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Foliations
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Leafwise Homotopy Equivalence and Rational Pontrjagin Classes

Paul Baum and Alain Connes

§ 1. Introduction

Let V, V' be C^∞ manifolds[1] and let $f: V' \to V$ be a homotopy equivalence. Denote the i-th rational Pontrjagin classes by $P_i(V)$, $P_i(V')$.

$$P_i(V) \in H^{4i}(V; Q)$$
$$P_i(V') \in H^{4i}(V'; Q)$$

If f is homotopic to a homeomorphism, then the theorem of S. P. Novikov [16] applies to give

$$f^* P_i(V) = P_i(V') \qquad i = 1, 2, \cdots$$

where $f^*: H^{4i}(V; Q) \to H^{4i}(V'; Q)$ is the map of rational cohomology determined by f.

Thus whether or not $f^*: H^*(V; Q) \to H^*(V'; Q)$ preserves the rational Pontrjagin classes may be viewed as the first obstruction to deforming f to a homeomorphism. The first examples of homotopy equivalences which do not preserve the rational Pontrjagin classes were given by I. Tamura [17] and also, independently, by Shimada and R. Thom. These examples are reviewed in Section 5 below.

In this note we shall outline a proof that a leafwise homotopy equivalence of compact foliated manifolds does preserve the rational Pontrjagin classes if one of the foliations has negatively curved leaves. The precise statement of our result is given in Section 2 below. Of course, this raises the question of whether a homotopy equivalence satis-

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[1] V and V' are assumed to be C^∞, Hausdorff, second countable, and without boundary.

fying our hypothesis is homotopic to a homeomorphism.

The case of negatively curved leaves, which we study, contrasts quite sharply with the case of positively curved leaves. The examples of Tamura-Shimada-Thom [17] are 4–sphere bundles over the 4–sphere. These manifolds are foliated (in fact fibred over S^4) with positively curved leaves and the homotopy equivalences considered are leafwise but do not preserve the rational Pontrjagin classes.

The positively and negatively curved cases are unified by a conjecture which we call the *Novikov conjecture for foliations*. This conjecture is stated in Section 6 below.

§ 2. Statement of Theorem

A foliation is a pair (V, F) with V a C^∞ manifold and F a C^∞ integrable sub-vector-bundle of the tangent bundle TV. Given two points x, y of V set

$$x \sim y \quad (F)$$

if x and y are on the same leaf of the foliation. Let (V', F') be a second foliated manifold. A continuous map $f\colon V' \to V$ is leafwise if whenever $w \sim p\ (F')$ then $fw \sim fp\ (F)$. Two leafwise maps $f_0, f_1\colon V' \to V$ are *leafwise homotopic* if there exists a homotopy f_t, $0 \leqq t \leqq 1$, such that for all $p \in V'$ and all $t \in [0, 1]$

$$f_0(p) \sim f_t(p) \quad (F).$$

The notation

$$f_0 \sim f_1 \quad (F', F)$$

will indicate that f_0 and f_1 are leafwise homotopic.

Denote the identity maps of V, V' by 1_V, $1_{V'}$. A leafwise map $f\colon V' \to V$ is a *leafwise homotopy equivalence* if there exists a leafwise map $g\colon V \to V'$ with

$$f \circ g \sim 1_V \quad (F, F)$$

and

$$g \circ f \sim 1_{V'} \quad (F', F').$$

Let $\langle\ ,\ \rangle$ be a C^∞ Euclidean structure for F. Thus $\langle\ ,\ \rangle$ assigns a positive definite real-valued inner product to each fibre F_x. $\langle\ ,\ \rangle$ restricts to give a C^∞ Riemannian metric for each leaf of the foliation (V, F).

(2.1) **Definition.** A foliation (V, F) has *negatively curved leaves* if there exists a C^∞ Euclidean structure for F such that each leaf has all sectional curvatures non-positive.

(2.2) **Definition.** Let $f\colon V' \to V$ be a continuous map. Denote the i-th rational Pontrjagin classes of V, V' by P_i, P'_i. *f preserves the rational Pontrjagin classes* if for $i=1, 2, \cdots, f^*(P_i)=P'_i$ where $f^*\colon H^{4i}(V; \boldsymbol{Q}) \to H^{4i}(V'; \boldsymbol{Q})$ is the map of rational cohomology determined by f.

(2.3) **Definition.** A foliation (V, F) is *orientable* if V is an orientable manifold and F is an orientable $\boldsymbol{R}$ vector bundle on V.

