ and convolutions preserve L^1 norms, it also follows from eq. (3.46) that

$$\|\lambda\|\geq \tfrac{1}{2}\mu(A). \tag{3.48}$$

Since $\lambda=(\lambda_1)_{[b]}$, eq. (3.16) implies that we also have

$$\sup_{j\in\mathbb{Z}} \lambda(j)\leq 6\varepsilon. \tag{3.49}$$

Eqs. (3.48)-(3.49) give

$$6\varepsilon k\geq \tfrac{1}{2}\mu(A), \tag{3.50}$$

where k is the number of elements in the support K of λ. Eqs. (3.44) and (3.50) give

$$k\geq k_0. \tag{3.51}$$

Eqs. (3.51), (3.41) and (3.42) give

$$k\mu(B)<\tfrac{1}{2}\mu(A). \tag{3.52}$$

Eq. (3.30), which was obtained from the requirement that $T(\lambda_1)f$ is small on A^c, now gives

$$\mu(S)<\tfrac{1}{2}\mu(A) \tag{3.53}$$

which contradicts eq. (3.47). □

4. *Constructions of the flow of weights*

In order to make our exposition reasonably self-contained, we give here three different constructions of the flow of weights, each of which will be used at some point in our argument. We also construct some measures on the resulting spaces. For the proofs of all statements in this section, see [**5**].

Discrete construction. Let M be a type III_0 factor acting on the Hilbert space H, with $T \in T(M)$, $T > 0$. (If $T(M) = \{0\}$ one is then forced to use one of the following two constructions.) Let ϕ be a faithful state on M, $\sigma_T^\phi = 1$. Let $(e_n)_{n\in\mathbb{Z}}$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z})$, and define

$$Se_n = e_{n+1}, \tag{4.1}$$

$$\rho_\lambda e_n = \lambda^n e_n, \tag{4.2}$$

where $n \in \mathbb{Z}$ and $\lambda = \exp(-2\pi/T)$. The equation $\omega_\lambda(x) = \mathrm{Trace}\, \rho_\lambda x$ defines a faithful semifinite normal weight ω_λ on $\mathscr{L}(\ell^2(\mathbb{Z}))$. We have

$$\sigma_t^{\omega_\lambda}(S) = \rho_\lambda^{it} S \rho_\lambda^{-it} = \lambda^{it} S, \tag{4.3}$$

and

$$\omega_\lambda(SAS^*) = \lambda\omega_\lambda(A). \tag{4.4}$$

Let

$$\tilde{H} = \ell^2(\mathbb{Z}) \otimes H, \tag{4.5}$$

$$\tilde{M} = \mathscr{L}(\ell^2(\mathbb{Z})) \otimes M, \tag{4.6}$$

$$\tilde{\phi} = \omega_\lambda \otimes \phi, \tag{4.7}$$

$$\tilde{S} = S \otimes 1. \tag{4.8}$$

Then $\sigma_t^{\tilde{\phi}} = \sigma_t^{\omega_\lambda} \otimes \sigma_t^\phi$ and it follows from eq. (4.3) that the automorphism $\theta = \mathrm{Ad}\, \tilde{S}$ leaves the centralizer $N = \tilde{M}_{\tilde{\phi}}$ invariant. Eqs. (4.4), (4.7) and (4.8) give

$$\tilde{\phi} \circ \theta = \lambda \tilde{\phi}. \tag{4.9}$$

The centre C of N is isomorphic to $L^\infty(B, \nu_B)$ where (B, ν_B) is a Lebesgue measure space. θ then defines an automorphism, which we shall also denote by θ, of (B, ν_B). The flow of weights for M is the flow (X, ν, F_t) built over the base transformation (B, ν_B, θ) with a ceiling function of constant height $2\pi/T$. Furthermore

$$N = \int_B^\oplus N(b)\, d\nu_B(b), \tag{4.10}$$

where the $N(b)$ are type II_∞ factors. If M is injective then the $N(b)$ are all (isomorphic to) the unique injective II_∞ factor $R_{0,1}$ (see [4]). One can then write

$$N = R_{0,1} \otimes L^\infty(B, \nu_B) = \int_B^\oplus R_{0,1}\, d\nu_B(b), \tag{4.11}$$

and

$$\tilde{\phi} = \tau \otimes \nu_B = \int^\oplus \tau_b\, d\nu_B(b), \tag{4.12}$$

where τ is a trace on $R_{0,1}$ and τ_b is a trace on $N(b)$.

We now assign to certain positive operators in N, measures μ on B, $\mu < \nu_B$. Let

$$A = \int_B^\oplus A(b)\, d\nu_B(b) \in N_+ \tag{4.13}$$

be such that

$$\tilde{\phi}(A)=\int_B \tau_b(A(b))\, d\nu_B(b)<\infty. \tag{4.14}$$

The equation

$$\mu_A(c)=\tilde{\phi}(Ac), \qquad c\in C \tag{4.15}$$

defines a finite measure $\mu_A < \nu_B$. It follows from eqs. (4.14) and (4.15) that

$$(d\mu_A/d\nu_B)(b)=\tau_b(A(b)). \tag{4.16}$$

It follows from eq. (4.16) that if, e, f are $\tilde{\phi}$-finite projections in N, then $\mu_e=\mu_f$ if and only if $e\sim f$ in N. Since $N(b)\sim N(b)\otimes R_{0,1}$ one can construct families of projections $e_b(\alpha)\in N(b)$, $\alpha>0$ such that

$$\tau_b(e_b(\alpha))=\alpha, \tag{4.17}$$

and $e_b(f(b))$ is a ν_B-measurable family of projections for any $f\in L^1_+(B, \nu_B)$. It follows that every finite measure $\mu<\nu_B$ occurs as μ_e for some e, namely

$$e=\int_B^{\oplus} e_b(d\mu/d\nu_B)\, d\nu_B. \tag{4.18}$$

Continuous construction. Let M be a type III_0 factor acting on the Hilbert space H, ϕ a faithful state on M. On $L^2(\mathbb{R})$ we define

$$(V_s f)(t)=f(t-s), \tag{4.19}$$

and

$$(\rho f)(t)=e^t f(t). \tag{4.20}$$

The equation $\omega(x)=\mathrm{Trace}\,\rho x$ defines a faithful semifinite normal weight ω on $\mathscr{L}(L^2(\mathbb{R}))$. We have

$$\sigma_t^{\omega}(V_s)=e^{-its}V_s, \tag{4.21}$$

and

$$\omega(V_sAV_s^*)=e^t\omega(A). \tag{4.22}$$

Let

$$\tilde{H}=L^2(\mathbb{R})\otimes H, \tag{4.23}$$

$$\tilde{M}=\mathscr{L}(L^2(\mathbb{R}))\otimes M, \tag{4.24}$$

$$\tilde{\omega}=\omega\otimes\phi, \tag{4.25}$$

$$W_s=V_s\otimes 1. \tag{4.26}$$

Then $\theta_s N\theta_s^{-1}=N$ where $N=\tilde{M}_{\tilde{\omega}}$ and $\theta_s=\mathrm{Ad}\ W_s$. We have

$$N=\int_X^{\oplus} N(x)\, d\nu(x), \tag{4.27}$$

where the $N(x)$ are type II_∞ factors, and the centre C of N is isomorphic to $L^\infty(X, \nu)$ where (X, ν) is a Lebesgue measure space. The automorphisms θ_s define a flow F_s on (X, ν) which is the flow of weights for M. If M is injective then

$N(x) \sim R_{0,1}$ [4], and one can write

$$N = R_{0,1} \otimes L^{\infty}(X, \nu) = \int_X^{\oplus} R_{0,1}\, d\nu(x) \tag{4.28}$$

and

$$\tilde{\omega} = \tau \otimes \nu = \int^{\oplus} \tau\, d\nu \tag{4.29}$$

where τ is a trace on $R_{0,1}$.

The construction of measures in the continuous construction is quite analogous to the discrete case, except that they occur on the flow space X rather than the base space B. Let

$$A = \int_X^{\oplus} A(x)\, d\nu(x) \in N_+ \tag{4.30}$$

be such that

$$\tilde{\omega}(A) = \int_X \tau_x(A(x))\, d\nu(x) < \infty. \tag{4.31}$$

Then the equation

$$\mu_A(c) = \tilde{\omega}(Ac), \qquad c \in C \tag{4.32}$$

defines a finite measure $\mu_A < \nu$ such that

$$(d\mu_A/d\nu)(x) = \tau_b(A(x)). \tag{4.33}$$

If e, f are $\tilde{\omega}$-finite projections in N, then $\mu_e = \mu_f$ if and only if $e \sim f$ in N. As in the discrete case, every finite measure $\mu < \nu$ occurs as μ_e for some projection e. We shall need the following lemma.

