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Cyclic Cohomology and Noncommutative Differential Geometry

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Cyclic cohomology appeared independently from two different streams of ideas, algebraic K-theory and noncommutative differential geometry. I shall try to explain in this paper the meaning of noncommutative differential geometry. Its main object is a new notion of space. The need for considering such spaces and developing for them the analogues of the usual tools of differential geometry is best understood in the following two examples. In both, one tries to prove a result of classical differential geometry, and a heuristic proof is possible provided one accepts the new notion of space.

First example.

THEOREM (LICHNEROWICZ, 1961). *If M is a compact spin manifold whose $\hat{A}$ genus is nonzero, then it is impossible to endow M with a Riemannian metric of strictly positive scalar curvature.*

The proof of the result uses a simple global idea. By the Lichnerowicz identity, the square of the Dirac operator is $\nabla^*\nabla + \frac{1}{4}\chi$ where $\nabla^*\nabla$ is a positive operator and χ is the scalar curvature. Thus for $\chi > 0$, the Dirac operator has index equal to zero. But by the index theorem index (Dirac) $= \hat{A}(M) \neq 0$. Q.E.D.

A stronger result about the nonexistence of metrics with positive scalar curvature is the following

THEOREM [14]. *Let M be a compact oriented manifold with $\hat{A}(M) \neq 0$. Then there is no integrable spin subbundle F of TM with strictly positive scalar curvature.*

Let me give a heuristic proof of this result which will work when we get the right tools. The idea is the following: Given an integrable subbundle F of the tangent bundle of M, one can a priori integrate it and get a foliation of M which creates a new space B of leaves of this foliated manifold. (See Figure 1.)

Now $\hat{A}(M)$ is the index of the Dirac operator, at least if M is spin, or, equivalently, it is the pushforward $\pi!(L)$ of the trivial line bundle L on M by the map $\pi\colon M \to \text{pt}$. As $\pi = \pi_1 \circ \pi_2$, where π_2 is the projection of M on the space

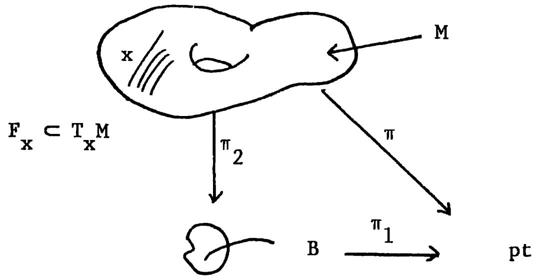

FIGURE 1

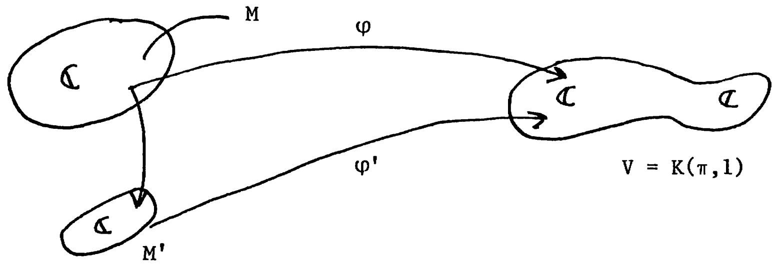

FIGURE 2

of leaves B, one has $\pi!(L) = \pi_1!(\pi_2!(L))$, but $\pi_2!(L) \in K(B)$ is the index of the family of Dirac operators along the leaves and hence is zero since the scalar curvature of leaves is strictly positive. This reasoning does work if one has just a fibration; one then applies the index theorem for families. However, in general, given an integrable subbundle F it is impossible to decide whether it creates a fibration or a foliation. For instance, on the two torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ the equation $dy = \theta\, dx$ defines a fibration iff θ is rational. Thus it is impossible to restrict to the case of fibrations, and one needs to handle spaces such as the space B of leaves of an arbitrary foliation. One needs new tools to understand and use such spaces because when just viewed as ordinary topological spaces they are of no use; in general they would carry the coarse topology and $K(B)$ would be trivial.