(2.4) **Theorem.** *Let (V, F) and (V', F') be C^∞ orientable foliations with V and V' compact. Let $f\colon V' \to V$ be a leafwise homotopy equivalence. Assume that (V, F) has negatively curved leaves. Then f preserves the rational Pontrjagin classes.*

§ 3. Outline of Proof

The proof of Theorem (2.4) uses ideas of G. Lusztig [13], G. G. Kasparov [11] [12] and A. S. Miscenko [14] [15]. In fact, the proof is done by suitably adapting their methods to the foliation context. There are two parts to the proof. The first part is analytic and uses the K theory of C^* algebras. The second part is a translation from an essentially analytic conclusion to topology. The negative curvature hypothesis is used *only* in the second part.

To begin the proof, let π be the fundamental groupoid along the leaves of (V, F). A point of π is an equivalence class of continuous paths

$$\gamma\colon [0, 1] \longrightarrow V$$

such that for all $t \in [0, 1]$

$$\gamma(0) \sim \gamma(t) \quad (F).$$

Thus γ stays within one leaf of the foliation. Two such paths γ_0, γ_1 are identified if

$$\gamma_0(0)=\gamma_1(0)$$
$$\gamma_0(1)=\gamma_1(1)$$

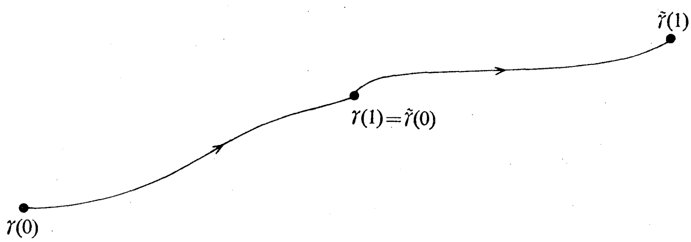
and there is a homotopy γ_t from γ_0 to γ_1 with each γ_t staying within the same leaf of F and with endpoints fixed.

$$\gamma_t(0)=\gamma_0(0) \qquad t \in [0, 1]$$
$$\gamma_t(1)=\gamma_0(1) \qquad t \in [0, 1]$$

π is then a C^∞ manifold which might not be Hausdorff. π comes equipped with two maps to V.

$$\pi \underset{r}{\overset{s}{\rightrightarrows}} V$$
$$s(\gamma)=\gamma(0)$$
$$r(\gamma)=\gamma(1)$$

π is a groupoid since two points γ, $\tilde{\gamma}$ of π can be composed if $r(\gamma)=s(\tilde{\gamma})$.



The composition, denoted $\gamma\tilde{\gamma}$ has $s(\gamma\tilde{\gamma})=\gamma(0)$ and $r(\gamma\tilde{\gamma})=\tilde{\gamma}(1)$. From π a C^* algebra, denoted $C^*_\pi(V, F)$, is constructed by the same method used in [6] [7] to define $C^*(V, F)$. The only difference is that now the groupoid π is used instead of the holonomy groupoid. Except for this change the construction is the same, and this construction is given in the appendix.

A leafwise homotopy equivalence $f\colon V' \to V$ gives a Morita equivalence of the groupoids π', π. This establishes an isomorphism of the K theory of the C^* algebras $C^*_{\pi'}(V', F')$, $C^*_\pi(V, F)$. This isomorphism will be denoted by

(3.1) $$\hat{f}\colon K_* C^*_{\pi'}(V', F') \longrightarrow K_* C^*_\pi(V, F).$$

By hypothesis (V, F) and (V', F') are orientable. For $p \in V'$ let $\mathscr{L}'_p(\mathscr{L}_{fp})$ be the leaf containing p (fp). The compactness of V and V' implies that $f\colon \mathscr{L}'_p \to \mathscr{L}_{fp}$ is proper. Choose the orientations of F and F' so that for each leaf $\mathscr{L}'_p$, $f\colon \mathscr{L}'_p \to \mathscr{L}_{fp}$ is orientation preserving.

For simplicity, assume that the leaves of (V, F) and $(V' F')$ are even dimensional. Choose C^∞ Euclidean structures for F, F' and let $d+\delta$, $d'+\delta'$ be the Hirzebruch signature operators along the leaves. Restricted to a leaf $\mathscr{L}_x$, $d+\delta$ is the Hirzebruch signature operator of $\mathscr{L}_x$. The index of $d+\delta$, denoted Index $(d+\delta)$, is an element of $K_0 C^*_\pi(V, F)$.