LEMMA 4.1. *Let (Y, σ) be a measure space, and let e_y be a σ-measurable family of $\tilde{\omega}$-finite positive operators in N such that*

$$e = \int_Y e_y\, d\sigma(y) \tag{4.34}$$

is $\tilde{\omega}$-finite. Then

$$\mu_e = \int_Y \mu_{e_y}\, d\sigma(y). \tag{4.35}$$

Proof. We have

$$e = \int_X^{\oplus} e(x)\, d\nu(x) \tag{4.36}$$

and

$$e_y = \int_X^{\oplus} e_y\, d\nu(x). \tag{4.37}$$

Eqs. (4.34), (4.36) and (4.37) imply that

$$e(x) = \int_Y e_y(x)\, d\sigma(y) \qquad \text{(a.e. } \nu). \tag{4.38}$$

Eq. (4.38) implies eq. (4.35). □

Lacunary construction. Let M be a factor of type III_0, ϕ a faithful lacunary weight of infinite multiplicity on M (lacunary means that 1 is an isolated point in $\mathrm{Sp}\,\Delta_\phi$). Then

$$M_\phi = \int_B^\oplus M(b)\, d\nu_B(b), \tag{4.39}$$

where the $M(b)$ are type II_∞ factors, and the centre C_ϕ of the centralizer M_ϕ is isomorphic to $L^\infty(B, \nu_B)$ where (B, ν_B) is a Lebesgue measure space. If M is injective then $M(b) \sim R_{0,1}$. There exists $\rho \in C_\phi$, $0 < \rho \leq \lambda_0$ for some $\lambda_0 < 1$, and a unitary $U \in M$ such that

$$\phi(UxU^*) = \phi(\rho x), \qquad x \in M, \tag{4.40}$$

$$M = \{M_\phi, U\}'', \tag{4.41}$$

and

$$UM_\phi U^* = M_\phi. \tag{4.42}$$

Then $\theta = \mathrm{Ad}\, U$ defines an automorphism, which we also denote by θ, of (B, ν_B). The flow of weights for M is the flow $(X, \nu, \mathcal{F}_s, s \in \mathbb{R}_+^*)$ built over (B, ν_B, θ) with the ceiling function ρ, where

$$X = \{(b, t) \colon b \in B, 1 \geq t > \rho(b)\} \tag{4.43}$$

and $\mathcal{F}_s(b, t) = (b, e^{-s}t)$if $1 \geq e^{-s}t > \rho(b)$ with the obvious extension to other values of s.

We again construct certain measures on the flow space X. Let $\psi \in M_*^+$. Then there exists $h \in M_\phi^+$ such that $s(h)\rho \leq h \leq 1$, $1 - h$ is non-singular, and there exists a unitary $u \in M$ such that

$$\psi(x) = \phi(hE(uxu^*)), \tag{4.44}$$

where E is the conditional expectation from M onto M_ϕ. We have

$$h = \int_B^\oplus h_b\, d\nu_B(b) \tag{4.45}$$

where $s(h_b)\rho(b) \leq h_b \leq 1$. Let $f \in L^\infty(X, \nu)$. Then the operator

$$hf(h) = \int_B^\oplus h_b f_b(h_b)\, d\nu_B(b), \tag{4.46}$$

where $f_b(t) = f(b, t)$, is well-defined. The equation

$$\mu_\psi(f) = \phi(hf(h)) \tag{4.47}$$

defines a measure μ_ψ on the flow space X (which we will sometimes write as μ_h). (Eqs. (4.47) and (4.32) are related as follows. Let $\psi = \tilde{\omega}_e$, $e \in \tilde{M}_{\tilde{\omega}}$. Then $\mu_\psi = \mu_{e'}$.) Some terminology is helpful at this point.

Definition 4.2. Let (X, ν, F_t) be a flow. We call the measure μ on X *smooth* if $\mu < \nu$, and *smoothable* if

$$f * \mu = \int_{-\infty}^{\infty} dt\, f(t) F_t \mu \tag{4.48}$$

is smooth for all $f \in L_+^1(\mathbb{R}, dt)$.

The finite weight ψ is integrable if and only if μ_ψ is smooth. The smoothable measures are precisely the measures of the form $\int \mu_b \, d\nu_B(b)$ with respect to some base and ceiling function construction of the flow. It is then obvious from eqs. (4.45)-(4.47) that given any smoothable measure μ, one can choose h_b, $b \in B$ so that $\phi(hf(h)) = \mu(f)$, $f \in L^\infty(X, \nu)$. I.e. every smoothable measure μ is of the form μ_ψ for some $\psi \in M_*^+$.

5. *Comparison of finite periodic weights*

Let M be a type III$_0$ factor with $T \in T(M)$ for some $0 < T < \infty$, and let ϕ be a faithful state on M, $\sigma_T^\phi = 1$. Let λ, $\tilde{M}$, $\tilde{\phi}$, $\tilde{S}$, $\tilde{M}_{\tilde{\phi}}$, $C_{\tilde{\phi}}$, θ, B, ν_B be as in eqs. (4.1)-(4.12). To each finite weight $\psi \in \tilde{M}_{\tilde{\phi}}^T$ (see definition 5.1) we associate a finite measure μ_ψ on B, $\mu_\psi < \nu_B$. We establish a number of properties of the map $\psi \to \mu_\psi$. In particular lemmas 5.7, 5.8 and 5.9 are required for the proof that AT implies ITPFI in the discrete case $T(M) \neq \{0\}$ (see lemma 8.2).

If ψ is a weight on M such that $\sigma_T^\psi = 1$, then $(D\psi : D\phi)_T = e^{i\alpha} s(\psi)$ where $0 \leq \alpha < 2\pi$, and $(D\psi : D\phi)_t$, $t \in \mathbb{R}$ is the cocyle Radon-Nikodym derivative (see [5, pp. 478-479]). The weight $\theta = \psi \oplus \phi$ will satisfy $\sigma_T^\theta = 1$ if and only if $\alpha = 0$. Note that for $\beta > 0$ we have

$$(D(\beta\psi) : D\phi)_T = \beta^{iT} (D\psi : D\phi)_T, \tag{5.1}$$

so that for some $\lambda < \beta \leq 1$ we have $(D(\beta\psi) : D\phi)_T = s(\psi)$.

Definition 5.1. Let ω be a faithful weight on the von Neumann algebra $\mathcal{A}$ such that $\sigma_T^\omega = 1$ for some $0 < T < \infty$. Then $\mathcal{A}_\omega^T$ denotes the set of all finite weights ψ on $\mathcal{A}$ such that $(D\psi : D\omega)_T = s(\psi)$. If u is a partial isometry in $\mathcal{A}$ with $uu^* \in \mathcal{A}_\omega$ then the equation $\psi(x) = \omega(uxu^*)$, $x \in \mathcal{A}_+$ defines a weight ψ with support u^*u. We write $\psi = \omega_u$.

Definition 5.2. Let ω and ψ be weights on a von Neumann algebra $\mathcal{A}$. We say that ω and ψ are *equivalent* and write $\omega \sim \psi$ if there exists a partial isometry $u \in \mathcal{A}$ such that $uu^* = s(\omega)$, $u^*u = s(\psi)$ and $\psi = \omega_u$.

LEMMA 5.3. *Let M, T, ϕ, $\tilde{M}$, $\tilde{\phi}$ be as above. Let $\psi \in \tilde{M}_{\tilde{\phi}}^T$. Then there exists a projection $e \in \tilde{M}_{\tilde{\phi}}$ such that $\psi \sim \tilde{\phi}_e$. If e, f are projections in $\tilde{M}_{\tilde{\phi}}$ then $\tilde{\phi}_e \sim \tilde{\phi}_f$ if and only if $e \sim f$ in $\tilde{M}_{\tilde{\phi}}$.*

Proof. The assertion $\tilde{\phi}_e \sim \tilde{\phi}_f$ if and only if $e \sim f$ in $\tilde{M}_{\tilde{\phi}}$ is lemma 1.4(d) of [5]. To prove the first assertion, consider the weight θ defined on $P = \tilde{M} \otimes F_2$ by

$$\theta\left(\sum_{i,j=1}^{2} x_{ij} \otimes e_{ij} \right) = \tilde{\phi}(x_{11}) + \psi(x_{22}). \tag{5.2}$$

From ([2, lemma 1.2.2]) and ([5, pp. 478-9]) we have

$$\sigma_t^\theta(x \otimes e_{11}) = \sigma_t^{\tilde{\phi}}(x) \otimes e_{11}, \tag{5.3}$$

$$\sigma_t^\theta(s(\psi) \otimes e_{21}) = u_t \otimes e_{21}, \tag{5.4}$$

and

$$\sigma_t^\theta(x \otimes e_{22}) = \sigma_t^\psi(x) \otimes e_{22} \tag{5.5}$$

for all $x \in M$, where $u_t = (D\psi : D\tilde{\phi})_t$, $t \in \mathbb{R}$. By [5, lemma 1.4(b)]) we have $\psi \sim \tilde{\phi}_e$

for some e if and only if

$$s(\psi)\otimes e_{22}\prec 1\otimes e_{11}(P_\theta), \tag{5.6}$$

(i.e. $s(\psi)\otimes e_{22}$ is equivalent in the centralizer P_θ to a sub-projection of $1\otimes e_{11}$). Since $\tilde{M}_{\tilde{\phi}}$ is properly infinite (see eq. (4.10)), there exist projections $e_j\in\tilde{M}_{\tilde{\phi}}$, $j\in\mathbb{Z}$ such that $e_je_k=0$ if $j\neq k$, $e_j\sim e_k$ and $\sum e_j=1$. It follows from eq. (5.3) that $e_j\otimes e_{11}\in P_\theta$, $e_j\otimes e_{11}\sim e_k\otimes e_{11}(P_\theta)$, and hence that $1\otimes e_{11}=\sum_{j\in\mathbb{Z}}e_j\otimes e_{11}$ is a properly infinite projection in P_θ. Hence to prove eq. (5.6) it suffices to show that given any projection $f\leq s(\psi)$, $f\neq 0$, $f\in\tilde{M}_\psi$ there exists $y\in P_\theta$ such that

$$(f\otimes e_{22})y(1\otimes e_{11})\neq 0. \tag{5.7}$$

Since $\sigma_T^\theta=1$ it follows from eq. (5.4) that u_t is periodic with period T and hence

$$u_t=\sum_{k\in\mathbb{Z}}u^{(k)}\,e^{i2\pi kt/T}, \tag{5.8}$$

where

$$u^{(k)}\otimes e_{21}=\frac{1}{T}\int_0^T dt\,\sigma_t^\theta(s(\psi)\otimes e_{21})\,e^{-i2\pi kt/T}. \tag{5.9}$$