Second example. We now pass from the space of leaves of a foliation to another example related to discrete groups. It comes from a problem stated by Novikov—the homotopy invariance of the higher signatures. Let M be a compact oriented manifold and φ a map from M to a $K(\pi, 1)$ space V. For instance, one could take for φ the map which classifies the universal cover of M. For each cohomology class $\omega \in H^*(V, \mathbf{C}) = H^*(\pi, \mathbf{C})$, the higher signature of the pair (M, φ) is given by the scalar $\langle \mathcal{L}_M \cdot \varphi^*(\omega), [M] \rangle$ where $\mathcal{L}_M$ is the L genus of M and $\varphi^*(\omega)$ the pullback of ω by φ. The problem is the following: Is the well-defined number above a homotopy invariant of the pair (M, φ)? (See Figure 2.)

When $V = \text{pt}$, one gets the ordinary signature of M, which is a homotopy invariant. By the work of Wall and Miscenko, on equivariant surgery theory, one can assign a π-equivariant signature to the covering $\tilde{M}$ of M pullback by φ of the universal cover $\tilde{V}$ of V. Moreover, this equivariant signature belongs (neglecting torsion) to the Witt group of the group ring $\mathbf{C}\pi$ and is a homotopy invariant, $\mathrm{Signature}_\pi(M) \in \mathrm{Witt}(\mathbf{C}\pi)$. When π is *commutative*, one can prove the homotopy invariance of higher signatures as follows. There is indeed a *space* assigned to the group π, the space of characters, i.e., the dual $\hat{\pi}$, which is Hausdorff and compact, finite-dimensional if π is finitely generated. Then the group ring $\mathbf{C}\pi$ embeds as a subring of the ring $C(\hat{\pi})$ of continuous functions on $\hat{\pi}$:

$$\mathbf{C}\pi \subset C(\hat{\pi}).$$

The diagonalization of quadratic forms on $C(\hat{\pi})$ yields a map from the Witt group of $\mathbf{C}\pi$ to the K^0 group of $\hat{\pi}$:

$$\mathrm{Witt}\, \mathbf{C}\pi \to K^0(\hat{\pi}).$$

Now any

$$\omega \in H^n(V, \mathbf{C}) = H^n(\pi, \mathbf{C})$$

is represented by a group cocycle $\omega(g^1, \dots, g^n)$ totally antisymmetric in the g^i's. One then defines uniquely a current C on $\hat{\pi}$ by the equality:

$$\langle c, f^0\, df^1 \wedge \cdots \wedge df^n \rangle = \sum_{\prod_0^n g^i = 1} \hat{f}^0(g^0)\hat{f}^1(g^1) \cdots \hat{f}^n(g^n)\omega(g^1, \dots, g^n)$$

where the f^i are functions on $\hat{\pi}$ so that their Fourier transform $\hat{f}^i$ are functions on the group π itself. The current C is closed because ω is a group cocycle.

The main lemma, then, which is a corollary of the index theorem for families, says that if you pair C with the Chern character of the equivariant signature you get the higher signature:

$$\langle C, \mathrm{Ch}(\mathrm{Signature}_\pi(\tilde{M}))\rangle = \langle \mathcal{L}_M \cdot \varphi^*(\omega), [M]\rangle.$$

Thus the right-hand side is a homotopy invariant. Q.E.D.

In general, *when π is not commutative*, there is no interesting space of characters and one cannot really talk about the dual of π as a space. However, and this will be the key to this discussion, one can assign a noncommutative C^*-algebra to π; it is the completion of the group ring $\mathbf{C}\pi$ acting in the Hilbert space $l^2(\pi)$.