(3.2) $$\text{Index}\,(d+\delta) \in K_0 C_\pi^*(V, F)$$

The analytic part of the proof of Theorem 2.4 is:

(3.3) **Proposition.** Index $(d+\delta)$ *and* Index $(d'+\delta')$ *correspond under the isomorphism* $\hat{f}$: $K_0 C_{\pi'}^*(V', F') \to K_0 C_\pi^*(V, F)$.

(3.4) $$\hat{f}\,\text{Index}\,(d'+\delta') = \text{Index}\,(d+\delta).$$

Proposition (3.3) is proved by using the method of G. Lusztig [13], adapted to the context of Hilbert C^* modules[2]. Proposition (3.3) is quite general and does not use the negative curvature hypothesis. Next, consider the commutative diagram

(3.5) $$\begin{array}{ccc} K_0 C_{\pi'}^*(V', F') & \xrightarrow{\hat{f}} & K_0 C_\pi^*(V, F) \\ \uparrow \iota' & & \uparrow \iota \\ K^0(F') & \xrightarrow{f_!} & K^0(F) \\ \downarrow \tau' & & \downarrow \tau \\ H^*(V'; Q) & \xrightarrow[f_*]{} & H^*(V; Q). \end{array}$$

In this diagram ι: $K^0(F) \to K_0 C_\pi^*(V, F)$ assigns to a symbol σ, the index of the pseudo-differential operator $\mathscr{D}_\sigma$ along the leaves associated to σ.

(3.6) $$\iota(\sigma) = \text{Index}\,(\mathscr{D}_\sigma) \qquad \sigma \in K^0(F)$$

τ: $K^0(F) \to H^*(V; Q)$ is

(3.7) $$\tau(\sigma) = (-1)^{l(l+1)/2} T^{-1}\,\text{ch}\,(\sigma) \cup \text{Td}\,(C \otimes_R F) \qquad \sigma \in K^0(F)$$

where l is the dimension of the leaves of F, ch; $K^0(F) \to H_c^*(F; Q)$ is the Chern character[3], T: $H^*(V; Q) \to H_c^*(F; Q)$ is the Thom isomorphism, and Td $(C \otimes_R F)$ is the Todd class of the complexification of F.

(3.8) $$\text{Td}\,(C \otimes_R F) \in H^*(V; Q)$$

For $f_!$: $K^0(F') \to K^0(F)$ in diagram (3.5) recall [2] that if M_1 and M_2 are C^∞ manifolds and h: $M_1 \to M_2$ is any continuous map, then there is a homomorphism

[2] J. Kaminker and J. Miller have also generalized Lusztig's method to Hilbert C^* modules.

[3] $K^0(F)$ is the K theory of F with compact supports. $H^*(F; Q)$ is the rational cohomology of F with compact supports. Let BF, SF be the unit ball and unit sphere bundles of F. Since V is compact $H_c^*(F; Q) = H^*(BF, SF; Q)$.

(3.9) $$h_!;\ K^0(TM_1)\longrightarrow K^0(TM_2).$$

Since f: $V'\to V$ is leafwise the homomorphisms

(3.10) $$f_!:\ K^0(T\mathscr{L}'_p)\longrightarrow K^0(T\mathscr{L}_{fp})$$

fit together to give

(3.11) $$f_!:\ K^0(F')\longrightarrow K^0(F).$$

Finally, f_*: $H^*(V', Q)\to H^*(V; Q)$ is the Gysin "push-forward" map in rational cohomology. f_* depends on the choice of orientation for V, V', and this choice is assumed made so that f is orientation preserving.

This completes the description of the commutative diagram (3.5). The commutativity of the diagram requires proof, but we omit the details of this argument.

Let $\sigma(d+\delta)$ and $\sigma(d'+\delta')$ be the symbols of $d+\delta$ and $d'+\delta'$.

$$\sigma(d+\delta)\in K^0(F)$$
$$\sigma(d'+\delta')\in K^0(F')$$

From (3.4) one would like to conclude that in $K^0(F)$ there is the equality

(3.12) $$f_!\sigma(d'+\delta')=\sigma(d+\delta).$$

(3.12) is clearly implied by (3.4), the commutativity of (3.5), and

(3.13) **Proposition.** *Let (V, F) be a C^∞ foliation with V compact and with negatively curved leaves. Then*

$$\iota:\ K^*(F)\longrightarrow K_*C^*_\pi(V, F)$$

is injective.

