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Entire Cyclic Cohomology of Banach Algebras and Characters of θ-Summable Fredholm Modules

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Abstract. We define, using cocycles with infinite support in the fundamental (b, B) bicomplex of cyclic cohomology, a $\mathbb{Z}/2$ graded cohomology of entire functions on a Banach algebra, which pairs with topological K-Theory. We then construct, using an algebra of operator-valued distributions with support in $\mathbb{R}_+$, a canonical entire cocycle $Ch(\mathscr{H}, D)$ on A for every θ-summable Fredholm module $(\mathscr{H}, D)$ over a Banach algebra A.

Key words. Infinite dimensional analysis, cyclic cohomology, θ-summable Fredholm modules, K homology.

1. Introduction

We showed in [4] that in order to handle infinite-dimensional spaces such as those occurring in constructive quantum field theory, or the noncommutative spaces which are duals of nonamenable discrete groups (cf. [4], [6]), it is necessary to consider Fredholm modules $(\mathscr{H}, D)$ which are no longer finitely summable (i.e. $\mathrm{Trace}((1 + D^2)^{-p}) < \infty$ for some finite p) but satisfy the weaker condition of θ-summability: $\mathrm{Trace}(e^{-tD^2}) < \infty$.

Our aim in this paper is to show that the construction of [2] of the Chern character of finitely summable Fredholm modules can be extended to the new class of θ-summable Fredholm modules. To achieve this goal one needs

(1) to extend cyclic cohomology to incorporate infinite-dimensional cycles,
(2) to extend to the new theory the pairing with topological K theory
(3) to define the character of θ-summable Fredholm modules by explicit formulae,
(4) to show that the index of the Fredholm module with coefficients in a K-theory class is given by the value of its character on that class.

While ordinary cyclic cohomology $H^n_\lambda(A)$ of an algebra A is based on monomials of degree $n + 1$, the new theory is based (with A a complex Banach algebra) on entire functions on A. More specifically, one considers in the fundamental (b, B) bicomplex ([2] p. 123) of cyclic cohomology, the cochains $(\phi_{2n})_{n\in\mathbb{N}}$ or $(\phi_{2n+1})_{n\in\mathbb{N}}$ such that

$$\sum \frac{\|\phi_{2n}\|}{n!} r^n \langle \infty \quad \forall r \rangle 0.$$

Thus, to each even cochain corresponds an entire function

$$F_\phi(x) = \Sigma(-1)^n \frac{\phi_{2n}(x, \ldots, x)}{n!}$$

on A. In Section 2 we show that if we endow the universal differential algebra ΩA with the norms $\| \quad \|_r$ of Arveson [1] and the corresponding inductive limit topology, the following three notions are identical.

(a) Normalized entire cocycles (ϕ_{2n}) in the (b, B) bicomplex.

(b) Continuous functionals μ on ΩA such that:

$$\mu(\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1) = \tfrac{1}{2}(-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2) \quad \forall \omega_j \in \Omega^{\partial_j}, j = 1, 2.$$

(c) Continuous traces τ on the algebra $\mathscr{E}A = QA \times_\sigma \mathbb{Z}/2$ such that $\tau \circ \hat{\sigma} = -\tau$.

In (c), we consider the free product algebra $QA = A * A$ of A by itself with the topology it inherits from the canonical linear isomorphism $QA \simeq \Omega A$ ([3]), and let $\mathscr{E}A$ be the crossed product of QA by its canonical involution σ [11].

A pair (Ω, μ) of a graded differential algebra Ω with $\Omega^0 = A$ and a functional μ satisfying (b) is an infinite-dimensional cycle over A, cycles of dimension n in the sense of [2] are special examples of such objects.

Interpretation (c) is most useful for actual construction of infinite-dimensional cycles such as the character of a θ-summable Fredholm module.

The pairing with K-theory is given by the formulae

$$e \in \operatorname{Proj} A \to F_\phi(e) = \tau\left(\frac{Fe}{\sqrt{1-(qe)^2}}\right),$$

where the equality is between the notions (a) and (c) and $F \in \mathscr{E}A$, $F^2 = 1$.

The construction of the character occupies Sections 3 and 4 of the paper, and Section 5 is the proof of the estimates showing that this character is an entire cocycle. It relies heavily on the existence of an algebra of convolution of operator-valued distributions $T(s)$, $s \in [0, +\infty[$ satisfying suitable analyticity and Schatten class properties. Finally, in Section 7 we prove the relevant index formula. This paper can be considered as an improvement of the basic tools [2] of noncommutative differential geometry, necessary (by [4]) to handle several really important examples which are 'infinite dimensional'.

1. Entire Cyclic Cohomology of Banach Algebras

Let A be a unital Banach algebra over $\mathbb{C}$. Let us recall the construction ([2], p. 119) of the fundamental (b, B) bicomplex of cyclic cohomology. For any positive integer $n \in \mathbb{N}$, one lets $C^n(A, A^*)$ be the space of continuous $n + 1$ linear forms ϕ on A. For $n < 0$ one sets $C^n = \{0\}$. One defines two differentials b, B as follows:

(1) $b\colon C^n \to C^{n+1}$,

$$(b\phi)(a^0, \ldots, a^{n+1}) = \sum_0^n (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1}\phi(a^{n+1}a^0, \ldots, a^n),$$

(2) $B: C^n \to C^{n-1}$, $(B\phi) = AB_0\phi$,

where

$$(B_0\phi)(a^0, \dots, a^{n-1}) = \phi(1, a^0, \dots, a^{n-1}) - (-1)^n \phi(a^0, \dots, a^{n-1}, 1) \quad \forall \phi \in C^n,$$

$$(A\psi)(a^0, \dots, a^{n-1}) = \sum_0^{n-1} (-1)^{(n-1)j} \psi(a^j, a^{j+1}, \dots, a^{j-1}) \quad \forall \psi \in C^{n-1}.$$

By [2], Lemma 30, one has $b^2 = B^2 = 0$ and $bB = -Bb$ so that one obtains a bicomplex $(C^{n,m}; d_1, d_2)$, where $C^{n,m} = C^{n-m}$ for any $n, m \in \mathbb{Z}$,

$$d_1\phi = (n - m + 1)b\phi \quad \forall \phi \in C^{n,m}, \qquad d_2\phi = \frac{1}{n-m} B\phi \quad \forall \phi \in C^{n,m}$$

(cf. [2], p. 123). The main lemma (36, p. 121) of [2] asserts that the b cohomology of the complex Ker B/Im B is zero, so that the spectral sequence associated to the first filtration has the E_2 term equal to 0. Since the bicomplex $C^{n,m}$ has support in $\{(n, m), (n + m) \geqslant 0\}$ this spectral sequence does not converge in general when we take cochains with *finite* support, and by [2], Theorem 40, the cohomology of the bicomplex, when taken with finite supports, is exactly the periodic cyclic cohomology $H^*(A)$. If we take cochains with arbitrary supports, without any control of their growth, then by the above lemma we get a trivial cohomology. (This statement is dual to the μ-torsion property of cyclic homology [7], p. 403.) It turns out, however, that provided we control the growth of $\|\phi_m\|$ in a cochain (ϕ_{2n}) or (ϕ_{2n+1}) of the (b, B) bicomplex, we then get the relevant cohomology to analyze infinite-dimensional spaces and cycles. Because of the periodicity $C^{n,m} \to C^{n+1,m+1}$ in the bicomplex (b, B) it is convenient, following C. Kassel [8], to just work with

$$C^{\mathrm{ev}} = \{(\phi_{2n})_{n \in \mathbb{N}}, \phi_{2n} \in C^{2n} \, \forall n \in \mathbb{N}\}$$

and

$$C^{\mathrm{odd}} = \{(\phi_{2n+1})_{n \in \mathbb{N}}, \phi_{2n+1} \in C^{2n+1}, \forall n \in \mathbb{N}\}$$

and the boundary operator $\partial = d_1 + d_2$ which maps C^{ev} to C^{odd} and C^{odd} to C^{ev}. We shall enforce the following growth condition:

DEFINITION 1. An even (resp. odd) cochain $(\phi_{2n})_{n \in \mathbb{N}} \in C^{\mathrm{ev}}$ (resp. $(\phi_{2n+1})_{n \in \mathbb{N}} \in C^{\mathrm{odd}}$) is called *entire* iff the radius of convergence of $\Sigma \|\phi_{2n}\| (z^n/n!)$ (resp. $\Sigma \|\phi_{2n+1}\| z^n/n!$) is infinity.