Now

$$\begin{aligned}0\neq\sigma_t^\theta(f\otimes e_{21})&=\sigma_t^\theta((f\otimes e_{22})(s(\psi)\otimes e_{21}))\\&=fu_t\otimes e_{21}\end{aligned} \tag{5.10}$$

and hence $fu^{(K)}\neq 0$ for some $K\in\mathbb{Z}$. Eq. (5.9) gives

$$\sigma_t^\theta(u^{(K)}\otimes e_{21})=e^{i2\pi Kt/T}u^{(K)}\otimes e_{21}. \tag{5.11}$$

It follows from eq. (4.3) that the unitary $\tilde{S}$ defined by eq. (4.8) satisfies

$$\sigma_t^{\tilde{\theta}}(\tilde{S})=e^{-i2\pi t/T}\tilde{S}, \tag{5.12}$$

since $\lambda^{it}=e^{-i2\pi t/T}$. Define

$$y=(u^{(K)}\otimes e_{21})(\tilde{S}^{-K}\otimes e_{11}). \tag{5.13}$$

Eqs. (5.11)-(5.13) give $\sigma_t^\theta(y)=y$, hence $y\in P_\theta$. Since $\tilde{S}$ is unitary and $fu^{(K)}\neq 0$, eq. (5.7) is satisfied. □

Definition 5.4. Let $\tilde{M}$, $\tilde{\phi}$ be as above, $\psi\in\tilde{M}_{\tilde{\phi}}^{T}$. We define the measure μ_ψ associated with ψ as the measure μ_e defined by eq. (4.15) where $\psi\sim\tilde{\phi}_e$.

LEMMA 5.5. *Let $\psi_1,\psi_2\in\tilde{M}_{\tilde{\phi}}^{T}$. Then $\mu_{\psi_1}=\mu_{\psi_2}$ if and only if $\psi_1\sim\psi_2$.*

Proof. The lemma follows immediately from lemma 5.3 and the fact that $\mu_e=\mu_f$ if and only if $e\sim f$ in $\tilde{M}_{\tilde{\phi}}$ (see eq. (4.16)). □

LEMMA 5.6. *Let $\psi_1,\psi_2\in\tilde{M}_{\tilde{\phi}}^{T}$ be such that $s(\psi_1)s(\psi_2)=0$. Then $\mu_{\psi_1+\psi_2}=\mu_{\psi_1}+\mu_{\psi_2}$.*

Proof. Note that eq. (4.16) implies that if $e=e_1+e_2$ where $e,e_1,e_2\in\tilde{M}_{\tilde{\phi}}^{+}$, then $\mu_e=\mu_{e_1}+\mu_{e_2}$. By lemma 5.3 we have $\psi_j=\tilde{\phi}_{u_j}$, $j=1,2$ where, since $\tilde{M}_{\tilde{\phi}}$ is properly infinite, we can choose u_1 and u_2 such that $e_1e_2=0$ where $e_j=u_j^*u_j$. Then $u=u_1+u_2$ is a partial isometry such that $\psi_1+\psi_2=\tilde{\phi}_u$, and $u^*u=e_1+e_2$. Since $\tilde{\phi}_u\sim\tilde{\phi}_{u^*u}$ and $\tilde{\phi}_{u_j}\sim\tilde{\phi}_{e_j}$ the result follows. □

LEMMA 5.7. *Let $\psi\in\tilde{M}_{\tilde{\phi}}^{T}$. Let $\mu_1,\ldots,\mu_n$ be measures on B such that $\mu_\psi=\sum_{j=1}^n\mu_j$. Then there exist orthogonal projections $e_1,\ldots,e_n$ such that $s(\psi)=\sum_{j=1}^n e_j$ and $\mu_{\psi_{e_j}}=\mu_j$.*

Proof. Since every $\mu < \nu_B$ occurs as μ_ρ for some $\rho \in \tilde{M}_\Phi^T$ (see eq. (4.33)), there exist mutually orthogonal projections $f_1, \ldots, f_n \in \tilde{M}$ and $\rho_1, \ldots, \rho_n \in \tilde{M}_\Phi^T$ such that

$$s(\rho_j) = f_j \tag{5.14}$$

and

$$\mu_{\rho_j} = \mu_j, \qquad j = 1, \ldots, n. \tag{5.15}$$

By lemma 5.5 and 5.6, $\rho = \sum \rho_j \sim \psi$. Hence there is a partial isometry u with $u^*u = s(\rho) = \sum f_j$, $uu^* = s(\psi)$, and $\psi = \rho_u$. Then $e_j = uf_j$ are the desired projections. □

LEMMA 5.8. *Let $\psi \in \tilde{M}_\Phi^T$, $\psi \sim \tilde{\phi}_e$. Then*

$$\theta\mu_\psi = \lambda\mu_{\lambda^{-1}\psi} = \lambda\mu_{\theta(e)}.$$

Proof. We have $\mu_\psi = \mu_e$. Let $c \in C_{\tilde{\phi}}$. Using eqs. (4.9) and (4.15) we have

$$\begin{aligned}(\theta\mu_\psi)(c) &= \mu_e(\theta^{-1}(c)) = \tilde{\phi}(e\theta^{-1}(c))\\ &= (\tilde{\phi} \circ \theta)(\theta(e)c) = \lambda\tilde{\phi}(\theta(e)c) = \lambda\mu_{\theta(e)}(c).\end{aligned} \tag{5.16}$$

It remains only to prove that $\mu_{\lambda\psi} = \mu_{\theta^{-1}(e)}$. Let u be a partial isometry in $\tilde{M}$ such that $\psi = \tilde{\phi}_u$ and $uu^* = e$. Then

$$(\lambda\psi)(x) = \lambda\tilde{\phi}(uxu^*) = \tilde{\phi}(\theta^{-1}(uxu^*)), \qquad x \in \tilde{M}. \tag{5.17}$$

Since $\theta = \mathrm{Ad}\, \tilde{S}$ we get

$$(\lambda\psi)(x) = \tilde{\phi}(\tilde{S}uxu^*\tilde{S}^*) = \tilde{\phi}_{\tilde{S}u}(x). \tag{5.18}$$

But $\tilde{\phi}_\omega \sim \tilde{\phi}_{\omega\omega^*}$ and $\tilde{S}uu^*\tilde{S}^* = \theta^{-1}(e)$. □

LEMMA 5.9. *Let $\psi_1 \in \tilde{M}_\Phi^T$, $\psi_1 \neq 0$, and $\varepsilon > 0$. Let $\mu_2 \neq 0$ be a finite measure on B, $\mu_2 < \nu_B$. Then there exists $\psi_2 \in \tilde{M}_\Phi^T$ such that*

(i) $s(\psi_2) = s(\psi_1)$;

(ii) $\mu_{\psi_2} = \mu_2$; *and*

(iii) $\|\psi_1 - \psi_2\| \le \|\mu_{\psi_1} - \mu_2\| + \varepsilon$.

Proof. Write $\mu_1 = \mu_{\psi_1}$ and $f_j = d\mu_j/d\nu_B$, $j = 1, 2$. By a routine argument one can choose a family of projections $e(\alpha) \in R_{0,1}$, $0 < \alpha < \infty$ such that

$$e(\alpha)e(\beta) = e(\alpha) \qquad \text{if } \alpha < \beta \tag{5.19}$$

and

$$\tau(e(\alpha)) = \alpha \tag{5.20}$$

where τ is the trace on $R_{0,1}$ such that $\tilde{\phi} = \tau \otimes \nu_B$ (see eqs. (4.11) and (4.12)). Then $\mu_j = \mu_{e_j}$, $j = 1, 2$ where

$$e_j = \int_B^\oplus e(f_j(b))\, d\nu_B(b). \tag{5.21}$$

Let

$$B_+ = \{b : f_2(b) > f_1(b)\}, \tag{5.22}$$

$$B_- = \{b : f_2(b) < f_1(b)\}, \tag{5.23}$$

$$B_0 = \{b : f_2(b) = f_1(b)\}. \tag{5.24}$$

Then

$$e_j = e_{j+} + e_{j-} + e_{j0}, \tag{5.25}$$

where

$$e_{j+} = \int_{B_+}^{\oplus} e(f_j(b))\, d\nu_B(b) \tag{5.26}$$

etc. Let u be a partial isometry in $\tilde{M}$ such that $\psi_1 = \tilde{\phi}_u$ where $uu^* = e_1$, $u^*u = s(\psi_1)$.

Case (i). $\nu_B(B_+) = \nu_B(B_-) = 0$. Take $\psi_2 = \psi_1$.