The proof of (3.13) is summarized in Section 4 below.

Granted the validity of (3.12), return to diagram (3.5) and observe (as in [3]) that

(3.14) $$\tau\sigma(d+\delta)=\boldsymbol{L}(F)$$

(3.15) $$\tau'\sigma(d'+\delta')=\boldsymbol{L}(F').$$

Here $\boldsymbol{L}$ denotes the $\boldsymbol{L}$ polynomial in the Pontrjagin classes. (3.12), (3.14) and (3.15) imply:

(3.16) $$f_*\boldsymbol{L}(F')=\boldsymbol{L}(F)$$

which immediately reformulates to

$$f^*L(F)=L(F'). \tag{3.17}$$

Let ν, ν' be the normal bundles of the foliations (V, F), (V', F'). For $x \in V$,

$$\nu_x = T_x V / F_x. \tag{3.18}$$

At the level of topological microbundles, the pull-back via f of ν is ν'.

$$f^*\nu=\nu' \tag{3.19}$$

Hence by Novikov's theorem [16]:

$$f^*L(\nu)=L(\nu') \tag{3.20}$$

(3.20) and (3.13) combine to yield

$$f^*L(TV)=L(TV'). \tag{3.21}$$

(3.21) implies that f preserves the rational Pontrjagin classes.

§ 4. Proof of injectivity

To prove Proposition (3.13) it is convenient (although not essential) to assume that (V, F) has even dimensional leaves and that F is a Spin^c vector bundle on V. The Thom isomorphism in K theory [1] then applies to give an isomorphism

$$K^0(F) \cong K^0(V). \tag{4.1}$$

$C(V)$ denotes the C^* algebra of all continuous complex-valued functions on V. There is the standard isomorphism

$$K^0(V) \cong K_0 C(V). \tag{4.2}$$

With the identifications (4.1) and (4.2), the homomorphism ι: $K^0(F) \to K_0 C^*_\pi(V, F)$ of Proposition (3.13) becomes

$$K_0 C(V) \longrightarrow K_0 C^*_\pi(V, F). \tag{4.3}$$

By assumption F has a Spin^c structure. The Dirac operator D along the leaves of (V, F) gives an element $[D]$ of $KK_0(C(V), C^*_\pi(V, F))$

$$[D] \in KK_0(C(V), C^*_\pi(V, F)). \tag{4.4}$$

Here KK is the bivariant K theory of G.G. Kasparov [11]. Quite generally, for two C^* algebras A, B Kasparov's group $KK(A, B)$ comes equipped with a map

$$(4.5) \qquad \theta: KK(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(K_*A, K_*B).$$

So in our context we have

$$(4.6) \qquad \theta: KK_0(C(V), C^*_\pi(V,F)) \longrightarrow \mathrm{Hom}_Z(K_0C(V), K_0C^*_\pi(V, F)).$$

This map takes $[D] \in KK_0(C(V), C^*_\pi(V, F))$ to $\iota \in \mathrm{Hom}\,(K_0(C(V)), K_0C^*_\pi(V, F))$

$$(4.7) \qquad \theta[D] = \iota.$$

Hence to prove Proposition (3.13) it will suffice to construct an element Δ in $KK_0(C^*_\pi(V, F), C(V))$ with

$$(4.8) \qquad [D] \underset{C^*_\pi(V,F)}{\otimes} \Delta = 1_{C(V)}$$

where $\otimes_{C^*(V,F)}$ denotes the Kasparov product:

$$(4.9) \qquad KK(C(V), C^*_\pi(V, F)) \otimes KK(C^*_\pi(V, F), C(V)) \longrightarrow KK(C(V), C(V))$$

The desired element Δ in $KK(C^*_\pi(V, F), C(V))$ is obtained by the "Dual Dirac" construction of A. S. Miscenko [15]. This construction uses the negative curvature hypothesis.