Here for any m and $\phi \in C^m$, the norm $\|\phi\|$ is the Banach space norm:

$$\|\phi\| = \sup\{|\phi(a^0, \dots, a^m)|; \|a^j\| \leqslant 1\}.$$

It follows, in particular, that any *even* cochain $(\phi_{2n}) \in C^{\mathrm{ev}}$ which is entire, defines an entire function F_ϕ on the Banach space A by

$$F_\phi(x) = \sum_{n=0}^{\infty} (-1)^n \phi_{2n}(x, \dots, x)/n!$$

Next, let $q \in \mathbb{N}$ and $A_q = M_q \otimes A = M_q(A)$ be the Banach algebra of $q \times q$ matrices over A. For any $\phi \in C^m$, let ϕ^q be the natural extension $\phi^q = \mathrm{Tr}\ \# \phi$ of ϕ to $M_q(A)$ (cf. [2], p. 108), i.e. by definition one has

$$\phi^q(\mu^0 \otimes a^0, \ldots, \mu^m \otimes a^m) = \mathrm{Trace}\,(\mu^0 \ldots \mu^m)\phi(a^0, \ldots, a^m),$$

where $\mu^j \in M_q(\mathbb{C})$ and $a^j \in A$. Then one has

LEMMA 2. (1) *For any entire even (resp. odd) cochain* (ϕ_{2n}) *(resp.* (ϕ_{2n+1})*) on* A *the cochain* (ϕ^q_{2n}) *(resp.* (ϕ^q_{2n+1})*) on* A_q *is also entire.*

(2) *The map* $\phi \to \phi^q$ *is a morphism of the complexes of entire cochains.*

Proof. (1) One has an inequality of the form $\|\phi^q\| \leqslant q^m \|\phi\|$ for $\phi \in C^m$, hence the answer.

(2) It is an immediate check (cf. [2]). □

LEMMA 3. *If* ϕ *is an even (resp. odd) entire cochain, then so is* $(d_1 + d_2)\phi = \partial\phi$.

Proof. For $\phi_m \in C^m$ one has $\|b\phi_m\| \leqslant (m+2)\|\phi_m\|$ and $\|B_0\phi_m\| \leqslant 2\|\phi_m\|$, $\|AB_0\phi_m\| \leqslant 2m\|\phi_m\|$, thus the conclusion. □

DEFINITION 4. Let A be a Banach algebra, then the *entire* cyclic cohomology of A is the cohomology of the short complex:

$$C^{\mathrm{ev}}_\varepsilon(\mathrm{A}) \to C^{\mathrm{odd}}_\varepsilon(\mathrm{A}) \to C^{\mathrm{ev}}_\varepsilon(\mathrm{A})$$

of entire cochains in A.

We thus have two groups $H^{\mathrm{ev}}_\varepsilon(A)$ and $H^{\mathrm{odd}}_\varepsilon(A)$. There is an obvious map from $H(A)$ to $H_\varepsilon(A)$, where $H(A)$ (cf. [2], Theorem 40) is the periodic cyclic cohomology of A. We also have a natural filtration of H_ε by the dimensions of the cochains, where (ϕ_{2n}) say, is of dimension $\leqslant k$ if $\phi_{2n} = 0\ \forall n,\ 2n > q$. However, unlike what happens for H, this filtration does not, in general, exhaust all of H_ε, in fact it exhausts exactly the image of $H(A)$ in $H_\varepsilon(A)$.

Let us now compute H_ε for the simplest case, i.e. when $A = \mathbb{C}$ is the trivial Banach algebra. An element of $C^{\mathrm{ev}}_\varepsilon$ is given by an infinite sequence $(\lambda_{2n})_{n\in\mathbb{N}}$, $\lambda_{2n} \in \mathbb{C}$ such that $\Sigma|\lambda_{2n}|(z^n/n!) < \infty$ for any z and, similarly, for $C^{\mathrm{odd}}_\varepsilon$. The boundary $\partial = d_1 + d_2$ of (λ_{2n}) is 0, since both b and B are 0 on even cochains. For m odd and $\phi_m \in C^m$, $\phi(a^0, \ldots, a^m) = \lambda a^0 \ldots a^m$ one has

$$(b\phi)(a^0, \ldots, a^{m+1}) = \lambda a^0 \ldots a^{m+1}, \qquad (B\phi)(a^0, \ldots, a^{m-1}) = 2m\lambda a^0 \ldots a^{m-1},$$

thus

$$(d_1\phi)(a^0, \ldots, a^{m+1}) = \lambda(m+1)a^0 \ldots a^{m+1}, \quad (d_2\phi)(a^0, \ldots, a^{m-1}) = 2\lambda a^0 \ldots a^{m-1}.$$

So the boundary $\partial(\lambda)$ of an odd cochain (λ_{2n+1}) is given by $\partial(\lambda)_{2n} = 2n\lambda_{2n-1} + 2\lambda_{2n+1}$. Thus, $\partial(\lambda) = 0$ means that $\lambda_{2n+1} = (-1)^n n! \lambda_1$ and, hence, is possible only if $\lambda = 0$, for $\lambda \in C^{\mathrm{odd}}_\varepsilon$. Moreover, for any $(\lambda_{2n}) \in C^{\mathrm{ev}}_\varepsilon$, the series $\sigma(\lambda) = \Sigma_0^\infty (-1)^n (\lambda_{2n}/n!)$ is convergent and $\sigma(\lambda) = 0$ iff $\lambda \in \partial C^{\mathrm{odd}}_\varepsilon$. Thus we have

PROPOSITION 5. *One has $H_\varepsilon^{\mathrm{odd}}(\mathbb{C}) = \{0\}$ and $H_\varepsilon^{\mathrm{ev}}(\mathbb{C}) = \mathbb{C}$ with isomorphism given by*

$$\sigma((\phi_{2n})) = \sum_0^\infty \frac{(-1)^n}{n!}\phi_{2n}(1,\dots,1).$$

Let us now go back to the general case; we shall say that a cocycle (ϕ_{2n}) (resp. (ϕ_{2n+1})) is *normalized* iff for any m one has

$$B_0\phi_m = \frac{1}{m}AB_0\phi_m. \tag{$*$}$$

In other words, the cochain $B_0\phi_m$ is already cyclic: $B_0\phi_m \in C_\lambda^{m-1}$ so that $(1/m)A(B_0\phi_m) = B_0\phi_m$. Only the normalized cocycles have a natural interpretation in terms of the universal differential algebra ΩA and the algebra QA.