Case (ii). $\nu_B(B_+)$, $\nu_B(B_-) > 0$. Let ω be a partial isometry in $\tilde{M}$ mapping the non-zero projection $e_{2+} - e_{1+}$ onto the non-zero projection $e_{1-} - e_{2-}$. Define $\psi_2 = \psi_u$ where

$$\psi = \tilde{\phi}_{e_1 - (e_{1-} - e_{2-})} + (\tilde{\phi}_{e_{2+} - e_{1+}})_\omega. \tag{5.27}$$

Then $\psi \sim \tilde{\phi}_{e_2}$ so that $\mu_\psi = \mu_2$. Furthermore $s(\psi) = e_1$, $s(\psi_2) = u^*u = s(\psi_1)$, and

$$\|\psi_2 - \psi_1\| = \|\psi - \tilde{\phi}_{e_1}\|. \tag{5.28}$$

We have

$$\psi - \tilde{\phi}_{e_1} = (\tilde{\phi}_{e_{2+} - e_{1+}})_\omega - \tilde{\phi}_{e_{1-} - e_{2-}}. \tag{5.29}$$

Since $\|\tilde{\phi}_f\| = \tilde{\phi}(f) = \mu_f(B)$ we obtain

$$\begin{aligned} \|\psi_2 - \psi_1\| &\le \|\tilde{\phi}_{e_{2+} - e_{1+}}\| + \|\tilde{\phi}_{e_{1-} - e_{2-}}\| \\ &= (\mu_2 - \mu_1)(B_+) + (\mu_1 - \mu_2)(B_-) = \|\mu_1 - \mu_2\|. \end{aligned} \tag{5.30}$$

Case (iii). $\nu_B(B_+) = 0$, $\nu_B(B_-) \neq 0$. Choose $g(b)$, $b \in B$ such that

$$0 \le g(b) \le f_2(b) \tag{5.31}$$

and

$$0 < \int_B g(b)\, d\nu_B(b) < \tfrac{1}{2}\varepsilon. \tag{5.32}$$

Let

$$g = \int_B^{\oplus} e(g(b))\, d\nu_B(b). \tag{5.33}$$

Let ω be a partial isometry in $\tilde{M}$ such that $\omega^*\omega = g$, $\omega\omega^* = e_1 - e_2 + g$. Define $\psi_2 = \psi_u$ where

$$\psi = \tilde{\phi}_{e_2 - g} + (\tilde{\phi}_g)\omega. \tag{5.34}$$

Then $\psi \sim \tilde{\phi}_{e_2}$ so that $\mu_\psi = \mu_2$. Furthermore $s(\psi) = e_1$ so that $s(\psi_2) = u^*u = s(\psi_1)$, and

$$\|\psi_2 - \psi_1\| = \|\psi - \tilde{\phi}_{e_1}\|. \tag{5.35}$$

Since

$$\tilde{\phi}_{e_1} - \psi = \tilde{\phi}_g + \tilde{\phi}_{e_1 - e_2} - (\tilde{\phi}_g)_\omega \tag{5.36}$$

and

$$\|\tilde{\phi}_g\| = \tilde{\phi}(g) = \int g(b) d\nu_B < \tfrac{1}{2}\varepsilon \tag{5.37}$$

we obtain $\|\psi_2 - \psi_1\| \le \|\mu_1 - \mu_2\| + \varepsilon$.

Case (iv). $\nu_B(B_+) \neq 0$, $\nu_B(B_-) = 0$. The argument is similar to case (iii). □

6. *Comparison of finite weights: the general case*

We extend the results of § 5 to the general case. This is straightforward except for lemma 5.9, where we now use the lacunary construction (see lemma 6.4). In this section M is a type III_0 factor, and we follow the notation of § 4.

LEMMA 6.1. *Let* $\psi \in M_*^+$, *then* $\mu_{\lambda\psi} = \lambda \mathcal{F}_{\lambda^{-1}}^M \mu_\psi$. *If* $\psi_1, \psi_2 \in M_*^+$ *then* $\psi_1 \sim \psi_2 \Leftrightarrow \mu_{\psi_1} = \mu_{\psi_2}$, *and* $\mu_{\psi_1+\psi_2} = \mu_{\psi_1} + \mu_{\psi_2}$ *if* $s(\psi_1)s(\psi_2) = 0$.

Proof: This is corollary 1.13 (ii) of [5]. □

LEMMA 6.2. *Let* $\psi \in M_*^+$, $\mu_\psi = \sum_{j=1}^n \mu_j$. *Then there exist orthogonal projections* $e_1, \ldots, e_n$ *such that* $s(\psi) = \sum_{j=1}^n e_j$ *and* $\psi = \sum_{j=1}^n \psi_{e_j}$ *where* $\mu_{\psi_{e_j}} = \mu_j$.

Proof. Since every smoothable measure occurs as μ_χ for some χ (see eq. (4.48) *et seq.*) the proof of lemma 5.7 holds verbatim. □

The next lemma is a technical result needed in the proof of lemma 6.4.

LEMMA 6.3. *Let* σ_1, σ_2 *be non-atomic measures on* $I = [0, 1], \sigma_1(I) = \sigma_2(I) < \infty$. *Let* $F_j(x) = \sigma_j([0, x])$, *and let*

$$S_j = \{x \in I\colon F_j(y) = F_j(x) \text{ implies } y = x\},$$

$j = 1, 2$. *Then* $\sigma_j(I \setminus S_j) = 0$ *and the equation* $F_1(\gamma(x)) = F_2(x)$ *defines a monotonic bijection* $\gamma\colon S_1 \to S_2$ *satisfying*

$$\int_0^1 f(\gamma(t))\, d\sigma_1(t) = \int_0^1 f(t)\, d\sigma_2(t)$$

for all $f \in L^1(I, \sigma_2)$. *Furthermore*

$$\|\sigma_1 - \sigma_2\| \geq \int_0^1 |t - \gamma(t)|\, d\sigma_1(t). \tag{6.1}$$

Proof. To prove eq. (6.1) consider the function f defined by

$$f'(t) = \begin{cases} +1 & \text{if } t > \gamma(t) \\ 0 & \text{if } t = \gamma(t), \\ -1 & \text{if } t < \gamma(t), \end{cases} \tag{6.2}$$

and $f(0) = 0$. Since $\|f\|_\infty \leq 1$ we have

$$\begin{aligned} \|\sigma_1 - \sigma_2\| &\geq |\sigma_1(f) - \sigma_2(f)| \\ &= \left| \int [f(t) - f(\gamma(t))]\, d\sigma_1(t) \right| \\ &= \int |t - \gamma(t)|\, d\sigma_1(t). \end{aligned} \tag{6.3}$$

All other properties are obvious. □

LEMMA 6.4. *Let* ψ_1 *be a finite integrable weight on M, and let* μ_2 *be a smooth measure on* X (*i.e.* $\mu_2 < \nu$, *see definition* 4.2).*Then there exists* $\psi_2 \in M_*^+$ *such that*:

(i) $s(\psi_2) = s(\psi_1)$;

(ii) $\mu_{\psi_2} = \mu_2$; *and*

(iii) $\|\psi_1 - \psi_2\| \leq 5\|\mu_{\psi_1} - \mu_2\|$.

Proof. We use the lacunary state construction of the flow of weights (eqs. (4.39)–(4.48)) where we choose the lacunary state ϕ so that $\frac{1}{2} \leq \rho < 1$. We have

$$\psi_1(x) = \phi(h_1 E(uxu^*)), \tag{6.4}$$

where $h_1 \in M_\phi^+$ so that

$$h_1 = \int_B^{\oplus} d\nu_B(b) \int_{\rho(b)}^{1_\oplus} t\, d\sigma_{1,b}(t), \tag{6.5}$$

where we can choose the measures $\sigma_{1,b}$ so that for $f \in L^1(X, \mu_1)$ we have

$$\mu_1(f) = \int_B d\nu_B(b) \int_{\rho(b)}^{1} tf(b, t)\, d\sigma_{1,b}(t) \tag{6.6}$$

where $\mu_1 = \mu_{\psi_1}$. By the remark following definition 4.2, we can choose measures $\sigma_{2,b}$ such that

$$\mu_2(f) = \int_B d\nu_B(b) \int_{\rho(b)}^{1} tf(b, t)\, d\sigma_{2,b}(t). \tag{6.7}$$

We begin by altering the measures slightly so that lemma 6.3 can be used. Let

$$B_+ = \{b \in B: \sigma_{1,b}([\rho(b), 1]) \geq \sigma_{2,b}([\rho(b), 1])\}, \tag{6.8}$$

and

$$B_- = B \setminus B_+. \tag{6.9}$$

For $b \in B_+$ define $t_{2,b} = 1$ and

$$t_{1,b} = \sup\{\rho(b) \leq t \leq 1: \sigma_{1,b}([\rho(b), t]) = \sigma_{2,b}([\rho(b), 1])\}, \tag{6.10}$$

and for $b \in B_-$ define $t_{1,b} = 1$ and

$$t_{2,b} = \sup\{\rho(b) \leq t \leq 1: \sigma_{2,b}([\rho(b), t]) = \sigma_{1,b}([\rho(b), 1])\}. \tag{6.11}$$

Define μ_1', μ_2' by

$$(d\mu_j'/d\mu_j)(b, t) = \chi_{[\rho(b), t_{j,b}]}(t). \tag{6.12}$$

Then

$$\begin{aligned} \|\mu_1 - \mu_1'\| &= \mu_1(\chi_{[t_{1,b}, 1]}) \\ &\leq \nu_B \circ \sigma_1(\chi_{[t_{1,b}, 1]}) = (\nu_B \circ \sigma_1 - \nu_B \circ \sigma_2)(\chi_{B_+}) \\ &\leq \|\nu_B \circ \sigma_1 - \nu_B \circ \sigma_2\| \leq 2\|\mu_1 - \mu_2\|, \end{aligned} \tag{6.13}$$

where the last inequality follows from the fact that $d\mu_j/d\nu_B \circ \sigma_j = t \geq \frac{1}{2}$. Similarly

$$\|\mu_2 - \mu_2'\| \leq 2\|\mu_1 - \mu_2\|. \tag{6.14}$$

We can now write $\mu_j' = \nu_B \circ \sigma_j'$, $j = 1, 2$ where

$$\sigma_{1,b}'([\rho(b), 1]) = \sigma_{2,b}'([\rho(b), 1]), \qquad b \in B. \tag{6.15}$$