To construct the "Dual Dirac" element Δ, as in Section 3 above let r, s be the two maps of π to V.

$$\pi \underset{s}{\overset{r}{\rightrightarrows}} V$$

$$s(\gamma) = \gamma(0)$$

$$r(\gamma) = \gamma(1)$$

On V there are the two $\frac{1}{2}$-Spin bundles associated to the Spin^c structure of F. Denote these by S^+, S^-. By hypothesis an Euclidean structure $\langle\ ,\ \rangle$ has been chosen for F such that each leaf has all sectional curvatures non-positive. For $x \in V$, let $\mathscr{H}^\pm_x = L^2(r^{-1}(x), S^\pm_x)$ be the Hilbert space of all L^2 functions from $r^{-1}(x)$ to $S^\pm_x$.

$$(4.10) \qquad r^{-1}(x) = \{\gamma \in \pi \mid r(\gamma) = x\}$$

$\mathscr{H}_x^+$ is then a field of Hilbert spaces on V and so gives rise to a $C(V)$ Hilbert C^* module. Moreover, $C_\pi^*(V, F)$ operates (on the left) on this Hilbert C^* module by convolution. Similarly set $\mathscr{H}_x^- = L^2(r^{-1}(x), S_x^-)$.

Define an operator $T_x: \mathscr{H}_x^+ \to \mathscr{H}_x^-$ as follows. For $x \in V$, let $\mathscr{L}_x$ be the leaf containing x. Then s maps $r^{-1}(x)$ onto $\mathscr{L}_x$.

$$\text{(4.11)} \qquad s: r^{-1}(x) \longrightarrow \mathscr{L}_x$$

and thus $r^{-1}(x)$ is the universal covering space of $\mathscr{L}_x$. The Riemannian metric of non-positive curvature on $\mathscr{L}_x$ lifts to give a Riemannian metric of non-positive curvature on $r^{-1}(x)$. Hence $r^{-1}(x)$ is a complete simply connected Riemannian manifold of non-negative curvature. In $r^{-1}(x)$ there is a distinguished point, namely the point 1_x where 1_x denotes the constant path at x. Given any other point γ in $r^{-1}(x)$ there is a unique geodesic in $r^{-1}(x)$ from γ to 1_x. Let $\xi(\gamma)$ be the unit tangent vector to this geodesic at 1_x. $\xi(\gamma)$ may be viewed as an element of F_x, since F_x is identified with the tangent space of $r^{-1}(x)$ at 1_x.

$$\text{(4.12)} \qquad \xi(\gamma) \in F_x \qquad \gamma \in r^{-1}(x),\ \gamma \neq 1_x$$

Choose a small compact ball in $r^{-1}(x)$ centered at 1_x. Let $\alpha: r^{-1}(x) \to [0, 1]$ be a C^∞ function which is 1 outside the ball and is zero on a neighborhood of 1_x. Then $T_x: \mathscr{H}_x^+ \to \mathscr{H}_x^-$ is defined by

$$\text{(4.13)} \qquad (T_x u)\gamma = \alpha(\gamma)\xi(\gamma)\cdot u(\gamma) \qquad x \in V,\ \gamma \in r^{-1}(x),\ u \in \mathscr{H}_x^+$$

where $\cdot$ denotes the Clifford multiplication

$$\text{(4.14)} \qquad F_x \otimes S_x^+ \longrightarrow S_x^-.$$

The triple $(\mathscr{H}_x^+, \mathscr{H}_x^-, T_x)$ determines an element Δ in the Kasparov group $KK_0(C_\pi^*(V, F), C(V))$. The proof of Proposition (3.13) is now completed by showing that in $KK(C(V), C(V))$ there is the equality:

$$\text{(4.15)} \qquad |D| \underset{C_\pi^*(V,F)}{\otimes} \Delta = 1_{C(V)}$$

§ 5. Examples of Tamura-Shimada-Thom

$\boldsymbol{H}$ denotes the quaternions. View S^4 as $\boldsymbol{H}P^1$, the quaternion projective line. Over $\boldsymbol{H}P^1$ there is the canonical bundle E. As an $\boldsymbol{H}$ vector bundle E has fibre dimension one. As an $\boldsymbol{R}$ vector bundle E has fibre dimension 4 and

$$\text{(5.1)} \qquad P_1(E)[S^4] = -2.$$

In (5.1) $P_1(E)$ is the Pontrjagin class of the $\boldsymbol{R}$ vector bundle E.

For each integer k let E_k be the $\boldsymbol{R}$ vector bundle on S^4 which is the pull-back of E via a map $S^4 \to S^4$ of degree k. Then:

(5.2) $$P_1(E_k)[S^4] = -2k$$

1 denotes the trivial $\boldsymbol{R}$ vector bundle on S^4, of fibre dimension one.

(5.3) $$1 = S^4 \times \boldsymbol{R}$$

Choose an Euclidean structure $\langle\ ,\ \rangle$ for $E_k \oplus 1$ and let $W_k = S(E_k \oplus 1)$ be the unit sphere bundle of $E_k \oplus 1$.