LEMMA 6. *For every entire cocycle there is a normalized cohomologous entire cocycle.*

Proof. Let $\phi_m \in C^m$ be such that $b\phi_m \in \operatorname{Im} B$, $B\phi_m \in \operatorname{Im} b$ (which is the case for the components of a cocycle) we shall construct $\psi \in C^{m-1}$ with

(a) $B\psi = 0$,

(b) $B_0 b\psi = B_0\phi_m - (1/m)AB_0\phi_m$,

(c) $\|\psi\| \leqslant 9m\|\phi_m\|$

It follows that $\phi'_m = \phi_m - b\psi$ satisfies

$$B_0\phi'_m = B_0\phi_m - B_0 b\psi = \frac{1}{m}AB_0\phi_m = \frac{1}{m}AB_0\phi'_m$$

(since $Bb\psi = -bB\psi = 0$), so that $(\phi_{2n}) \to (\phi'_{2n})$ and $(\phi_{2n+1}) \to (\phi'_{2n+1})$ are the required normalizations. Now let $\theta = B_0\phi_m - (1/m)AB_0\phi_m$. One has $A\theta = 0$ so there is a canonical $\tilde{\theta} \in C^{m-1}$,

$$\tilde{\theta} = \frac{-1}{m}\sum_0^{m-1}(k+1)\varepsilon(\lambda)^k\theta^{\lambda k} \quad ([3]),$$

$$\|\tilde{\theta}\| \leqslant \frac{m+1}{2}\|\theta\|, \qquad D\tilde{\theta} = \theta, \quad \text{with } D\tilde{\theta} = \tilde{\theta} - \varepsilon(\lambda)\tilde{\theta}^\lambda \quad ([3]).$$

Let us show that $B_0 b\tilde{\theta} = \theta$. Since $D = B_0 b + b'B_0$ ([2] p. 117), we want to show that $b'B_0\tilde{\theta} = 0$. One has ([2], p. 117)

$$(B_0\tilde{\theta})(a^0,\dots,a^{m-2}) = (-1)^{m-2}\theta(a^0,\dots,a^{m-2},1),$$

$$\begin{aligned}(b'B_0\tilde{\theta})&(a^0,\dots,a^{m-1})\\ &= (-1)^{m-1}(b\phi_m)(1,a^0,\dots,a^{m-1},1) - (b\phi_m)(a^0,\dots,a^{m-1},1,1) +\\ &\quad + b(B\phi_m)(a^0,\dots,a^{m-1},1) = 0,\end{aligned}$$

since $b\phi_m \in \operatorname{Im} B$ and $bB\phi_m = 0$. Next, since $b'B_0\tilde{\theta} = 0$, there exists a canonical

$\theta' \in C^{m-3}$ such that $b'\theta' = B_0\tilde{\theta}$. One has

$$\theta'(a^0, \ldots, a^{m-3}) = (B\tilde{\theta})(a^0, \ldots, a^{m-3}, 1)$$

so that

$$\|\theta'\| \leqslant \|B_0\tilde{\theta}\| \leqslant \|D\tilde{\theta}\| = \|\theta\| \leqslant 4\|\phi_m\|.$$

Let $\theta'' \in C^{m-2}$ be such that $AB_0\theta'' = A\theta'$, $\|\theta''\| \leqslant 2\|\theta'\|$. To construct θ'', one can use a linear form L on the Banach space A such that $\|L\| = 1$, $L(1) = 1$ and use the formula of [2], p. 117, corollary 31. Now one has

$$B\theta'' = A\theta', \qquad Bb\theta'' = -bB\theta'' = -bA\theta' = -Ab'\theta' = -AB_0\tilde{\theta} = -B\tilde{\theta}.$$

Thus, $\psi = \tilde{\theta} + b\theta''$ satisfies

(a) $B\psi = 0$,

(b) $B_0 b\psi = B_0 b\tilde{\theta} = \theta$,

(c) $\|\psi\| \leqslant \|\tilde{\theta}\| + \|b\theta''\|$

$$\leqslant \frac{m+1}{2}\|\theta\| + (m-1)\|\theta''\| \leqslant (m+1)\|\phi_m\| + 8(m-1)\|\phi_m\|$$

$$\leqslant 9m\|\phi_m\|. \qquad \square$$

LEMMA 7. *Let $(\phi_{2n})_{n\in\mathbb{N}}$ be a normalized entire cocycle on A, then if $\phi \in \operatorname{Im}\partial \subset C_\varepsilon^{\mathrm{ev}}$, one has*

$$\sum_0^\infty \frac{(-1)^n}{n!}\phi_{2n}(e, \ldots, e) = 0$$

for any idempotent $e \in A$.

Proof. Let $(\psi_{2n+1}) \in C_\varepsilon^{\mathrm{odd}}$ be such that $\partial\psi = \phi$. Thus for each n,

$$\phi_{2n} = 2nb\psi_{2n-1} + \frac{1}{2n+1}B\psi_{2n+1}.$$

Now since ϕ is normalized, $B_0\phi_{2n} \in C_\lambda^{2n}$ is cyclic so that

$$B_0 b\psi_{2n-1} = \frac{1}{2n}B_0\phi_{2n}$$

is cyclic for any n. Let

$$\alpha_n = (B_0\psi_{2n+1})(e, \ldots, e) = \frac{1}{2n+1}B\psi_{2n+1}(e, \ldots, e).$$

One has, since $e^2 = e$, that

$$\alpha_n = (b'B_0\psi_{2n+1})(e, \ldots, e) = ((D - B_0 b)\psi_{2n+1})(e, \ldots, e)$$
$$= (D\psi_{2n+1})(e, \ldots, e) = 2\psi_{2n+1}(e, \ldots, e).$$

Also

$$(b\psi_{2n+1})(e,\ldots,e) = \psi_{2n+1}(e,\ldots,e) = \tfrac{1}{2}\alpha_n,$$

Thus

$$\phi_{2n}(e,\ldots,e) = 2n\tfrac{1}{2}\alpha_{n-1} + \alpha_n,$$

so that

$$\sum_0^\infty \frac{(-1)^n}{n!}\phi_{2n}(e,\ldots,e) = \sum_0^\infty \frac{(-1)^n}{n!}(n\alpha_{n-1} + \alpha_n) = 0. \qquad \square$$

THEOREM 8. *Let* $\phi = (\phi_{2n})_{n\in\mathbb{N}}$ *be an entire normalized cocycle on* A, *and*

$$F_\phi,\ F_\phi(x) = \Sigma(-1)^n \frac{1}{n!}\phi_{2n}^q(x,\ldots,x)$$

the corresponding entire function on $M_\infty(A)$. *Then the restriction of* F_ϕ *to the idempotents* $e = e^2$, $e \in M_\infty(A)$ *defines an additive map:* $K_0(A) \to \mathbb{C}$. *The value* $\langle\phi, [e]\rangle$ *of* $F_\phi(e)$ *only depends upon the class of* ϕ *in* $H_\varepsilon^{\mathrm{ev}}(A)$.

Proof. Replacing A by $\tilde{A}$ and ϕ_{2n} by $\tilde{\phi}_{2n}$,

$$\tilde{\phi}_{2n}(x^0 + \lambda^0 1,\ldots,x^{2n} + \lambda^{2n}1) = \phi_{2n}(x^0,\ldots,x^{2n}) + \lambda^0 B_0\phi_{2n}(x^1,\ldots,x^{2n})$$

one can assume that each ϕ_{2n} vanishes if some x^i, $i > 0$ is equal to 1. We just need to show that the value of F_ϕ on $e \in \operatorname{Proj} M_q(A)$ only depends upon the connected component of e in $\operatorname{Proj} M_q(A)$. Since the map $\phi \to \phi^q$ is a morphism of complexes, we can assume that $q = 1$. Then let $t \to e(t)$ be a C^1 map of $[0, 1]$ to $\operatorname{Proj}(A)$. We want to show that $(\mathrm{d}/\mathrm{d}t)F_\phi(e(t)) = 0$. One has $(\mathrm{d}/\mathrm{d}t)(e(t)) = [a(t), e(t)]$, where $a(t) = (1 - 2e(t))(\mathrm{d}/\mathrm{d}t)e(t)$. We just need to compute $(\mathrm{d}/\mathrm{d}t)F_\phi(e(t))$ for $t = 0$, and we let $e = e(0)$, $a = a(0)$. We have:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_{2n}(e(t),\ldots,e(t))\right)_{t=0} = \sum_0^{2n+1}\phi_{2n}(e,\ldots,[a,e],\ldots,e)$$