A careful scrutiny of the two previous examples reveals that one needs, in order to proceed, a suitable generalization of the notion of space, which would allow one to handle both leaf spaces and duals of noncommutative groups as if they were ordinary spaces.

$$\begin{tikzcd}
 & \text{Category of sheaves on } X \\
\text{Space } X \arrow[ur, "\text{Grothendieck}"] \arrow[dr, "\text{Gelfand}"'] & \\
 & \text{Algebra of complex functions on } X
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{Topos} \\
\text{“Space'' } X \arrow[ur] \arrow[dr] & \\
 & \text{Noncommutative } C^*\text{-algebras}
\end{tikzcd}$$

The basic idea underlying the new notion of space discovered by Grothendieck —and which he named "topos"—is that in an ordinary topological space the main part is not so much played by the points and their proximity relations, but by the category of sheaves on the space. Indeed the original topological space can be recovered from this category, and, moreover, if one keeps only the truly relevant conditions satisfied by such categories one obtains the notion of topos which plays a fundamental implicit role in the new algebraic geometry. The new notion of space that we shall deal with is based on a similar idea, but assigns a specific role to the complex numbers $\mathbf{C}$ or, equivalently, to functional analysis. It goes back to Gelfand's theory of C^*-algebras. It asserts that a compact topological space X is characterized by the $*$-algebra $C(X)$ of complex-valued continuous functions on X and that such algebras are the most general *commutative* C^*-algebras. That there is no good reason to restrict oneself to commutative C^*-algebras versus noncommutative ones, goes back to the early development of quantum mechanics with the discovery by Heisenberg of *matrix mechanics.* In understanding, from a very positivistic point of view highly enforced by experimental evidence in spectroscopy, the interaction of matter with the radiation field, Heisenberg showed that the usual observables of classical mechanics have to be replaced by matrices which violate the commutativity of mutliplication. Thus the phase space of quantum particles is an early example of the new type of spaces that we shall deal with. To take this second idea of space further, we need many examples, each being used as a small laboratory in which to test ideas and to see what works. We summarize a few examples in the following table:

Space	*Algebra*
X	$C(X)$
$X = \hat{\pi}$ dual of	$C^*(\pi) \supset \mathbf{C}\pi$
a discrete group	(completion in $l^2(\pi)$)
$X = M/F$ leaf space	$C^*(M, F)$
Example: Kronecker foliation	$VU = (\exp 2\pi_i\theta)UV$
$X = \Omega/G$ orbit space	$C_0(\Omega) \rtimes G$ crossed product

We have already discussed the first example. The second comes from foliations. There is a very natural C^*-algebra coming from operators which differentiate only in the leaf direction, and are elliptic in that direction. These turn out to have natural parametrices; they are invertible modulo operators which are smoothing in the leaf direction. These operators constitute a C^*-algebra, $C^*(M, F)$. An example would be to take the Kronecker foliation of the two torus, which is induced by the equation $dy = \theta\, dx$ where θ is irrational. In that case you get a C^*-algebra generated by two unitary elements which do not commute, but do commute up to a phase $\lambda = \exp 2\pi i\theta$. This is an algebra with which one may do many computations, exactly as if one were computing with the ordinary functions on the two torus using Fourier analysis.

Another very important example was discovered by Bellissard [6] from solid state physics and the quantum Hall effect. In the study of disordered systems, the Hamiltonian H_ω is labelled by a parameter $\omega \in \Omega$. Moreover, H_ω fails to commute with $H_{T_x(\omega)}$ where T is the action of the translation group on the parameter space Ω. Thus the translates of the Hamiltonian generate a noncommutative C^*-algebra, which corresponds to the "energy spectrum" of the system.

Given these examples one needs the right tools. The first comes from my original field of study: "von Neumann algebras." These algebras together constitute exactly the noncommutative analogue of measure theory. Their classification and understanding have now reached a fairly complete and satisfactory stage.