Each W_k is a 4-sphere bundle over S^4. Let ρ: $W_k \to S^4$ be the projection. The evident section s: $S^4 \to W_k$ applied to the fundamental cycle of S^4 gives $s_*[S^4] \in H_4(W_k; \boldsymbol{Z})$. Now $H_4(W_k; \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z}$ with the generators given by $s_*[S^4]$ and $[\Sigma]$ where $[\Sigma]$ is the fundamental cycle of a fibre of the projection ρ: $W_k \to S^4$. Then:

(5.4) $$P_1(TW_k)s_*[S^4] = -2k$$

(5.5) $$P_1(TW_k)[\Sigma] = 0$$

According to a result of James and Whitehead [10] among the W_k there are only thirteen different fibre homotopy types. From (5.4) and (5.5) it is clear that a fibrewise homotopy equivalence from W_{k_1} to W_{k_2} $(k_1 \neq k_2)$ cannot preserve the rational Pontrjagin classes. Hence by the result of James and Whitehead there are many fibrewise homotopy equivalences among the W_k which do not preserve the rational Pontrjagin classes.

With E_k as above let $V_k = E_k \oplus 1$. Given a fibrewise homotopy equivalence between two W_k, extend this radially to obtain a fibrewise homotopy equivalence between the corresponding V_k. These fibrewise homotopy equivalences among the V_k are proper and do not preserve the rational Pontrjagin classes. Note that since V_k is a vector bundle over S^4, V_k can be given a Riemannian metric such that each fibre has zero curvature. This shows that the compactness hypothesis in Theorem (2.4) is really needed, and cannot be replaced by the condition that the leafwise homotopy equivalence f: $V \to V'$ is proper.

§ 6. Novikov conjecture for foliations

As above, (V, F) is a C^∞ foliation and π is the fundamental groupoid along the leaves. $B\pi$ denotes the classifying space of the topological groupoid π. Since V is the units of π there is a canonical map [9]:

(6.1) $$\lambda: V \longrightarrow B\pi$$

π is itself a principal π-bundle over V and the map λ of (6.1) is the classifying map of this principal π-bundle.

(6.2) **Conjecture.** Let (V, F) and (V', F') be orientable C^∞ foliations with V, V' compact. Let $f\colon V' \to V$ be a leafwise homotopy equivalence. Choose orientations for V and V' so that f is orientation preserving. Then in $H_*(B\pi; \boldsymbol{Q})$ there is the equality:

$$\lambda_*(L(TV) \cap [V]) = \lambda_* f_*(L(TV') \cap [V']) \tag{6.3}$$

Remarks. In (6.3) $[V]$, $[V']$ are the fundamental cycles of V, V'. $\boldsymbol{L}$ denotes the total $\boldsymbol{L}$ polynomial in the Pontrjagin classes.

$$f_*\colon H_*(V'; \boldsymbol{Q}) \longrightarrow H_*(V; \boldsymbol{Q})$$
$$\lambda_*\colon H_*(V; \boldsymbol{Q}) \longrightarrow H_*(B\pi; \boldsymbol{Q})$$

are the maps of rational homology determined by f and λ. As usual $\cap$ denotes the cap product so that $L(TV) \cap [V]$ is the Poincaré dual of $L(TV)$.

Conjecture (6.2) is the Novikov conjecture for foliations. It is interesting to see what (6.2) becomes at the two extremes. One extreme is when each leaf is a point. For this case $f\colon V' \to V$ is a homeomorphism and (6.2) becomes Novikov's theorem [16]. The opposite extreme occurs when there is only one leaf, i.e. $F = TV$, $F' = TV'$. In this case $B\pi$ is homotopy equivalent to $B\pi_1$ where $\pi_1 = \pi_1(V)$ is the fundamental group of V. Hence at this extreme (6.2) becomes the Novikov conjecture on the homotopy invariance of higher signatures [5]. In general, (6.2) can be viewed as a statement which interpolates between the Novikov theorem and the Novikov conjecture.

Let $\mathscr{L}$ be a leaf of the foliation (V, F). $\pi_i(\mathscr{L})$ denotes the i-th homotopy group of $\mathscr{L}$. If for every leaf $\mathscr{L}$, $\pi_i(\mathscr{L}) = 0$ for all $i \geqq 2$, then $B\pi = V$. If V is compact and (V, F) has negatively curved leaves this is the case, so Theorem (2.4) is a special case of Conjecture (6.2).