Thus, by Lemma 7, in order to show that the above derivative vanishes, it is enough to prove that the following cocycle is a coboundary.

$$(\phi'_{2n})_{n\in\mathbb{N}}, \qquad \phi'_{2n}(x^0,\ldots,x^{2n}) = \sum_0^{2n+1}\phi_{2n}(x^0,\ldots,[a,x^j],\ldots,x^{2n}).$$

Let

$$\psi_{2n-1}(x^0,\ldots,x^{2n-1}) = \frac{1}{2n}\sum_0^{2n-1}(-1)^{j+1}\phi_{2n}(x^0,\ldots,x^j,a,x^{j+1},\ldots,x^{2n-1}).$$

Using the equality $B\psi_{2n-1} = A\theta_{2n-2}$, where

$$\theta_{2n-2}(x^0,\ldots,x^{2n-2}) = (B_0\phi_{2n})(a,x^0,\ldots,x^{2n-2})$$

one checks that for any n one has:

$$d_1\psi_{2n-1} + d_2\psi_{2n+1} = \phi'_{2n}.$$

2. Infinite Dimensional Cycles and Traces on the Algebras QA, $\mathscr{E}A$

In this section, we shall establish a canonical one-to-one correspondence between the following three notions on an algebra A:

(1) Cocycles with infinite support in the (b, B) bicomplex which satisfy the normalization condition of Lemma 6.

(2) Linear functionals $\int$ on the universal graded differential algebra ΩA such that

$$\int(\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1) = \tfrac{1}{2}(-1)^{\partial_1}\int \mathrm{d}\omega_1\,\mathrm{d}\omega_2$$

(3) Odd traces on the algebras QA, $\mathscr{E}A$ ([3]).

At first, this correspondence will be established at a purely algebraic level, then we shall have to translate in cases (2), (3) what *entire* cocycles give.

Thus, let A be an algebra over $\mathbb{C}$, and C^n be the space of $(n + 1)$ linear forms on A.

PROPOSITION 1. *Let* $(\psi_{2n})_{n\in\mathbb{N}}$, $\psi_{2n} \in C^{2n}$ *(resp.* $(\psi_{2n+1})_{n\in\mathbb{N}}$, $\psi_{2n+1} \in C^{2n+1}$*) be such that*

(a) $b\psi_m = B_0\psi_{m+2} \quad \forall m,$

(b) $B_0\psi_m = \dfrac{1}{m}AB_0\psi_m \quad \forall m.$

Then the functional μ *on* ΩA *given by*

(α) $\mu(a^0\mathrm{d}a^1\cdots\mathrm{d}a^m) = \psi_m(a^0, a^1, \ldots, a^m),$

(β) $\mu(\mathrm{d}a^1\cdots\mathrm{d}a^m) = (B_0\psi_m)(a^1, \ldots, a^m),$

(γ) $\mu(\omega) = 0$ *if* $\partial\omega$ *is odd (resp. even),*

satisfies the following equality:

$$\mu(\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1) = (-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2) \tag{0}$$

(i.e. *equality* (2) *without the factor of* $\tfrac{1}{2}$).

Proof. Let us prove the even case. Let us check that for $a \in A$, $\mathrm{d}a$ belongs to the centralizer of μ. The equality

$$\mu(\mathrm{d}a(\mathrm{d}a^1\cdots\mathrm{d}a^{2n-1})) = (-1)^{2n-1}\mu((\mathrm{d}a^1\cdots\mathrm{d}a^{2n-1})\,\mathrm{d}a)$$

follows from the cyclicity of $B_0\psi_{2n}$ (i.e. b)). One has $B_0\psi_{2n} = b\psi_{2n-2}$ so that $bB_0\psi_{2n} = 0$, and also $B_0b\psi_{2n} = 0$, since $b\psi_{2n}$ is cyclic. Thus, the equality $B_0b + b'B_0 = D$ ([2]) entails that:

$$\psi_{2n}(a^0, \ldots, a^{2n-1}, a) - (-1)^{2n}\psi_{2n}(a, a^0, \ldots, a^{2n-1}) +$$
$$+ (-1)^{2n}B_0\psi_{2n}(a\,a^0, a^1, \ldots, a^{2n-1}) = 0,$$

i.e. that

$$\mu(\mathrm{d}a(a^0\mathrm{d}a^1\cdots\mathrm{d}a^{2n-1}))=(-1)^{2n-1}\,\mu((a^0\,\mathrm{d}a^1\cdots\mathrm{d}a^{2n-1})\,\mathrm{d}a).$$

Thus, we have shown that any $\mathrm{d}\omega$ belongs to the centralizer of μ. Let us now show that

$$\mu(a\omega-\omega a)=\mu(\mathrm{d}a\,\mathrm{d}\omega)\quad\forall a\in A. \tag{$*$}$$

With $\omega=a^0\mathrm{d}a^1\cdots\mathrm{d}a^{2n}$ one has

$$\begin{aligned}\mu(\omega a)=\mu(a^0(\mathrm{d}a^1\cdots\mathrm{d}a^{2n})a)&=\psi_{2n}(a^0,a^1,\ldots,a^{2n-1},a^{2n}a)-\\&-\psi_{2n}(a^0,a^1,\ldots,a^{2n-1}a^{2n},a)+\cdots+\\&+(-1)^j\psi_{2n}(a^0,\ldots,a^{2n-j}a^{2n-j+1},\ldots,a)+\cdots+\\&+(-1)^{2n}\psi_{2n}(a^0a^1,\ldots,a).\end{aligned}$$

Thus

$$\begin{aligned}\mu(\omega a-a\omega)&=b\psi_{2n}(a^0,a^1,\ldots,a^{2n},a)=B_0\psi_{2n+2}(a^0,\ldots,a^{2n},a)\\&=\mu(\mathrm{d}\omega\,\mathrm{d}a)=-\mu(\mathrm{d}a\,\mathrm{d}\omega).\end{aligned}$$

Finally, we just need to check that if $\omega_1=a\,\mathrm{d}\omega$ is of degree ∂_1 with $a\in A$, and $\omega_2\in\Omega$ is of degree ∂_2, one has (0). Since $\mathrm{d}\omega$ is in the centralizer of μ we have

$$\begin{aligned}&\mu(\omega_1\omega_2-(-1)^{\partial_1\partial_2}\,\omega_2\omega_1)\\&\quad=\mu(a\,\mathrm{d}\omega\,\omega_2-(-1)^{\partial_1\partial_2}\,\omega_2 a\,\mathrm{d}\omega)\\&\quad=\mu(a\,\mathrm{d}\omega\,\omega_2-\mathrm{d}\omega(\omega_2 a)).\end{aligned}$$

Using ($*$) we get

$$\mu(\omega_1\omega_2-(-1)^{\partial_1\partial_2}\,\omega_2\omega_1)=\mu(\mathrm{d}a\,\mathrm{d}(\mathrm{d}\omega\omega_2))=(-1)^{\partial_1}\,\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2). \qquad \square$$