By lemma 6.3 there exists a measurable function $\gamma(b, t)$ such that for all $f \in L^1(X, \mu_1')$ we have

$$\int_{\rho(b)}^{1} \gamma(b, t) f(b, \gamma(b, t))\, d\sigma_{1,b}'(t) = \int_{\rho(b)}^{1} tf(b, t)\, d\sigma_{2,b}'(t), \tag{6.16}$$

for a.e. $b \in B$. We define

$$h_1' = \int_B^{\oplus} d\nu_B(b) \int_{\rho(b)}^{1_\oplus} t\, d\sigma_{1,b}'(t) \tag{6.17}$$

$$h_2' = \int_B^{\oplus} d\nu_B(b) \int_{\rho(b)}^{1_{\oplus}} \gamma(b,t)\, d\sigma_{1,b}'(t), \tag{6.18}$$

and

$$\psi_j'(x) = \phi(h_j' E(uxu^*)), \qquad j = 1, 2. \tag{6.19}$$

Using eq. (6.13) we obtain

$$\|\psi_1 - \psi_1'\| = \phi(|h_1 - h_1'|) = \|\mu_1 - \mu_1'\| \leq 2\|\mu_1 - \mu_2\|. \tag{6.20}$$

We also have

$$\begin{aligned} \|\psi_1' - \psi_2'\| &= \phi(|h_1' - h_2'|) \\ &= \int d\nu_B(b) \int |t - \gamma(b,t)|\, d\sigma_{1,b}'(t) \\ &\leq \int d\nu_B(b) \|\sigma_{1,b}' - \sigma_{2,b}'\| = \|\mu_1' - \mu_2'\|, \end{aligned} \tag{6.21}$$

where the last inequality follows from lemma 6.3.

Case (i). $\mu_1' = \mu_1$ and $\mu_2' = \mu_2$. Then $\psi_2 = \psi_2'$ satisfies the lemma.

Case (ii). $\mu_1 \neq \mu_1'$ and $\mu_2 \neq \mu_2'$. Let $h_1'' = h_1 - h_1'$. Then $s(h_1'')s(h_1') = 0$ and $s(h_1'') \neq 0$. Construct h_2'' such that $s(h_2'') = s(h_1'')$ and $\mu_{h_2''} = \mu_2 - \mu_2'$. Let $h_2 = h_2' + h_2''$ and define

$$\psi_2 = \phi(h_2 E(uxu^*)). \tag{6.22}$$

By construction $\mu_{\psi_2} = \mu_2$ and $s(\psi_2) = s(\psi_1)$. We have $\psi_2 = \psi_2' + \psi_2''$ where

$$\|\psi_2''\| = \phi(|h_2''|) = \|\mu_2 - \mu_2'\| \leq 2\|\mu_1 - \mu_2\| \tag{6.23}$$

(see eq. (6.14)). Eqs. (6.20), (6.21) and (6.23) give

$$\|\psi_1 - \psi_2\| \leq 5\|\mu_1 - \mu_2\|. \tag{6.24}$$

Case (iii). $\mu_1 = \mu_1'$ and $\mu_2 \neq \mu_2'$. Choose a projection $e \in M$ such that $0 < e < s(h_2')$, and

$$\phi(eh_2') \leq \|\mu_1 - \mu_2\|. \tag{6.25}$$

Choose h_2'' such that $s(h_2'') = e$ and

$$\mu_{h_2''} = \mu_2 - \mu_2' + \mu_{eh_2'}. \tag{6.26}$$

Then eqs. (6.14) and (6.25) give

$$\|\mu_{h_2''}\| \leq 3\|\mu_1 - \mu_2\|. \tag{6.27}$$

Define

$$\psi_2(x) = \phi((h_2'' + (1-e)h_2')E(uxu^*)). \tag{6.28}$$

By construction $\mu_{\psi_2} = \mu_2$ and $s(\psi_2) = s(\psi_1)$. Eqs. (6.25) and (6.27) give

$$\|\psi_2 - \psi_2'\| \leq 4\|\mu_1 - \mu_2\|. \tag{6.29}$$

Since $\psi_1 = \psi_1'$, eq. (6.21) now gives

$$\|\psi_1 - \psi_2\| \leq 5\|\mu_1 - \mu_2\|. \tag{6.30}$$

Case (iv). $\mu_1 \neq \mu_1'$ and $\mu_2 = \mu_2'$. Choose a projection $e \in M$ such that $0 < e < s(h_1') = s(h_2')$ and

$$\phi(eh_2') \leq \|\mu_1 - \mu_2\|. \tag{6.31}$$

Choose h_2'' such that $s(h_2'')=s(h_1)-s(h_1')+e$ and

$$\mu_{h_2''}=\mu_{eh_2'}. \tag{6.32}$$

Define ψ_2 by eq. (6.28). By construction $\mu_{\psi_2}=\mu_2$ and $s(\psi_2)=s(\psi_1)$. Eqs. (6.31), (6.32) and (6.28) give

$$\|\psi_2-\psi_2'\|\leq 2\|\mu_1-\mu_2\|. \tag{6.33}$$

Eqs. (6.20) and (6.21) now give

$$\|\psi_1-\psi_2\|\leq 5\|\mu_1-\mu_2\|. \tag{6.34}$$

□

7. *Product property and* ITPFI *factors*

We introduce the 'product property' (definition 7.1) which is a variation of Størmer's property of being 'asymptotically a product state' [**19**]. This is a technical property which is equivalent to ITPFI (corollary 7.4 and lemma 7.6). Its purpose is to simplify the task of verifying the ITPFI property by eliminating the iterative part of the argument.

Definition 7.1. Let M be a von Neumann algebra. A finite weight ϕ on M is said to have the *product property* if given $\varepsilon>0$, a strong neighbourhood V of 0, and $x_1,\ldots,x_n\in M$, there exists a finite type I factor $K\subset M$ and finite weights ϕ_1, ϕ_2 on K, $K^c=K'\cap M$ respectively such that:

(i) $x_j\in K+V$, $j=1,\ldots,n$, and

(ii) $\|\phi-\phi_1\otimes\phi_2\|<\varepsilon$.

If M has a faithful finite weight with the product property, then M is said to have the *product property.*

Remark 7.2. It follows immediately from the above definition that the product property implies approximately type I (see [**8**]). Clearly one can require that ϕ_1 and ϕ_2 are faithful, and (ii) can be replaced by

(ii)′ $\|\phi-\phi|_K\otimes\phi|_{K^c}\|<\varepsilon$.

If one were to study von Neumann algebras of the form $\otimes_\nu(\mathscr{A}_\nu,\phi_\nu)$ where the $\mathscr{A}_\nu$ are finite-dimensional matrix algebras, the appropriate product property would be to require only that K be a finite-dimensional subalgebra.

In order to prove that an ITPFI factor has the product property we will use the following martingale condition, which was introduced by Araki and Woods ([**1**, lemma 6.10]).

LEMMA 7.3. *Let* $M=\bigotimes_{k=1}^{\infty}(M_k,\phi_k)$ *be an* ITPFI *factor. For each* $n\in\mathcal{N}$ *there is a conditional expectation* $E_n:M\to M^{(n)}=\bigotimes_{k=1}^{n}M_k$ *such that*

(i) $E_n(x)\to x$ *strongly for all* $x\in M$;

(ii) $E_n(M_\phi)=M^{(n)}_{\phi^{(n)}}$ *where* $\phi=\bigotimes_{k=1}^{\infty}\phi_k$ *and* $\phi^{(n)}=\bigotimes_{k=1}^{n}\phi_k$; *and*

(iii) $E_nE_m=E_n$ *if* $n<m$.

Proof. Define E_n by the equation

$$(E_n(x)\alpha,\beta)=\left(x\left(\alpha\otimes\left(\bigotimes_{k=n+1}^{\infty}\Phi_k\right)\right),\beta\otimes\left(\bigotimes_{k=n+1}^{\infty}\Phi_k\right)\right) \tag{7.1}$$

for all $x \in M$, $\alpha, \beta \in \bigotimes_{k=1}^{n} H_k$ where M_k acts on H_k and $\phi_k(y) = (y\Phi_k, \Phi_k)$ for all $y \in M_k$. That E_n is a conditional expectation and properties (i) and (iii) follow directly from eq. (7.1) (compare [1, lemma 6.10]). Condition (ii) follows from

$$\sigma_t^{\phi^{(n)}}(E_n(x)) = E_n(\sigma_t^{\phi}(x)) \tag{7.2}$$

which in turn follows from the observation that

$$\Delta_\phi = \Delta_{\phi^{(n)}} \otimes \Delta_{(\bigotimes_{k=n+1}^{\infty} \phi_k)} \tag{7.3}$$

and a routine calculation using eq. (7.1). □

COROLLARY 7.4. *Any* ITPFI *factor has the product property.*

Proof. Lemma 7.3 implies that any product state $\otimes \phi_k$ has the product property. □

LEMMA 7.5. *Let M be a von Neumann algebra with the product property. Then any finite weight ψ has the product property.*