But what we need then is a little more than just measure theory; we need topology. I will now describe the basic tool in topology, first introduced by Grothendieck in algebraic geometry, and then by Atiyah for the purposes of topology. That tool is K-theory. There is a quite simple relation between complex vector bundles over the space X and projective modules over the algebra $A = C(X)$; this is the Serre-Swann theorem:

$$K^i(X) = K_i(A = C(X)).$$

It allows us to do K-theory of spaces by doing linear algebra where the field $\mathbf{C}$ is replaced by the ring A. Then the group of dimensions of finite projective modules is the K-group $K_0(A)$. The Bott periodicity theorem tells us that the K-groups of a C^*-algebra A are the homotopy groups of the gauge group, i.e., of the unitary group $\mathcal{U}$ of infinite matrices over A:

$$K_i(A) = \pi_{i+1}(\mathcal{U}).$$

Whenever a space is constructed by patching together two spaces, such that one has a short exact sequence of algebras, there is a corresponding long exact sequence of K-groups, which is shortened thanks to periodicity:

$$0 \to J \to A \to B \to 0 \Rightarrow \quad \begin{tikzcd}[column sep=small] & K_0(J) \arrow[r] & K_0(A) \arrow[dr] & \\ K_1(B) \arrow[ur] & & & K_0(B) \arrow[dl] \\ & K_1(A) \arrow[ul] & K_1(J) \arrow[l] & \end{tikzcd}$$

Moreover, there is a general principle which is absolutely crucial. Above, we used twice the index theorem for families. Now the principle is that a "space" X will be described by a noncommutative algebra A, and that when one has a family (D_x), $x \in X$ indexed by X, such as the family of leafwise Dirac operators indexed by the space of leaves, then the index of this family belongs to $K^0(X) = K_0(A)$. This principle is very important because it allows us to translate into K-theoretic terms the basic analytical properties such as:

- The vanishing of the index of the family of leafwise Dirac operators:

$$\mathrm{Index}(\mathrm{Dirac}_L)_{L \in M/F} = 0$$

when the scalar curvature of leaves is strictly positive.

• The homotopy invariance of the π-equivariant signature: $\mathrm{Signature}_\pi(\tilde{M}) \in K_0(C^*(\pi))$.

The first vanishing above takes place in the K-group $K_0(C^*(M,F))$. Both K-groups are countable abelian groups but are at first extremely mysterious objects, being defined through the above C^*-algebras. When dealing with ordinary spaces one gets some intuition about their K-groups, but this is less clear when dealing with C^*-algebras. The first real breakthrough which got everything started was done by Pimsner and Voiculescu [**26**] who, in particular, computed the K-groups for the Kronecker flow foliation discussed above. It allowed P. Baum and the author to guess what the answer should be in both general and geometric terms. The situation is described as follows: We construct both a geometric group, the K-homology of the classifying space, and a map μ to the K-group of the C^*-algebra. The classifying space makes sense in all the above situations since topologists have a way of making sense, up to homotopy, of spaces like the leaf space of a foliation or the orbit space of a group action. What they do is to amplify the space, say M, on which the group Γ acts, by crossing M with a contractible space $E\Gamma$ on which Γ acts freely; then the quotient $M \times_\Gamma E\Gamma$ makes sense and is "homotopic to M/Γ."