The above proof shows that, conversely, any functional μ on ΩA which is even (resp. odd) and satisfies

$$\mu(\omega_1\omega_2-(-1)^{\partial_1\partial_2}\,\omega_2\omega_1)=(-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2),$$

defines an even (resp. odd) normalized cochain (ψ_{2n}) (resp. (ψ_{2n+1})) such that $b\psi_m=B_0\psi_{m+2}$ for any m, by the equality

$$\psi_m(a^0,\ldots,a^m)=\mu(a^0\,\mathrm{d}a^1\cdots\mathrm{d}a^m).$$

Thus, since ΩA is the universal differential graded algebra over A, we see that

PROPOSITION 2. *Let (Ω, d) be a differential graded algebra such that $\Omega^0=A$, and μ an even (resp. odd) linear form on Ω such that*

$$\mu(\omega_1\omega_2-(-1)^{\partial_1\partial_2}\,\omega_2\omega_1)=(-1)^{\partial_1}\,\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2)\quad\forall\omega_i\in\Omega.$$

Then the equality

$$\psi_m(a^0,\ldots,a^m)=\mu(a^0\,\mathrm{d}a^1\cdots\mathrm{d}a^m)\quad\forall a^i\in A$$

defines a normalized even (resp. odd) cochain such that

$$b\psi_m = B_0\psi_{m+2} \quad \forall m.$$

In [2], we took as a starting point of cyclic cohomology the notion of cycle of dimension n, given by a differential graded algebra (Ω, d) as above and a homogeneous linear form $\int$ of degree n such that

$$(1) \quad \int \omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1 = 0 \quad \forall \omega_j \in \Omega^{\partial_j}, j = 1, 2,$$

$$(2) \quad \int \mathrm{d}\omega = 0 \quad \forall \omega \in \Omega^{n-1}.$$

The above Proposition 2 shows that in order to handle cocycles with infinite support, one has to replace conditions (1), (2) by the single condition

$$\int (\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1) = (-1)^{\partial_1}\int \mathrm{d}\omega_1\,\mathrm{d}\omega_2. \tag{0}$$

Note also that the condition $b\psi_m = B_0\psi_{m+2}$ which we use in Propositions 1 and 2 is slightly different from the cocycle condition of Section 1, namely: $\mathrm{d}_1\phi_m + \mathrm{d}_2\phi_{m+2} = 0$, the exact relation is given by

$$\phi_{2n} = (-1)^n(2n-1)\cdots 3.1\ \psi_{2n}, \tag{ev}$$

$$\phi_{2n+1} = (-1)^n(2n)(2n-2)\cdots 2.1\ \psi_{2n+1}. \tag{odd}$$

We let (cf. [5]) QA be the algebraic-free product of A by itself. It is generated as an algebra by the subalgebra $A = \{a * 1, a \in A\}$ and the elements $qa = a * 1 - 1 * a, a \in A$. Also we let $\sigma \in \mathrm{Aut}(QA)$ be the involution given by

$$\sigma(a) = a - q(a), \qquad \sigma(q(a)) = -q(a) \quad \forall a \in A,$$

and $\mathscr{E}A$ be the crossed product algebra $\mathscr{E}A = QA \times_\sigma \mathbb{Z}/2$ (cf. [11, 3]). We let F, $F^2 = 1$ be the canonical generator of $\mathscr{E}A$ over QA. We shall say that a linear form T on QA (resp. $\mathscr{E}A$) is odd iff $T \circ \sigma = -T$ (resp. $T \circ \hat{\sigma} = -T$, where $\hat{\sigma}$ is the involution dual to σ).

In [3] J. Cuntz and the author give the general form of odd traces on both QA and $\mathscr{E}A$, so it might appear at first sight that such traces are easy to construct and are not interesting from a cohomological point of view. It turns out, however, that the explicit construction of [3] is the translation of the triviality of the first spectral sequence of the (b, B) bicomplex (thm. 40 of [2]). Thus, provided we impose a suitable growth condition on components T_m of T (when A is a Banach algebra), this explicit construction becomes incompatible with the growth condition and does not exclude the existence of nontrivial traces. We first establish at a purely algebraic level, the correspondence between odd traces on $\mathscr{E}A$ (resp. QA) and even (resp. odd) functionals on ΩA satisfying (0). Let us recall (cf. [3]) that we can identify QA with ΩA as a linear space and obtain the product of QA by a simple formula. We let π be the linear bijection $\Omega A \to QA$ such that

$$\pi(a^0\,\mathrm{d}a^1 \cdots \mathrm{d}a^n) = a^0 q a^1 \cdots q a^n \quad \forall\, a^i \in A,$$

$$\pi(\mathrm{d}a^1 \cdots \mathrm{d}a^n) = q a^1 \cdots q a^n \quad \forall\, a^i \in A.$$

PROPOSITION 3. *Let T be an odd linear form on QA (resp. $\mathscr{E}A$) and μ_T the restriction of $T \circ \pi$ to Ω^{odd} (resp., the restriction to Ω^{ev} of the linear form $\omega \to T(F\pi(\omega))$. The map $T \to \mu_T$ is a canonical bijection between odd traces on QA (resp. $\mathscr{E}A$) and odd (resp. even) functionals μ on ΩA such that*

$$\mu(\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\omega_2\omega_1) = \tfrac{1}{2}(-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2). \tag{0'}$$

Proof. First let T be an odd trace on QA and $\mu = T \circ \pi$. Let us check (0′). We can assume that the degrees ∂_i of ω_i are $\partial_1 \equiv 1$ (2), $\partial_2 \equiv 0$ (2). One has

$$T(\pi(\omega_1)\pi(\omega_2)) = T(\pi(\omega_2)\pi(\omega_1)),$$

i.e.

$$T \circ \pi(\omega_1\omega_2) + T \circ \pi(\omega_1\,\mathrm{d}\omega_2) = T \circ \pi(\omega_2\omega_1).$$

Now

$$T \circ \pi(\omega_1\,\mathrm{d}\omega_2) = -T(\sigma(\pi(\omega_1\,\mathrm{d}\omega_2)))$$

and

$$\sigma(\pi(\omega_1\,\mathrm{d}\omega_2)) = \pi(\omega_1\,\mathrm{d}\omega_2 - \mathrm{d}\omega_1\,\mathrm{d}\omega_2),$$

since on ΩA as a linear space, $\sigma(\omega) = (-1)^{\partial\omega}(\omega - \mathrm{d}\omega)$, thus

$$\mu(\omega_1\omega_2 - \omega_2\omega_1) = -T \circ \pi(\omega_1\,\mathrm{d}\omega_2) = -\tfrac{1}{2}T \circ \pi(\mathrm{d}\omega_1\,\mathrm{d}\omega_2) = \tfrac{1}{2}(-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2).$$

Conversely, let μ be a functional on Ω^{odd} satisfying (0′). Then let T be the linear form on QA given by:

(1) $T(\pi(\omega)) = \mu(\omega)$ if $\partial\omega \equiv 1 \pmod 2$,

(2) $T(\pi(\omega)) = \tfrac{1}{2}\mu(\mathrm{d}\omega)$ if $\partial\omega \equiv 0 \pmod 2$.

We have to check that

$$T(\pi(\omega_1)\pi(\omega_2)) = T(\pi(\omega_2)\pi(\omega_1)) \quad \forall\, \omega_1, \omega_2 \in \Omega.$$

There are three cases to consider, which we label by the degrees modulo 2 of ω_1 and ω_2:

(0,0). One has

$$\pi(\omega_1\omega_2) = \pi(\omega_1)\pi(\omega_2), \qquad \pi(\omega_2\omega_1) = \pi(\omega_2)\pi(\omega_1)$$

thus one just needs the equality

$$\mu(\mathrm{d}(\omega_1\omega_2)) = \mu(\mathrm{d}(\omega_2\omega_1))$$

which follows from

$$\mu((\mathrm{d}\omega_1)\omega_2) = \mu(\omega_2\,\mathrm{d}\omega_1) \quad \text{and} \quad \mu(\omega_1\,\mathrm{d}\omega_2) = \mu((\mathrm{d}\omega_2)\omega_1),$$

by (0)′.