Proof. Let $\varepsilon > 0$, V a strong neighbourhood of 0, and $x_1, \ldots, x_n \in M$. Let ϕ be a faithful finite weight on M with the product property. It follows from the Hahn-Banach theorem that the set of all finite weights χ such that $\chi \leq \lambda\phi$ for some $\lambda > 0$, is norm-dense in M_*^+. It then follows from [**17**, theorem 1.24.3, p. 76] that there exists $h \in M^+$ such that

$$|\psi(a) - \phi(hah)| \leq \varepsilon \|a\|, \tag{7.4}$$

for all $a \in M$. By assumption there exist a finite type I factor $K \subset M$, $k \in K$, and finite weights ϕ_1, ϕ_2 such that

$$x_j \in K + V, \qquad j = 1, \ldots, n, \tag{7.5}$$

$$\phi(|h-k|^2) \leq \varepsilon^2 \|h\|^{-2} \|\phi\|^{-1}, \tag{7.6}$$

$$\|\phi - \phi_1 \otimes \phi_2\| \leq \varepsilon \|h\|^{-2}, \tag{7.7}$$

and

$$\|k\| \leq \|h\|. \tag{7.8}$$

That one can achieve eq. (7.8) follows, as in the proof of the Kaplansky density theorem, by approximating an element $h' \in M$ such that $h = 2h'(1+(h')^2)^{-1}$ (see [7, p. 44]). Since

$$\phi(hah) = \phi(kak) + \phi((h-k)ah) + \phi(ka(h-k)),$$

it follows from the Cauchy-Schwarz inequality and eq. (7.6) that

$$|\phi(hah) - \phi(kak)| \leq 2\varepsilon \|a\|. \tag{7.9}$$

Eqs. (7.4), (7.7) and (7.9) give

$$\|\psi - \psi_1 \otimes \psi_2\| \leq 4\varepsilon, \tag{7.10}$$

where $\psi_1(a) = \phi_1(kak)$, $a \in K$, and $\psi_2(a) = \phi_2(a)$, $a \in K^c$. □

LEMMA 7.6. *Let M be a properly infinite von Neumann algebra with the product property. Then M is* ITPFI.

Proof. Let ψ be a faithful state on M. Let $(x_j)_{j \in \mathbb{N}}$ be dense in the unit ball M_1 of M, let V_j be a sequence of strong neighbourhoods of 0 decreasing to 0, and let

$\eta_n > 0$ with

$$\sum_{n\in\mathbb{N}} \eta_n < 1. \tag{7.11}$$

We will construct a sequence of mutually commuting finite type I factors K_i and faithful states ϕ_i on K_i such that

$$x_j \in K^{(n)} + V_n, \qquad j = 1, \dots, n \tag{7.12}$$

where

$$K^{(n)} = \left(\bigotimes_{i=1}^{n} K_i\right) \otimes 1, \tag{7.13}$$

and

$$\|\psi|_{(K^{(n-1)})^c} - \phi_n \otimes \psi|_{(K^{(n)})^c}\| < \eta_n, \tag{7.14}$$

where $(K^{(n)})^c = (K^{(n)})' \cap M$. It will then follow that $M \sim \bigotimes_{n\in\mathbb{N}} (K_n, \phi_n)$.

By assumption there is a finite type I factor K_1 and a faithful state ϕ_1 such that eqs. (7.12) and (7.14) are satisfied for $n = 1$ (use remark 7.2). Now assume that $K_1, \dots, K_n$ and $\phi_1, \dots, \phi_n$ have been chosen so that eqs. (7.12) and (7.14) are satisfied. Let $e_{ij}^{(n)}$, $i, j = 1, \dots, k_n$ be a complete set of matrix units for $K^{(n)}$, and let $e = e_{11}^{(n)}$. Since M is properly infinite, $e \sim 1$ and $M_e = eMe \sim M$. In particular M_e has the product property. Let W be a strong neighbourhood of 0 such that the sum of any k_n^2 elements from W must lie in V_{n+1}. There exists a finite type I subfactor L_{n+1} of M_e, $y_{k,i,j}^{(n)} \in L_{n+1}$, $i, j = 1, \dots, k_n$, $k = 1, \dots, n+1$, and a state ϕ_{n+1} such that

$$e_{i1}^{(n)} y_{k,ij}^{(n)} e_{1j}^{(n)} - e_{ii}^{(n)} x_k e_{jj}^{(n)} \in W, \tag{7.15}$$

and eq. (7.14) is satisfied with n replaced by $n+1$ (where we have used the canonical identification of M_e with $K^{(n)c}$). Let

$$y_k^{(n)} = \sum_{i,j=1}^{k_n} e_{i1}^{(n)} y_{k,ij}^{(n)} e_{1j}^{(n)} \in K^{(n+1)}, \tag{7.16}$$

$k = 1, \dots, n+1$, where K_{n+1} is the finite type I subfactor of M obtained from the canonical embedding of L_{n+1}. Then

$$y_k^{(n)} - x_k \in V_{n+1}$$

so that eq. (7.12) is satisfied. Now let $\phi = \bigotimes_{k=1}^{\infty} \phi_k$. It follows from eqs. (7.11) and (7.14) that $\|\phi - \psi\| < 1$. Hence the representation π_ϕ of the UHF C^*-algebra $\mathscr{A} = \bigcup_{j\in\mathbb{N}} K_j$ induced by ϕ is unitarily equivalent to π_ψ (see for example [**16**, theorem 2.7]). Eq. (7.12) implies that $\pi_\psi(\mathscr{A})'' = M$.

Remark 7.7. By further arguments, which we omit, one can show that in the general case the product property implies ITPFI.

8. ITPFI *factors and* AT *flows*

In this section we prove our major result, namely the equivalence of the ITPFI and AT properties (theorem 8.3). The proof that ITPFI implies AT (lemma 8.1) is rather straightforward. It follows from the known existence of a factor martingale (lemma

7.3), together with the relationship between certain positive operators in the centralizer and finite measures on the flow space (see eqs. (4.13)-(4.18) and (4.30)-(4.33)). The converse is obtained by proving that AT of the flow of weights implies the product property (lemma 8.2).

LEMMA 8.1. *Let M be an* ITPFI *factor of type* III_0. *Then the flow of weights for M is* AT.

Proof. Since the proof for the discrete case $T(M)\neq\{0\}$ is much more transparent and, furthermore, motivates the proof for the general case, we present it first.

Case (i). $T(M)\neq\{0\}$. Let $T\in T(M)$, $T\neq 0$. Using either lemma 11.2 of [**1**] or results from [**2**] we can write $M=\bigotimes_{k=1}^{\infty}(M_k,\phi_k)$ where $\sigma_T^{\phi_k}=1$ for all k. We will use the discrete construction of the flow of weights here. Let λ, $\tilde{M}$, $\tilde{\phi}$, θ, B and ν_B be as in eqs. (4.1)-(4.12). We will prove that the base transformation θ is AT. By lemma 2.5 this is equivalent to the flow being AT.

Let $\varepsilon>0$, and let $\mu_1,\dots,\mu_n$ be finite measures on the base space B, $\mu_1,\dots,\mu_n<\nu_B$. Using eqs. (4.15) and (4.18) we obtain projections $e_j\in\tilde{M}_{\tilde{\phi}}$ such that $\mu_{e_j}=\mu_j$, $j=1,\dots,n$. It follows from our lemma 7.3 and lemma 2.3 of [**14**] that there exist $m<\infty$ and positive operators $f_j\in M^{(m)}_{\phi^{(m)}}$ where $M^{(m)}=\bigotimes_{k=0}^{m}M_k$, $\phi^{(m)}=\omega_\lambda\otimes(\bigotimes_{k=1}^{m}\phi_k)$ and $M_0=\mathscr{L}(\ell^2(\mathbb{Z}))$, such that

$$\tilde{\phi}(|e_j-f_j|)\leq\varepsilon,\qquad j=1,\dots,n. \tag{8.1}$$

Since e_j, $f_j\in M_\phi$ we have the inequality

$$|\tilde{\phi}((e_j-f_j)c)|\leq\|c\|\tilde{\phi}(|e_j-f_j|),\qquad c\in M,$$

(see for example, [7, lemma 11, p. 63]. It now follows from eqs. (8.1) and (4.15) that

$$\|\mu_{e_j}-\mu_{f_j}\|\leq\varepsilon,\qquad j=1,\dots,n. \tag{8.2}$$

The proof will be completed by showing that

$$f_j=\sum_{k\in\mathbb{Z}}\sum_{l=1}^{N}c_{jkl}e_{jkl},\qquad j=1,\dots,n, \tag{8.3}$$

where $c_{jkl}\geq 0$, e_{jkl} are minimal projections in $M^{(m)}_{\phi^{(m)}}$, $e_{jkl}\sim\theta^k e$ where e is a fixed minimal projection in $M^{(m)}_{\phi^{(m)}}$, and $N<\infty$. Eqs. (8.3) and (4.15) and lemma 5.8 then give

$$\mu_{f_j}=\sum_{k\in\mathbb{Z}}c_{jk}\lambda^{-k}\theta^k\mu_e,\qquad j=1,\dots,n, \tag{8.4}$$

where $c_{jk}=\sum_l c_{jkl}$. The AT now follows directly from eqs. (8.2) and (8.4). Eq. (8.3) will follow directly from the structure of $M^{(m)}_{\phi^{(m)}}$ which we now determine.