Let $n = \dim V$, $l = \dim_{\boldsymbol{R}}(F_x)$, $q = n - l$. Let Γ_q be the Haefliger groupoid [8] of all germs of homeomorphisms of $\boldsymbol{R}^q$. The Haefliger classifying map [8]:

$$V \longrightarrow B\Gamma_q \tag{6.4}$$

for the foliation (V, F) factors through $B\pi$.

$$V \xrightarrow{\lambda} B\pi \longrightarrow B\Gamma_q \tag{6.5}$$

Given the hypothesis of (6.2) there is then a commutative diagram

$$\begin{array}{ccccc} V' & \xrightarrow{\lambda'} & B\pi' & \longrightarrow & B\Gamma_q \\ f\downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow[\lambda]{} & B\pi & \longrightarrow & B\Gamma_q \end{array} \tag{6.6}$$

in which the right vertical arrow is the identity map of $B\Gamma_q$.

If (6.2) is valid, then the following conjecture is also valid. The hypothesis is the same as in (6.2).

$$h\colon V \longrightarrow B\Gamma_q$$
$$h'\colon V' \longrightarrow B\Gamma_q$$

are the Haefliger classifying maps.

(6.7) **Conjecture.** Let (V, F), (V', F') and $f\colon V \to V'$ be as in (6.2). Let α be any element of $H^*(B\Gamma_q; \boldsymbol{C})$. Then

$$(h^*\alpha \cup \boldsymbol{L}(TV))[V] = (h'^*\alpha \cup \boldsymbol{L}(TV'))[V']. \tag{6.8}$$

Remarks. Due to Proposition (3.3), Conjecture (6.2) is implied by the isomorphism conjecture of [4]. Let (V, F) be any C^∞ foliation. In [4] a geometric K theory group $K^*_\pi(V, F)$ is introduced, together with a natural map

$$\mu\colon K^*_\pi(V, F) \longrightarrow K_* C^*_\pi(V, F)$$

which is conjectured to be an isomorphism. Conjecture (6.2) is implied by Proposition (3.3) and the rational injectivity of μ.

If (V, F) has negatively curved leaves, then $K^*_\pi(V, F)$ is isomorphic to $K^*(F)$ and μ becomes $\iota\colon K^*(F) \to K_* C^*_\pi(V, F)$. So in the negatively curved case Proposition (3.13) verifies the injectivity of μ.

Appendix: $C^*_\pi(V, F)$

Let W be a finite dimensional vector space over $\boldsymbol{R}$. $\beta(W)$ is the set of all ordered bases of W.

(A1) **Definition.** A $\frac{1}{2}$-*density* on W is a function $\phi\colon \beta(W) \to \boldsymbol{C}$ such that for $\beta_1, \beta_2 \in \beta(W)$

$$\phi(\beta_2) = |\det(\beta_2, \beta_1)|^{1/2} \phi(\beta_1) \tag{A2}$$

where $\det(\beta_2, \beta_1)$ is the determinant of the matrix which expresses β_2 in terms of β_1.

$\Omega^{1/2}(W)$ denotes the $\boldsymbol{C}$ vector space of all $\frac{1}{2}$-densities on W.

(A3) $$\dim_C \Omega^{1/2}(W)=1$$

For any C^∞ manifold X, $\Omega^{1/2}(X)$ denotes the C^∞ line bundle on X whose fibre at $p \in X$ is $\Omega^{1/2}(T_pX)$:

(A4) $$\Omega^{1/2}(X)_p=\Omega^{1/2}(T_pX)$$

$L^2(X)$ is the Hilbert space of all square-summable sections of $\Omega^{1/2}(X)$. (Note that it is not necessary to choose a measure or a Riemannian metric on X.)

Given a C^∞ foliation (V, F) let π be the fundamental groupoid along the leaves. As in Section 3 above, π maps to V by s and r.