(1,0). One has

$$\pi(\omega_1)\pi(\omega_2) = \pi(\omega_1\omega_2 + \omega_1\mathrm{d}\omega_2), \qquad \pi(\omega_2)\pi(\omega_1) = \pi(\omega_2\omega_1)$$

thus one has to show that

$$\mu(\omega_1\omega_2 - \omega_2\omega_1) + \tfrac{1}{2}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2) = 0$$

which follows from (0′).

(1,1). As above, we have to show that

$$T\circ\pi(\omega_1\omega_2 + \omega_1\,\mathrm{d}\omega_2) = T\circ\pi(\omega_2\omega_1 + \omega_2\,\mathrm{d}\omega_1),$$

i.e. that

$$\tfrac{1}{2}\mu(\mathrm{d}(\omega_1\omega_2)) + \mu(\omega_1\,\mathrm{d}\omega_2) = \tfrac{1}{2}\mu(\mathrm{d}(\omega_2\omega_1)) + \mu(\omega_2\,\mathrm{d}\omega_1).$$

But this follows from:

$$\mu((\mathrm{d}\omega_1)\omega_2) = \mu(\omega_2(\mathrm{d}\omega_1)),$$

$$\mu(\omega_1\,\mathrm{d}\omega_2) = \mu((\mathrm{d}\omega_2)\omega_1).$$

The case of $\mathscr{E}A$ and even μ's is treated similarly. □

It we put together Propositions 1, 2 and 3, we see that we get at a purely algebraic level, the identity between:

(1) Normalized cocycles $(\phi_{2n})_{n\in\mathbb{N}}$ as in Section 1,

(2) Linear functionals μ on $\Omega^{\mathrm{ev}}A$ such that

$$\mu(\omega_1\omega_2 - (-1)^{\partial_1\partial_2}\,\omega_2\omega_1) = (-1)^{\partial_1}\mu(\mathrm{d}\omega_1\,\mathrm{d}\omega_2),$$

(3) Odd traces on $\mathscr{E}A = QA \times_\sigma \mathbb{Z}/2$.

Starting with a cocycle $(\phi_{2n})_{n\in\mathbb{N}}$ (resp. $(\phi_{2n+1})_{n\in\mathbb{N}}$) as in Section 1, the corresponding functional μ on ΩA is given by Proposition 1 with components

$$\psi_{2n} = (-1)^n(1.3\cdots(2n-1))^{-1}\,\phi_{2n}\ (\text{resp. } \psi_{2n-1} = (-1)^n\,(2.4\cdots 2n)^{-1}\,\phi_{2n-1}).$$

Thus, if A is a Banach algebra and we endow (cf. [1], p. 262) ΩA with the Banach algebra norms

$$\left\|\sum_0^\infty \omega_k\right\|_r = \sum_0^\infty r^k\|\omega_k\|_\pi,$$

where $\|\ \ \|_\pi$ is the projective-tensor product norm on tensor powers of A, we get

PROPOSITION 4. *Let A be a Banach algebra. Then Propositions* 1 *and* 2 *establish* *a canonical bijective correspondence between normalized entire cocycles on A and linear forms on ΩA satisfying* (0′) *which are continuous for all the norms* $\|\ \ \|_r$ on ΩA.

Proof. One has

$$\frac{\|\phi_{2n}\|}{n!} \sim 2^n \|\psi_{2n}\|. \qquad \square$$

A similar statement holds for QA and $\mathscr{E}A$, moreover, if we translate the pairing (Theorem 8) at the level of $QA, \mathscr{E}A$ we obtain the following, where $QA \simeq \Omega A$ is endowed with the inductive limit topology of the norms $\|\ \ \|_r$, and $\mathscr{E}A \simeq QA \oplus QA$ (as linear space) of the direct sum topology:

THEOREM 5. *Let τ be a continuous odd trace on $\mathscr{E}A$, then the map of $K_0(A)$ to $\mathbb{C}$ given by Theorem* 8 *and the entire even cocycle associated to τ is obtained by the formula*

$$e \in \text{Proj } A \to \tau\left(F \frac{e}{\sqrt{1-(qe)^2}}\right).$$

Proof. Up to an overall normalization constant, the entire cocycle ϕ associated to τ has components ϕ_{2n} given by

$$\phi_{2n}(a^0, \ldots, a^{2n}) = (-1)^n 2^{-n}(2n-1) \cdots 3.1 \quad \tau(Fa^0 q(a^1) \cdots q(a^{2n})).$$

Thus, the answer follows from the formula giving F_ϕ.

3. The Algebra $\mathscr{L}$ of Operator Valued Distributions

In this section we shall introduce an algebra of operator valued distributions which will play an important technical role in the estimates of the character of θ-summable Fredholm modules.

We let $\mathscr{H}$ be a Hilbert space. By an operator valued distribution we mean a norm continuous linear map T from the Schwarz space $S(\mathbb{R})$ (with its usual nuclear space topology) to the Banach space $\mathscr{L}(\mathscr{H})$ of bounded operators in $\mathscr{H}$. Thus, there exists by hypothesis a continuous seminorm p on $S(\mathbb{R})$ such that $\|T(f)\| \leqslant p(f)\,\forall f \in S(\mathbb{R})$. We let $\mathscr{L}$ be the space of operator-valued distributions T which satisfy the following properties:

(1) Support $T \subset \mathbb{R}^+ = [0, +\infty[$.

(2) There exists $r > 0$ and an analytic operator valued function $t(z)$ for $z \in C = \bigcup_{s>0} sU$, where U is the disk with center at 1 and radius r such that

  (a) $t(s) = T(s)$ on $]0, +\infty[$,

  (b) the function

$$h(p) = \sup_{z \in (1/p)U} \|t(z)\|_p, \quad p \in\,]1, +\infty[$$

  is majorized by a polynomial in p for $p \to \infty$.

In (2), the norm $\|t(z)\|_p$ is the Banach space norm [10, 2] of the Schatten class $\mathscr{L}^p(\mathscr{H})$. In particular we see that $t(1) \in \mathscr{L}^1(\mathscr{H})$ is a trace class operator. The operator valued analytic function $t(z)$ is, of course, uniquely determined by the distribution T and we shall use the abuse of notation $T(z)$ instead of $t(z)$. Two distributions $T_1, T_2 \in \mathscr{L}$ such that $T_1(z) = T_2(z) \ \forall z \in]0, +\infty[$ differ by a distribution with support the origin, of the form $\Sigma a_k \delta_0^{(k)}$, where $a_k \in \mathscr{L}(\mathscr{H})$, and $\delta_0^{(k)}$ is the kth derivative of the Dirac mass δ_0 at the origin.

LEMMA 1. (a) *Let $T \in \mathscr{L}$ then the derivative $T' = (\mathrm{d}/\mathrm{d}s)T$ also belongs to $\mathscr{L}$.*

(b) *Let $T \in \mathscr{L}$, there exists an integer q and $S \in \mathscr{L}$ such that $T - S^{(q)}$ has support $\{0\}$ and that*

$$\sup_p \sup_{z \in (1/p)U} \|S(z)\|_p < \infty,$$

where $U = \{z \in \mathbb{C}, |z-1| \leqslant r\}$.

Proof. (a) By definition, $T'(f) = -T(f') \ \forall f \in S(\mathbb{R})$, so that T' is an operator-valued distribution satisfying property (1). Let r and U be as in (2) for T and let $r' = r/2$, then by Cauchy's theorem, the operator $T'(z)$ for $z \in 1/p\, U'$, $U' = \{z \in \mathbb{C}, |z-1| \leqslant r'\}$ is of the form $\int_{u \in (1/p)U} T(u)\, \mathrm{d}\mu(u)$, where μ has total mass less than $2p/r$, thus

$$\sup_{z \in (1/p)U'} \|T'(z)\|_p \leqslant \frac{2p}{r} \sup_{z \in (1/p)U} \|T(z)\|_p,$$

which proves that T' satisfies property (2).