$$K_*(\text{Classifying space}) \xrightarrow{\mu} K(C^*\text{-algebra})$$

The map μ is difficult to construct [**5**, **4**, **12**, **24**] and even when one deals with a one point space, its mere existence is the Atiyah-Singer index theorem [**5**]. It is essentially a Poincaré duality map to the extent that it reverses functorialities. The main problem of the theory is to handle this map μ; all computations so far indicate that it is a bijection [**4**, **23**, **24**, **27**]. An important tool developed by the Russian school, by Miscenko and Kasparov in particular, and also by Atiyah, Brown, Douglas, and Fillmore (cf. [**23**, **24**, **1**, **7**]), is K-homology for C^*-algebras. Since this theory played a crucial role in the understanding of the analogue of de Rham's theory of currents for the above spaces, I shall sketch it briefly. For ordinary spaces, K-homology is defined, using duality, by a general theorem which states that given any cohomology theory (such as K-theory) there is a corresponding homology theory, called here K-homology. One wants to realize this homology theory concretely. It is quite striking that if one was very conservative and wanted to stick to ordinary spaces, not accepting "spaces," one would not be able to describe the theory K-homology (X) (there is a K_{even} and K_{odd}) as homotopy classes of maps from spaces $Z_{\text{even}}, Z_{\text{odd}}$ to the space X. However, with "spaces" this is possible; Z_{ev} is obtained by glueing together two contractible "spaces," and the C^*-algebra $C(Z_{\text{ev}})$ is the noncommutative algebra A_{ev} of pairs of operators (x,y) in Hilbert space $\mathfrak{h}$ whose difference $x-y$ is a compact operator. Similarly $C(Z_{\text{odd}}) = A_{\text{odd}}$, which also appears in *Beyond Affine Lie Algebras*, by I. Frenkel, is the algebra of 2×2 matrices (x_{ij}) of operators, such that x_{12} and x_{21} are compact. Of course a "continuous map" from Z_{ev} to X is given by a homomorphism from $C(X)$ to $C(Z_{\text{ev}})$, i.e., a homomorphism from the C^*-algebra $A = C(X)$ to A_{ev}. This is called a *Fredholm module* over

A because it amounts to giving a $\mathbf{Z}/2$ graded Hilbert space, $\mathfrak{h} \oplus \mathfrak{h}$ with grading $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, with a left A-module structure such that

(1) $\varepsilon a = a\varepsilon \; \forall a \in A$,

(2) $[F, a]$ is compact $\forall a \in A$ where $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

There is a similar notion of odd Fredholm module. On any even-dimensional compact spin^c manifold, the module of L^2 spinors, with $\mathbf{Z}/2$ grading given by the γ_5 matrix [**5**] and operator F given by the phase $F = D|D|^{-1}$ of the Dirac operator, is a Fredholm module which represents the *fundamental class* of the manifold in K-homology [**5**]. If one puts together this notion of a Fredholm module with the ideas of Helton and Howe, Carey and Pincus [**18, 9**] on operators commuting modulo trace ideals, one is led to the noncommutative analogue of de Rham's theory: cyclic cohomology. Helton and Howe associated to any operator T, normal modulo trace class operators, a de Rham current on $\mathbf{R}^2$ with boundary carried by the essential spectrum of T. Their work was very inspiring because it showed that the calculus of differential forms could be born from purely operator theoretic considerations in Hilbert space. This is what is done in [**11**]; given a Fredholm module over A, one can define *differential forms* on the corresponding "space," not by using local charts and patching these together but directly as operators in $\mathfrak{h}$. It is exactly the same step as the replacement, in quantum mechanics, of Poisson brackets by commutators. Thus

$$da = i[F, a] \quad \forall a \in A$$

defines the differential of a function. The forms of degree q are obtained as sums of products of 1-forms: $\Omega^q = \{\sum x^0\, dx^1 \cdots dx^q,\ x^j \in A\}$. In this way, one gets a graded differential algebra; the product is the product of operators and the differential is given by

$$d\omega = i(F\omega - (-1)^q \omega F) \quad \text{ for } \omega \in \Omega^q.$$

One has $d^2 = 0$, and the main point is to obtain an integration of forms $\omega \to \int \omega \in \mathbf{C}$ satisfying $\int d\omega = 0$ and $\int \omega_2\omega_1 = (-1)^{q_1 q_2} \int \omega_1\omega_2$.

The formula which works is quite simple: $\int \omega = \mathrm{Trace}(\varepsilon\omega)$. This is where the *dimension* appears, the trace only makes sense if ω is a *trace class* operator. By the Holder inequality this holds, for any $\omega \in \Omega^n$, provided $[F, a] \in \mathcal{L}^n \; \forall a \in A$. Here, for every *real* number $p \in [1, \infty]$, $\mathcal{L}^p$ is the ideal of compact operators T with $\sum \lambda_q(|T|)^p < \infty$, where $\lambda_q(|T|)$ is the qth eigenvalue of the absolute value of T. The *dimension* of a Fredholm module over an algebra is the infimum of the p's for which $[F, a] \in \mathcal{L}^p \; \forall a \in A$. For the fundamental class of a manifold M described above, it yields the dimension of M. In general it need not be an integer. Given an even Fredholm module of dimension p on A one can integrate only the forms $\omega \in \Omega^n$ of degree $\geq p$. Moreover, odd forms have integral 0. Thus the above construction yields for each even integer $n \geq p$, the functional τ_n called the n-dimensional character of the Fredholm module:

$$\tau_n(a^0, \ldots, a^n) = \int a^0\, da^1 \cdots da^n \quad \forall a^i \in A.$$

Carefully analyzing these functionals led me to discover cyclic cohomology in 1981. It was discovered independently from algebraic K-theory by Feigin and Tsigan [**17**, **30**], replacing group homology by Lie alagebra homology in the basic construction of Quillen's algebraic K-theory. It also appeared, at least in implicit form, in the work of Hsiang and Staffeld on the algebraic K-theory of spaces [**20**]. It is of course quite striking that from different streams of ideas one gets to the same theory: *cyclic cohomology.*

A crucial and simple lemma is the following.

LEMMA. *Let $\mathcal{A}$ be an algebra and τ an $(n+1)$-linear map $\mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}\to\mathbf{C}$ such that*

(1) $\tau(a^1,\dots,a^n,a^0)=(-1)^n\tau(a^0,\dots,a^n)\ \forall a^i\in\mathcal{A}$;

(2) $\sum_0^n(-1)^j\tau(a^0,\dots,a^ja^{j+1},\dots,a^{n+1})+(-1)^{n+1}\tau(a^{n+1}a^0,\dots,a^n)=0\ \forall a^i \in\mathcal{A}$.

Then the map $e\in\mathcal{A}$, $e^2=e\to\tau(e,\dots,e)$, gives a morphism of $K_0(\mathcal{A})$ to $\mathbf{C}$.

In fact $K_0(\mathcal{A})$ is generated by idempotents $e^2=e$ in matrices over $\mathcal{A}$, $M_q(\mathcal{A})=M_q(\mathbf{C})\otimes\mathcal{A}$, and one has to extend τ to $M_q(\mathcal{A})$ by the equality:

$$\tau_q(m^0\otimes a^0,\dots,m^n\otimes a^n)=\mathrm{Trace}(m^0\cdots m^n)\tau(a^0,\dots,a^n) \qquad (*)$$

$\forall m^j\in M_q(\mathbf{C})$, $a^j\in\mathcal{A}$.

Here are a few examples of functionals τ satisfying (1) and (2):

EXAMPLE α. Let $\mathcal{A}=C^\infty(M)$, the algebra of smooth functions on a compact manifold, and C a closed current on M of dimension k. Then $\tau(f^0,\dots,f^k)=\langle C,f^0\,df^1\wedge\cdots\wedge df^k\rangle\ \forall f^j\in\mathcal{A}$ has exactly the properties (1), (2) of a cyclic cocycle. In fact τ satisfies $\tau^\sigma=\mathrm{sign}(\sigma)\tau$ for *any* permutation of $\{0,1,\dots,k\}$, but since $\mathrm{Trace}(m^0\cdots m^k)$ is invariant only under *cyclic* permutations it is only (1) which is satisfied by all τ_q. One has $K_0(\mathcal{A})=K^0(M)$ and the lemma gives back the ordinary Chern character, viewed as a pairing with the homology of M.