(A5) $$\pi \underset{r}{\overset{s}{\rightrightarrows}} V$$

On π, let $\Omega(\pi)$ be the C^∞ line bundle whose fibre at $\gamma \in \pi$ is:

(A6) $$\Omega(\pi)_\gamma=\Omega^{1/2}(F_{s\gamma})\otimes\Omega^{1/2}(F_{r\gamma})$$

$C_c^\infty(\Omega(\pi))$ is the space of all C^∞ sections of $\Omega(\pi)$ with compact support[4). $C_c^\infty(\Omega(\pi))$ is an algebra with involution:

(A7) $$(\Psi_1+\Psi_2)(\gamma)=\Psi_1\gamma+\Psi_2\gamma, \qquad \Psi_1, \Psi_2 \in C_c^\infty(\Omega(\pi))$$

(A8) $$(\Psi_1*\Psi_2)(\gamma)=\int_{\gamma_1\gamma_2=\gamma}(\Psi_1\gamma_1)(\Psi_2\gamma_2)$$

(A9) $$\Psi^*(\gamma)=\overline{\Psi(\gamma^{-1})}$$

For $x \in V$, $r^{-1}(x)=\{\gamma \in \pi \mid r\gamma=x\}$. Define a representation ρ_x of $C_c^\infty(\Omega(\pi))$ as bounded operators on the Hilbert space $L^2(r^{-1}x)$ by:

(A10) $$((\rho_x\Psi)\phi)\gamma=\int_{\gamma_1\gamma_2=\gamma}\Psi(\gamma_1)\phi(\gamma_2)$$
$$\Psi \in C_c^\infty(\Omega(\pi))$$
$$\phi \in L^2(r^{-1}x)$$

If x and y are on the same leaf of the foliation, then ρ_x is unitarily equivalent to ρ_y. $C_\pi^*(V, F)$ is, by definition, the completion of $C_c^\infty(\Omega(\pi))$ with respect to the norm

(A11) $$\|\Psi\|=\operatorname*{Sup}_{x\in V}\|\rho_x\Psi\|$$

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4) π might not be Hausdorff. For a careful explanation of details needed when π is not Hausdorff see [7].

where $\|\rho_x\Psi\|$ is the operator norm of the bounded operator $\rho_x\Psi$.

References

[1] M. F. Atiyah, Bott periodicity and the index of elliptic operators, Quart. J. J. Math. Oxford, **19** (1968), 113–140.
[2] M. F. Atiyah and I. M. Singer, The index of elliptic operators I, Ann. of Math. (2), **87** (1968), 484–530.
[3] ——, The index of elliptic operators III, Ann. of Math. (2), **87** (1968), 546–604.
[4] P. Baum and A. Connes, Geometric K theory for Lie groups and foliations, to appear in Proceeding of 1983 U.S.-Japan seminar Geometric Methods in Operator Algebras.
[5] S. E. Cappell, On homotopy invariance of higher signatures, Invent. Math., **33** (1976), 171–179.
[6] A. Connes, Sur la théorie non commutative de l'intégration, *Algèbres d'opérateurs,* Lecture Notes in Math. **725**, Springer Verlag, 1979, 19–143.
[7] ——, A survey of foliations and operator algebras, *Operator algebras and applications,* Proc. Symposia in Pure Math., **38** (1982), Part 1, 521–628.
[8] A. Haefliger, Homotopy and integrability, *Proc. Summer Institute on Manifolds,* Lecture Notes in Math. **197**, Springer Verlag, 1971, 133–163.
[9] ——, Groupoides d'holonomie et classifiants, Université de Genève preprint.
[10] I. James and J. H. C. Whitehead, The homotopy theory of sphere bundles over spheres I, Proc. Lond. Math. Soc., **4** (1954), 196–218.
[11] G. G. Kasparov, The operator K-functor and extensions of C^* algebras, Izv. Akad. Nauk SSSR, Ser. Mat., **44** (1980), 571–636=Math. USSR Izv., **16** (1981), 513–572.
[12] G. G. Kasparov, Group C^* algebras and higher signatures (conspectus), preprint, Chernogolovka, 1981.
[13] G. Lusztig, Novikov's higher signature and families of elliptic operators, J. Differential Geom., **7** (1971), 229–256.
[14] A. S. Miscenko, Homotopy invariants of nonsimply connected manifolds III: Higher signatures, Math. USSR Izv., **5** (1971), 1325–1364.
[15] A. S. Miscenko, Infinite dimensional representations of discrete groups and higher signatures, Math. USSR Izv., **8** (1974), 85–112.
[16] S. P. Novikov, Manifolds with free Abelian fundamental groups and their applications (Pontrjagin classes, smoothness, multidimensional knots) Izv. Akad. Nauk SSSR. Mat., **30** (1966), 207–246; English transl. Amer. Math. Soc. Transl. (2), **71** (1968), 1–42.
[17] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250–262.

Paul Baum
Mathematics Department
Box 1917
Brown University
Providence, Rhode Island 02912
U.S.A.

Alain Connes
IHES
35 route de Chartres
91440 Bures-sur-Yvette
France