(b) By hypothesis, there exists $C < \infty$ and $q \in \mathbb{N}$ such that, with the notations of (2), $h(p) \leqslant Cp^q$. Let T_k be, for $k = 0, 1, \ldots$, the operator-valued analytic function in $C = \bigcup_{s>0} sU$, defined inductively by $T_0(z) = T(z)$ and $T_{k+1}(z) = \int_1^z T_k(u)\, \mathrm{d}u$. For $z \in (1/p)U$ one has

$$\|T_{k+1}(z)\|_p \leqslant 2 \int_0^1 h_k\left(\left((1-t) + \frac{t}{p}\right)^{-1}\right) \mathrm{d}t$$

where

$$h_k(p) = \sup_{z \in (1/p)U} \|T_k(z)\|_p$$

(since $\|T_k(u)\|_p \leqslant \|T_k(u)\|_{p'}$ for $p' \leqslant p$).

Thus we see that h_k is of the order of p^{q-k} for $k < q$, and that h_q is of the order of $\log p$ while h_{q+1} is bounded. Then let S be the operator-valued distribution given by

$$S(f) = \int f(s) T_{q+1}(s)\, \mathrm{d}s \quad \forall f \in S(\mathbb{R})$$

It is well defined, since $\|T_{q+1}(s)\|$ is bounded on $[0, 1]$ and by a polynomial for large s. By construction the $q+1$th derivative of S agrees with T outside the origin, thus the conclusion. □

We can now show that $\mathscr{L}$ is an algebra under the convolution product, which at the formal level can be written

$$(T_1 * T_2)(s) = \int_0^s T_1(u) T_2(s-u)\,\mathrm{d}u.$$

More precisely, given $f \in S(\mathbb{R})$, one can find $a_n, b_n \in S(\mathbb{R})$ such that the restriction to $]-1, \infty[\times]-1, \infty[$ of the function $(s, u) \to f(s+u)$ is given by the convergent series $\Sigma a_n \otimes b_n$. Then $(T_1 * T_2)(f) = \Sigma T_1(a_n) T_2(b_n) \in \mathscr{L}(\mathscr{H})$.

LEMMA 2. *If* $T_1, T_2 \in \mathscr{L}$ *then* $T_1 * T_2 \in \mathscr{L}$.

Proof. By Lemma 1 one can assume that T_i is given by

$$T_i(f) = \int_0^\infty f(s) T_i(s)\,\mathrm{d}s,$$

where $T_i(s)$ is an analytic operator valued function in $C = \bigcup_{t>0}(tU)$, $U = \{z \in \mathbb{C}, |z-1| < r\}$, and where

$$C_i = \sup_p \sup_{(1/p)U} \|T_i(z)\|_p < \infty.$$

Then let $T(z) = \int_0^1 T_1(\lambda z) T_2((1-\lambda)z) z\,\mathrm{d}\lambda$. It is by construction an analytic operator-valued function defined in C. One has, for $z \in (1/p)U$, that

$$\lambda z \in \frac{1}{p_1} U, \qquad (1-\lambda)z \in \frac{1}{p_2} U$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

so that by Holder's inequality:

$$\|T_1(\lambda z) T_2((1-\lambda)z)\|_p \leqslant \|T_1(\lambda z)\|_{p_1} \|T_2((1-\lambda)z)\|_{p_2} \leqslant C_1 C_2,$$

thus we get, for any $z \in (1/p)U$:

$$\|T(z)\|_p \leqslant \int_0^1 \|T_1(\lambda z) T_2((1-\lambda)z)\|_p |z|\,\mathrm{d}\lambda \leqslant |z| C_1 C_2.$$

It follows that $T(z)$ defines an element of $\mathscr{L}$ and it coincides with the convolution product $T_1 * T_2$. □

Let $\lambda = \delta_0'$ be the derivative of the Dirac mass at 0. One has $\lambda \in \mathscr{L}$, and as an operator valued distribution, λ has a natural square root (for the convolution product) given by

$$T(s) = \frac{1}{\sqrt{\pi s}},$$

but this square root does not define an element of the algebra $\mathscr{L}$, since (when $\dim(\mathscr{H}) = \infty$) it fails to satisfy condition (2) above, because the identity operator does not belong to any $\mathscr{L}^p$. We thus need to adjoin the square root $\lambda^{1/2}$ of λ to $\mathscr{L}$ and for this we consider the algebra $\tilde{\mathscr{L}}$ of pairs (T_0, T_1) of elements of $\mathscr{L}$ with product given by:

$$(T_0, T_1)*(S_0, S_1) = (T_0S_0 + \lambda T_1S_1, T_0S_1 + T_1S_0).$$

Since λ belongs to the center of $\mathscr{L}$, one checks that the above product turns $\tilde{\mathscr{L}}$ into an algebra. This algebra $\tilde{\mathscr{L}}$ contains $\mathscr{L}$ (by the homomorphism $T \to (T, 0)$), and the central element $\lambda^{1/2} = (0, \delta_0)$ so that every element of $\tilde{\mathscr{L}}$ is of the form $A + B\lambda^{1/2}$ with $A, B \in \mathscr{L}$.

LEMMA 3. *The equality* $\tau(T_0, T_1) = \mathrm{trace}(T_1(1))$ *defines a trace on the algebra* $\tilde{\mathscr{L}}$.

Proof. By condition (2) we know that $T_1(1)$ belongs to $\mathscr{L}^1$ so that the trace is well defined. Since

$$(T_0, T_1)*(S_0, S_1) = (T_0S_0 + \lambda T_1S_1, T_0S_1 + T_1S_0),$$

it is enough to check that $T \to \mathrm{trace}\ T(1)$ is a trace on the algebra $\mathscr{L}$. The proof of Lemma 2 shows that for $T_i \in \mathscr{L}$ of the form $T_i(f) = \int f(s)T_i(s)\,\mathrm{d}s$, with

$$\sup_p \sup_{z \in (1/p)U} \|T_i(z)\|_p < \infty,$$

one has:

$$\mathrm{trace}(T_1 * T_2)(s) = \mathrm{trace}(T_2 * T_1)(s) \quad \forall s \in U.$$

Thus for any power of λ one has

$$\mathrm{trace}((\lambda^k T_1 * T_2)(1)) = \mathrm{trace}((\lambda^k T_2 * T_1)(1)).$$

Now by Lemma 1, to show that $\mathrm{trace}(S_1 * S_2)(1) = \mathrm{trace}(S_2 * S_1)(1)$, we can assume that $S_j = \lambda^{k_j}T_j + U_j$, where T_j is as above and U_j has support the origin. Thus, we just need to check, say, that $\mathrm{trace}(\lambda^{k_1}T_1U_2)(1) = \mathrm{trace}(U_2\lambda^{k_1}T_1)(1)$ which follows from $\mathrm{trace}(ab) = \mathrm{trace}(ba)$, $a \in \mathscr{L}(\mathscr{H})$, $b \in \mathscr{L}^1(\mathscr{H})$. □

4. Construction of the Character of a θ-Summable Fredholm Module

Let A be an algebra and $(\mathscr{H}, D, \varepsilon)$ an even θ-summable Fredholm module over A cf. [4], Def. 23. Thus by hypothesis $\varepsilon D = -D\varepsilon$ and $\exp(-tD^2)$ is of trace class for any positive t. Our aim is to construct, using D, an element F of the algebra $\tilde{\mathscr{L}}$ such that $F^2 = 1$, and to use the homomorphism $a \to a\delta_0$ of A in $\tilde{\mathscr{L}}$ as well as the trace τ to define the character of $(\mathscr{H}, D, \varepsilon)$.