Recall that if $\psi(x)=\mathrm{Trace}(\rho x)$, $x\in P$ is a weight on a type I factor P and $\rho\in P$, then $\sigma_t^\psi=\mathrm{Ad}\,\rho^{it}$ and hence $P_\psi=\{\rho\}'$. Thus the determination of the structure of $M^{(m)}_{\phi^{(m)}}$ becomes an elementary exercise in linear algebra. We have $\phi^{(m)}(x)=\mathrm{Trace}(\rho^{(m)}x)$, $x\in M^{(m)}$ where $\rho^{(m)}\in M^{(m)}$ has eigenvalues $\prod_{k=0}^{m}\lambda_{k,j(k)}$ where $\{\lambda_{kj}\}_{j=1,\dots,n_k}$ is the eigenvalue list of (M_k,ϕ_k) and $n_k<\infty$ for $k\neq 0$. Since $\sigma_T^{\phi_k}=1$, all ratios $\lambda_{kr}/\lambda_{ks}$ are precisely some integral power of λ. For $k=0$, each λ^p, $p\in\mathbb{Z}$ occurs precisely once as an eigenvalue. Hence the eigenvalues of $\rho^{(m)}$ are of the form $\alpha\lambda^s$ where α is fixed, and each $s \in \mathbb{Z}$ occurs precisely

$$N = \prod_{k=1}^{m} n_k \tag{8.5}$$

times. It follows that

$$M^{(m)}_{\phi^{(m)}} = \{\rho^{(m)}\}' = \oplus_{k \in \mathbb{Z}} M(k), \tag{8.6}$$

where $M(k) = F_N$, and eqs. (4.1), (4.2) and (4.8) and $\theta = \text{Ad}\,\tilde{S}$ imply that $\theta(M(k)) = M(k+1)$. Eq. (8.3) now results directly from diagonalizing f_j.

Case (ii) is the general case. We use here the continuous construction of the flow of weights. Let $\tilde{M}$, $\tilde{\omega}$, θ_s, X and ν be as in eqs. (4.19)-(4.29). Let $\varepsilon > 0$, and let $\mu_1, \dots, \mu_n$ be finite measures on the flow space X, $\mu_1, \dots, \mu_n \prec \nu$. Precisely as in case (i) there exist $e_j \in \tilde{M}_{\tilde{\omega}}$ such that $\mu_{e_j} = \mu_j$ (see eqs. (4.32), (4.33)), and positive operators $f_j \in M^{(m)}_{\phi^{(m)}}$ for some $m < \infty$, satisfying eq. (8.2). The same argument used above shows that this time we have

$$M^{(m)}_{\phi^{(m)}} = \int^{\oplus} dt\, M(t), \tag{8.7}$$

where $M(t) = F_N$ and N is again given by eq. (8.5). Let $f \in M^{(m)}_{\phi^{(m)}}$. Then

$$f = \int^{\oplus} dt f(t), \tag{8.8}$$

$$\theta_s f = \int^{\oplus} dt f(t-s), \tag{8.9}$$

and

$$\tilde{\omega}(f) = \int dt\, e^t \tau(f(t)), \tag{8.10}$$

where τ is the trace on F_N. Diagonalizing the positive operators f_j, we obtain

$$f_j = \sum_{l=1}^{N} f_{jl}, \tag{8.11}$$

where

$$f_{jl} = \int^{\oplus} dt f_{jl}(t), \tag{8.12}$$

and

$$f_{jl}(t) = c_{jl}(t) e_{jl}(t), \tag{8.13}$$

where $c_{jl}(t) \geq 0$ and $e_{jl}(t)$ are minimal projections in $M(t)$. Since $\tilde{\omega}(f_j) = \mu_{f_j}(x) < \infty$, eq. (8.10) implies that $e^t c_{jl}(t) \in L^1(\mathbb{R})$. Since the transitive action of $\mathbb{R}$ on $\mathbb{R}$ is AT, there exist $\lambda_{jl} \in L^1(\mathbb{R})$ and $c(t)$, $e^t c(t) \in L^1(\mathbb{R})$ such that

$$\left\| e^t c_{jl}(t) - \int ds\, \lambda_{jl}(s) e^{s+t} c(s+t) \right\|_1 \leq \varepsilon N^{-1}. \tag{8.14}$$

Let

$$F_{jl} = \int^{\oplus} dt\, c_{jl}(t) e(t), \tag{8.15}$$

where $e(t)$ is a measurable family of minimal projections in $M(t)$. Since $e(t) \sim e_{jl}(t)$ in $M(t)$ we have $F_{jl} \sim f_{jl}$ in $M^{(m)}_{\phi^{(m)}}$ and hence

$$\mu_{F_j} = \mu_{f_{jl}}. \tag{8.16}$$

Let

$$C = \int^{\oplus} dt\, c(t)e(t), \tag{8.17}$$

and

$$G_{jl} = \int^{\oplus} dt\, g_{jl}(t)e(t) \tag{8.18}$$

where

$$g_{jl}(t) = \int ds\, \lambda_{jl}(s)e^{s}c(s+t). \tag{8.19}$$

Then

$$G_{jl} = \int ds\, \lambda_{jl}(s)e^{s}\theta_s C. \tag{8.20}$$

Lemma 4.1 now gives

$$\mu_{G_j} = \int ds\, \lambda_{jl}(s)e^{s}\theta_s\mu_C. \tag{8.21}$$

Using eqs. (8.10), (8.14), (8.15), (8.18) and (8.19) we obtain

$$\|\mu_{F_{jl}} - \mu_{G_{jl}}\| = \int dt\, e^{t}|c_j(t) - g_{jl}(t)| \leq \varepsilon N^{-1}. \tag{8.22}$$

The AT of θ_s now follows from eqs. (8.2), (8.11), (8.16), (8.21) and (8.22). □

LEMMA 8.2. *Let M be a Krieger factor of type* III_0. *If the flow of weights is* AT, *then M has the product property.*

Proof. Let $\varepsilon > 0$, $x_1, \ldots, x_p \in M$, V a strong neighbourhood of 0 in M, and ϕ a faithful state on M. We give first an outline of the argument. Connes' martingale condition [3] gives the existence of a conditional expectation E onto a finite-dimensional subalgebra N such that $\phi \circ E = \phi$, and $x_1, \ldots, x_p \in N + V$. If N were a finite type I subfactor, the condition $\phi \circ E = \phi$ would imply that ϕ was a product state relative to $M = N \otimes N^c$ and the product property would be trivially satisfied. The strategy is to embed N in a finite type I subfactor P so that ϕ is approximately a product state. The basic idea is to use the AT condition to construct a new state ψ such that $\|\phi - \psi\| \leq \varepsilon$, and a set of matrix units for P which are eigenvectors of σ_t^{ψ} (which implies that ψ is a product state relative to $M = P \otimes P^c$). More precisely, one selects a minimal projection $e^{(k)}$ from each full matrix algebra in N. A measure μ_k is associated with each $e^{(k)}$. Using the AT condition the μ_k can be approximated by measures $\mu'_k = \sum \mu_{kl}$. The μ_{kl} determine both the desired subprojections of the $e^{(k)}$ (which are minimal projections in P), and the state ψ.

We give first the argument for the discrete case $T(M) \neq \{0\}$. The argument for the general case then proceeds in exactly the same way.

Case (i). There exists $T \in T(M)$, $T \neq 0$. Let ϕ be a faithful state on M such that $\sigma_T^\phi = 1$.

Step (i). By [3] there is a conditional expectation $E: M \to N$ where N is a finite-dimensional subalgebra of M such that

$$\phi(E(x)) = \phi(x) \qquad \text{for all } x \in M, \tag{8.23}$$

and

$$E(x_j) \in V + x_j, \qquad j = 1, \ldots, p. \tag{8.24}$$

We have

$$N = \bigoplus_{k=1}^{n} N_k, \tag{8.25}$$

where the N_k are type I_{m_k} factors. Let ϕ_k denote the restriction of ϕ to N_k. Then

$$\phi_k(x) = \mathrm{Trace}\, \rho_k x, \qquad x \in N_k, \tag{8.26}$$

where $\rho_k \in N_k$. Choose matrix units e_{ij}^k for N_k such that ρ_k is diagonal, i.e.

$$\rho_k e_{ij}^k = \lambda_{ki} e_{ij}^k \delta_{ij}, \qquad i, j = 1, \ldots, m_k \tag{8.27}$$

and

$$\lambda_{k1} \geq \lambda_{k2} \geq \cdots \geq \lambda_{km_k} > 0. \tag{8.28}$$

Then, since $\sigma_t^{\phi \circ E}(E(x)) = \sigma_t^\phi(E(x))$, we have

$$\sigma_t^\phi(e_{ij}^k) = \sigma_t^{\phi_k}(e_{ij}^k) = (\lambda_{ki}/\lambda_{kj})^{it} e_{ij}^k. \tag{8.29}$$

Step (ii). In order to use the AT condition, we construct the flow of weights as in § 4, eqs. (4.1)-(4.18). Let P_0 denote the projection onto the basis vector e_0 of $\ell^2(\mathbb{Z})$. Let

$$f^k = P_0 \otimes e^k, \qquad k = 1, \ldots, n \tag{8.30}$$

where $e^k = e_{11}^k$. Since $P_0 \in \mathscr{L}(\ell^2(\mathbb{Z}))_{\omega_\lambda}$ and $e^k \in M_\phi$ (see eq. (8.29)) we have $f^k \in \tilde{M}_{\tilde{\phi}}$. Since $\tilde{\phi}(f^k) = \phi(e^k) < \infty$, eq. (4.15) defines measures

$$\mu_k = \mu_{f^k}, \qquad k = 1, \ldots, n, \tag{8.31}$$

on the base space B such that $\mu_k < \nu_B$. Using the AT condition and a variant of remark 2.2(iii) we obtain a measure $\mu < \nu_B$ and integers $n_k(l)$, $k = 1, \ldots, n$, $l = -L, \ldots, L$ such that

$$\|\mu_k - \mu_k'\| \leq \tfrac{1}{2} n^{-1} m_k^{-1} \varepsilon, \qquad k = 1, \ldots, n \tag{8.32}$$

where the measures

$$\mu_k' = \sum_{l=-L}^{+L} n_k(l) \lambda^{-l} \theta^l \mu \tag{8.33}$$

are non-zero.