EXAMPLE β. Let π be a discrete group, $\mathcal{A}=\mathbf{C}\pi$ the group ring, and $\omega\in Z^n(\pi,\mathbf{C})$ a group cocycle suitably normalized so that $\omega(g^1,\dots,g^n)=0$ if $g^1\cdots g^n=1$. Then the equality

$$\tau(g^0,\dots,g^n)=0 \text{ if } g^0\cdots g^n\neq 1\quad \forall g^i\in\pi,$$
$$\tau(g^0,\dots,g^n)=\omega(g^1,\dots,g^n) \text{ if } g^0g^1\cdots g^n=1\quad \forall g^i\in\pi,$$

defines an n-cyclic cocycle τ on $\mathcal{A}$. Moreover, extending τ to infinite matrices over $\mathcal{A}$ one can show that

$$\langle\tau,\mathrm{Signature}_\pi(\tilde{M})\rangle=\langle\mathcal{L}_M\cdot\varphi^*(\omega),[M]\rangle$$

with the notations of the higher signature problem. The cyclic cohomology of group rings is computed by Burghelea in [**8**].

EXAMPLE γ. For each even $n\geq p$, the n-dimensional character τ_n of a Fredholm module over A is a *cyclic cocycle.* Moreover, the pairing with $K_0(A)$, $\langle\tau_n,e\rangle$, is given for any idempotent e by the *index of a Fredholm operator*, and, in particular, lands in $\mathbf{Z}\subset\mathbf{C}$. It corresponds to the $\mathbf{Z}$-valued pairing between K-theory and K-homology, which ensures that it is highly nontrivial.

Given any algebra $\mathcal{A}$, there is a trivial way to construct cyclic cocycles on $\mathcal{A}$, namely $\tau = b\varphi$ where $\varphi \in C_\lambda^{n-1}$ is an n-linear functional on $\mathcal{A}$ satisfying (1), and $b\varphi$ its Hochschild coboundary given by formula (2). The relevant group is the quotient $H_\lambda^n(\mathcal{A}) = Z_\lambda^n(\mathcal{A})/bC_\lambda^{n-1}$, where $Z_\lambda^n = \operatorname{Ker} b$, and is called *cyclic cohomology* of $\mathcal{A}$. It turns out that just by working with Example γ, of Fredholm modules, all the properties of cyclic cohomology fall into one's lap. First a Fredholm module has many characters τ_q, one for each even integer $q \geq p$, and it would be unreasonable to expect that $\tau_{q+2}, \tau_{q+4}, \ldots$ bring new information not contained in τ_q. Explicit computations show that there is a natural periodicity operator

$$S\colon H_\lambda^n(\mathcal{A}) \to H_\lambda^{n+2}(\mathcal{A})$$

given in fact by cup product by the generator of $H_\lambda^2(\mathbf{C})$ and such that $\tau_{q+2k} = S^k\tau_q$ in $H_\lambda^{q+2k}(\mathcal{A})$. Then, in order to find the smallest n for which τ_n is defined, one needs to determine the image of S. But by construction, the complex (C_λ^n, b) is a subcomplex of the Hochschild complex (C^n, b) where C^n is the space of all $(n+1)$-linear functionals on $\mathcal{A}$. It turns out that $\tau \in \operatorname{Im} S$ iff τ is trivial in the latter complex, whose cohomology $H^n(\mathcal{A}, \mathcal{A}^*)$, the Hochschild cohomology of $\mathcal{A}$ with coefficients in the bimodule of linear forms on $\mathcal{A}$, is computable by the general methods of homological algebra. The final point is the construction of a natural operator B from Hochschild cohomology $H^n(\mathcal{A}, \mathcal{A}^*)$ to $H_\lambda^{n-1}(\mathcal{A})$ and the proof of the exactness of the following sequence:

$$\cdots \to H_\lambda^n(\mathcal{A}) \xrightarrow{S} H_\lambda^{n+2}(\mathcal{A}) \xrightarrow{I} H^{n+2}(\mathcal{A}, A^*) \xrightarrow{B} H_\lambda^{n+1}(\mathcal{A}) \xrightarrow{S} H_\lambda^{n+3}(\mathcal{A}) \xrightarrow{I} H^{n+3}(\mathcal{A}, \mathcal{A}^*) \to \cdots$$

Thus Hochschild cohomology and cyclic cohomology from an exact couple which together with the associated spectral sequence becomes a basic tool to compute cyclic cohomology of algebras. The power of this tool is illustrated by two examples:

EXAMPLE a. Let M be a compact manifold, $\mathcal{A} = C^\infty(M)$. Imposing obvious continuity conditions to cochains one finds that the Hochschild cohomology groups $H^q(\mathcal{A}, \mathcal{A}^*)$ are identified with the space Ω_q of de Rham currents of dimension q on M. The map $I \circ B$ of the exact couple is the de Rham boundary d^t, and one gets

$$H_\lambda^q(\mathcal{A}) = \{\operatorname{Ker} d^t \subset \Omega_q\} + H_{q-2}(M, \mathbf{C}) + H_{q-4}(M, \mathbf{C}) + \cdots .$$

The de Rham homology of M identifies with the periodic cyclic cohomology of $\mathcal{A}$: $H_{\mathrm{Per}}^*(\mathcal{A}) = \varinjlim(H_\lambda^n(\mathcal{A}), S)$.

EXAMPLE b. Let (M, F) be a foliated manifold, $A = C^*(M, F)$ the corresponding C^*-algebra. In A there is a natural dense subalgebra $\mathcal{A}$ of smooth elements and one has to compute its cyclic cohomology. One has

$$H_{\mathrm{Per}}^*(\mathcal{A}) \cong H_\tau^*(\text{Classifying space})$$

where the right-hand side is the cohomology with complex coefficients of the classifying space of the holonomy groupoid or graph of the foliation. The index τ means that this cohomology is twisted by the orientation of the transverse bundle τ of the foliation. Using sheaves on M and the naturality of the construction of $\mathcal{A}$, one constructs a localization morphism λ_M, which is a far-reaching generalization of the Ruelle-Sullivan current:

$$\lambda_V : H^*_{\mathrm{Per}}(\mathcal{A}) \xrightarrow{\lambda_M} H^*_\tau(M, \mathbf{C})$$

and one reaches the following cohomological formulation of the longitudinal index theorem for foliations [**12**].

THEOREM. *Let (M, F) be a compact foliated manifold, D a longitudinal elliptic operator, and τ a cyclic cocycle on $\mathcal{A}$. Then*

$$\langle \tau, \mathrm{Index}(D) \rangle = \langle \lambda_M(\tau)\, \mathrm{Td}(F_{\mathbf{C}})\, \mathrm{Ch}\, \sigma_D, [M] \rangle$$

where σ_D is the longitudinal symbol of D.

There is, however, still a really hard step in order to use cyclic cohomology as ordinary de Rham theory for our "spaces"—such as the space of leaves of a foliation—and to prove Theorem 2 of this paper, for instance [**14**]. The point is that $\mathcal{A} \subset A$, in Example b, is not in general an isomorphism in K-theory, and the analytic information lies in $K(A)$ not $K(\mathcal{A})$. This problem is fully resolved in [**14**] for the transverse fundamental class of M/F and all classes coming by pullback of the Gelfand-Fuchs cohomology by the map $B(\text{Classifying space}) \to B\Gamma_q$.

The difficulty is that for a general foliation it is impossible to reduce the transverse structure group to a compact group. Equivalently, for a group of diffeomorphisms acting on a manifold, one cannot find an invariant Riemannian metric. The result implies, in particular, the Novikov conjecture for Gelfand-Fuchs cohomology classes on $B(\mathrm{Diff}\, N)$ for any N.

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Bibliographical comments. For a survey of cyclic cohomology and an extensive bibliography, see [**10**, **17**]. The computation of the homology of the traces on the universal differential algebra $\Omega(\mathcal{A})$ of an algebra $\mathcal{A}$ was explicitly done in [**11**, Theorem 33, p. 118]. Once dualized, it gives the computation [**22**] in terms of cyclic homology of the cohomology of the universal de Rham complex introduced by Karoubi in [**21**]. The computation of the homology of the Lie algebra of matrices in terms of cyclic homology is done in [**25**, **30**]. The idea of using the Lichnerowicz identity in a C^*-algebra context is due to J. Rosenberg [**27**].

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