Step (iii). We now modify the state ϕ. We first show that ϕ is determined by its restrictions to M_{e^k}, $k = 1, \ldots, n$ where, in effect, it is the conditional expectation. For $x \in M$ we have

$$E(x) = \sum x_{ij}^k e_{ij}^k, \tag{8.34}$$

where the numbers x_{ij}^k are determined by the equation

$$E(e_{ii}^k x e_{jj}^k) = x_{ij}^k e_{ij}^k. \tag{8.35}$$

Eqs. (8.23), (8.26), (8.27) and (8.34) give

$$\phi(x) = \sum \lambda_{ki} x_{ii}^k. \tag{8.36}$$

Since

$$\phi(e_{1i}^k x e_{i1}^k) = \phi(e_{1i}^k E(x) e_{i1}^k) = \lambda_{k1} x_{ii}^k, \tag{8.37}$$

we get

$$\phi(x) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} (\lambda_{ki}/\lambda_{k1}) \phi(e_{1i}^k x e_{i1}^k). \tag{8.38}$$

The desired alteration of ϕ will now be obtained by changing it on M_{e^k} and using eq. (8.38) to define the new state ψ on M. From eq. (8.32) and lemma 5.9 we obtain states $\tilde{\psi}_k$ such that

$$s(\tilde{\psi}_k) = f^k, \tag{8.39}$$

$$\mu_{\tilde{\psi}_k} = \mu'_k, \tag{8.40}$$

and

$$\|\tilde{\psi}_k - \tilde{\phi}_k\| \leq n^{-1} m_k^{-1} \varepsilon, \tag{8.41}$$

where $\tilde{\phi}_k$ is defined on $\tilde{M}_{f^k}$ by

$$\tilde{\phi}_k(P_0 \otimes x) = \phi_k(x). \tag{8.42}$$

We define ψ_k on M_{e^k} by

$$\psi_k(x) = \tilde{\psi}_k(P_0 \otimes x). \tag{8.43}$$

We define ψ on M by

$$\psi(x) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} (\lambda_{kj}/\lambda_{k1}) \psi_k(e_{1j}^k x e_{j1}^k). \tag{8.44}$$

Eqs. (8.28) and (8.41)-(8.44) give

$$\|\psi - \phi\| \leq \varepsilon. \tag{8.45}$$

Step (iv). We now embed N in a type I factor $P \subset M$ so that ψ is a product state relative to $M = P \otimes P^c$. Using eqs. (8.33), (8.40) and lemma 5.7 we obtain non-zero projections f_j^k, $j = 1, \ldots, q_k$, $k = 1, \ldots, n$ such that

$$f^k = \sum_{j=1}^{q_k} f_j^k, \tag{8.46}$$

$$\tilde{\psi}_k = \sum_{j=1}^{q_k} \tilde{\psi}_{kj}, \tag{8.47}$$

where

$$\tilde{\psi}_{kj} = (\tilde{\psi}_k)_{f_j^k}, \tag{8.48}$$

and integers s'_{kj} such that

$$\mu_{\tilde{\psi}_{kj}} = \lambda^{-s'_{kj}} \theta^{s_{kj}} \mu = \lambda^{-s_{kj}} \theta^{s_{kj}} \mu_{\tilde{\psi}_{11}}, \tag{8.49}$$

where $s_{kj} = s'_{kj} - s_{11}$. It now follows from lemmas 5.5 and 5.8 that

$$\tilde{\psi}_{kj} \sim \lambda^{s_{kj}} \tilde{\psi}_{11}. \tag{8.50}$$

Since P_0 is one-dimensional, we can define projections $e_j^k \in M$ by

$$f_j^k = P_0 \otimes e_j^k \tag{8.51}$$

and it then follows from eqs. (8.44), (8.47), (8.48), (8.50), (8.51) and lemma 1.2.3(b) of [2] that there exist partial isometries $u_{j1}^{k1} \in M$ such that

$$(u_{j1}^{k1})^* u_{j1}^{k1} = e_1^1, \tag{8.52}$$

$$u_{j1}^{k1}(u_{j1}^{k1})^* = e_j^k, \tag{8.53}$$

and

$$\sigma_t^\psi(u_{j1}^{k1}) = \lambda^{its_{kj}} u_{j1}^{k1}, \tag{8.54}$$

$j = 1, \ldots, q_k$, $k = 1, \ldots, n$. We extend the range of the index j to $1, \ldots, m_k q_k$ by defining

$$u_{j1}^{k1} = e_{l1}^k u_{t1}^{k1}, \tag{8.55}$$

where $j = (l-1)q_k + t$, $1 \le t < q_k$, $l = 1, \ldots, m_k$. Finally we obtain a complete set of matrix units by defining

$$u_{ij}^{kl} = u_{i1}^{k1}(u_{j1}^{l1})^* \tag{8.56}$$

where $i = 1, \ldots, q_k$, $j = 1, \ldots, q_l$, $k, l = 1, \ldots, n$. We can now define P to be the type I factor generated by the u_{ij}^{kl}. Eqs. (8.44) and (8.54)-(8.56) imply that the u_{ij}^{kl} are eigenvectors of σ_t^ψ. Hence $\sigma_t^\psi(P) = P$ which implies that ψ is a product state relative to $M = P \otimes P^c$.

Case (ii) is the general case. The argument is virtually identical. We use here the lacunary construction of the flow of weights. For this purpose we take ϕ to be a faithful lacunary integrable weight of infinite multiplicity. Basically the only change is that lemmas 5.5, 5.7, 5.8 and 5.9 are replaced by lemmas 6.1, 6.2 and 6.4.

Step (i) is identical. In step (ii) we now construct the flow of weights as in § 4, eqs. (4.39)-(4.48). In particular eq. (4.47) defines measures

$$\mu_k = \mu_{\phi_{e_k}}, \qquad k = 1, \ldots, n, \tag{8.57}$$

on the flow space X, $\mu_k < \nu$. Using the AT condition and again a variation on remark 2.2(iii) we obtain a measure $\mu < \nu$, $t_l \in \mathbb{R}_+^*$, $n_k(l) \in \mathbb{Z}$, $l = 1, \ldots, L$ such that

$$\|\mu_k - \mu'_k\| \le \tfrac{1}{5} n^{-1} m_k^{-1} \varepsilon, \qquad k = 1, \ldots, n, \tag{8.58}$$

where the measures

$$\mu'_k = \sum_{l=1}^{L} n_k(l) t^{-1} \mathscr{F}_{t_l}^M \mu \tag{8.59}$$

are non-zero.

Step (iii) is almost identical. We work directly on M and use lemma 6.4 to obtain states ψ_k on M_{e^k} satisfying

$$s(\psi_k) = e^k, \tag{8.60}$$

$$\mu_{\psi_k} = \mu'_k, \tag{8.61}$$

and

$$\|\psi_k - \phi_k\| \le n^{-1} m_k^{-1} \varepsilon. \tag{8.62}$$

ψ is again defined by eq. (8.44) and satisfies eq. (8.45). In step (iv) we begin by noting that the states ψ_k defined by eq. (8.60) are integrable since the measures μ'_k are smooth (see eq. (4.48) et seq.). Using eqs. (8.59), (8.61) and lemma 6.2 we obtain non-zero projections e_j^k, $j = 1, \ldots, q_k$, $k = 1, \ldots, n$ such that

$$e^k = \sum_{j=1}^{q_k} e_j^k \tag{8.63}$$

and

$$\psi_k = \sum_{j=1}^{q_k} \psi_{kj}, \tag{8.64}$$

where $\psi_{k_j} = (\psi_k)_{e_j^k}$, and $t'_{kj} \in \mathbb{R}$ such that

$$\mu_{\psi_{kj}} = e^{-t'_{kj}} \mathcal{F}^M_{t'_{kj}} \mu = e^{-t_{kj}} \mathcal{F}^M_{t_{kj}} \mu_{\psi_{11}}, \tag{8.65}$$

where $t_{kj} = t'_{kj} - t_{11}$. It now follows from lemma 6.1 that

$$\psi_{kj} \sim e^{-t_{kj}} \psi_{11}. \tag{8.66}$$

As before, it follows from eqs. (8.44), (8.66) and lemma 1.2.3(b) of [**2**] that there exist partial isometries u_{ij}^{kl} satisfying eqs. (8.52)-(8.56), and the type I factor P generated by the u_{ij}^{kl} has the desired properties. □

THEOREM 8.3. *Let M be a type* III_0 *injective factor. Then the following are equivalent.*
(i) *M is* ITPFI.
(ii) *The flow of weights for M is approximately transitive.*
(iii) *M has the product property.*

Proof. By [**4**] M is a Krieger factor. We have the following implications: (i)⇒(iii), corollary 7.4; (iii)⇒(i), lemma 7.6; (i)⇒(ii), lemma 8.1; and (ii)⇒(iii), lemma 8.2. □

Acknowledgement. Part of this work was carried out during the second named author's visit to the IHES, Bures-sur-Yvette and the University of Warwick. He would like to thank the Université de Paris VI, the Institut des Hautes Etudes Scientifiques, and the University of Warwick for their kind hospitality and financial support.

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