LEMMA 1. *One has, with* $s = \mathrm{Re}\ z > 0$, $p \in [1, \infty[$,

$$\|e^{-zD^2}\|_p = (\mathrm{trace}(e^{-spD^2}))^{1/p}, \ \|D\,e^{-zD^2}\|_p \leqslant s^{-1/2}\|e^{-(sp/2)D^2}\|_p.$$

Proof. One has

$$\|e^{-zD^2}\|_p = \|e^{-sD^2}\|_p = (\mathrm{trace}(e^{-psD^2}))^{1/p}.$$

To prove the second inequality it is enough to show that the operator norm of $\|D\,e^{-(s/2)D^2}\|$ is bounded by $1/\sqrt{s}$. But this follows from the inequality $x\,e^{-(s/2)x^2} \leqslant 1/\sqrt{s}$ for x real and positive. □

Lemma 1 shows that we can define an element N of $\mathscr{L}$ by the equality

$$N(f) = \frac{1}{\sqrt{\pi}} \int f(s) \frac{1}{\sqrt{s}} e^{-sD^2} \, ds, \quad f \in S(\mathbb{R}).$$

The integral makes sense since the operator norm of $1/\sqrt{s}\, e^{-sD^2}$ is integrable near the origin. We shall, however, also need to define the distribution DN which is formally given by

$$(DN)(f) = \frac{1}{\sqrt{\pi}} \int f(s) \frac{1}{\sqrt{s}} D \, e^{-sD^2} \, ds.$$

By Lemma 1, one has an analytic operator-valued function,

$$(DN)(z) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{z}} D \, e^{-zD^2},$$

defined for $\operatorname{Re} z > 0$ and such that $\sup_{(1/p)U} \|(DN)(z)\|_p$ is of the order of p, $p \to \infty$. However, since the operator norm of $(DN)(s)$ is of the order of $1/s$, $s \to 0$, and is not integrable, we have to be very careful in the definition of the distribution DN.

LEMMA 2. (a) *The Laplace transform of the distribution N is given by* $\int_0^\infty N(s) e^{-s\lambda} \, ds = (D^2 + \lambda)^{-1/2}$.

(b) *There exists a unique element of $\mathscr{L}$, noted DN, whose Laplace transform is equal to $D(D^2 + \lambda)^{-1/2}$, one has $(DN)(s) = DN(s)$ for any $s > 0$.*

Proof. (a) Follows from the equality

$$\int_0^\infty \frac{1}{\sqrt{\pi s}} e^{-s\alpha^2} e^{-s\lambda} \, ds = (\alpha^2 + \lambda)^{-1/2}.$$

(b) The uniqueness follows from [9], let us prove the existence. One has

$$\begin{aligned} &D(D^2 + \lambda)^{-1/2} - D(D^2 + 1)^{-1/2} \\ &= \frac{1}{\pi} \int_0^\infty D((D^2 + \lambda + \rho)^{-1} - (D^2 + 1 + \rho)^{-1}) \rho^{-1/2} \, d\rho \\ &= \frac{1}{\pi} (1 - \lambda) \int_0^\infty D(D^2 + \lambda + \rho)^{-1} (D^2 + 1 + \rho)^{-1} \rho^{-1/2} \, d\rho. \end{aligned}$$

Now $D(D^2 + 1)^{-1/2}$ is the Laplace transform of the element of $\mathscr{L}$ given by $D(D^2 + 1)^{-1/2} \delta_0$, thus we just have to show, using Lemma 1, that $\int_0^\infty D(D^2 + \lambda + \rho)^{-1} (D^2 + 1 + \rho)^{-1} \rho^{-1/2} \, d\rho$ is the Laplace transform of an element of $\mathscr{L}$. But $D(D^2 + 1 + \rho)^{-1} (D^2 + \lambda + \rho)^{-1}$ is the Laplace transform of $D(D^2 + 1 + \rho)^{-1} e^{-s(D^2 + \rho)}$ and it is enough to check that the operator norm of

$$T(s) = \int_0^\infty D(D^2 + 1 + \rho)^{-1} e^{-s(D^2 + \rho)} \rho^{-1/2} \, d\rho$$

is integrable near $s = 0$. One has:

$$\|T(s)\| \leqslant \int_0^\infty (1 + \rho)^{-1/2} e^{-s\rho} \rho^{-1/2} \, d\rho,$$

since $\|D(D^2+1+\rho)^{-1}\| \leqslant (1+\rho)^{-1/2}$, thus since

$$\int_0^\infty (1+\rho)^{-1/2}\,\mathrm{e}^{-s\rho}\rho^{-1/2}\,\mathrm{d}\rho \leqslant (3-\log s) = O(|\log s|)$$

when $s \to 0$ we see that the operator norm of $T(s)$ is integrable near 0. The same estimate works for the $\mathscr{L}^p$ norm $\|T(z)\|_p$ for $z \in (1/p)U$ and shows that $T \in \mathscr{L}$. □

PROPOSITION 3. *The equality $F = (DN, \varepsilon N)$ defines an element of $\tilde{\mathscr{L}}$ of square δ_0.*

Proof. By Lemma 2, the element $DN \in \mathscr{L}$ is well defined. Since ε anticommutes with D, DN anticommutes with εN so that the square is given by $F^2 = ((DN)^2 + \lambda N^2, 0)$. Now the Laplace transform of $(DN)^2$ is (Lemma 2) equal to $D^2(D^2+\lambda)^{-1}$ and that of λN^2 is $\lambda(D^2+\lambda)^{-1}$. Thus, the Laplace transform of $(DN)^2 + \lambda N^2$ is equal to 1 and we get $F^2 = (\delta_0, 0)$. □

Recall (cf. [11]) that $\mathscr{E}$A is the crossed product of QA by $\mathbb{Z}/2$ so that a homomorphism from $\mathscr{E}$A to an algebra B is given by

(1) a homomorphism from A to B,
(2) an element F of B of square 1.

DEFINITION 4. Let $(\mathscr{H}, D, \varepsilon)$ be a θ-summable Fredholm module over A, then its character is the odd trace T on εA given by $T(x) = \tau(\pi(x))$ where $\pi\colon \mathscr{E}A \to \tilde{\mathscr{L}}$ is the homomorphism given by $a \to a\delta_0$ and the element F.

The next lemma defines the *components* $\phi_{2n} = \Gamma(n+\frac{1}{2})\tau_{2n}$ of the character.

LEMMA 5. *For each n let τ_{2n} be the $2n+1$ linear form on* A,

$$\tau_{2n}(a^0, \ldots, a^{2n}) = \tau(Fa^0[F, a^1]\cdots[F, a^{2n}]).$$

(a) $\tau_0(a) = \dfrac{1}{\sqrt{\pi}}\mathrm{Trace}(\varepsilon a\,\mathrm{e}^{-D^2}) \quad \forall a \in A$

(b) $B_0\tau_{2n+2} = -2b\tau_{2n}$

(c) $d_1\phi_{2n} + \mathrm{d}_2\phi_{2n+2} = 0 \quad$ where $\phi_{2n} = \Gamma(n+\frac{1}{2})\tau_{2n}$.

Now it follows from the construction that each of the functionals τ_{2n} is norm continuous on A for the operator norm $a \to \|a\|$. However, one cannot estimate $\|\tau_{2n}\|$ using only the operator norm, so as to get an *entire* cocycle on A. One needs to use the finer norm $\|[D, a]\|$, $a \in$ A. This is the aim of the next section.

5. Estimate of the Character of a θ-Summable Fredholm Module

Let A be an algebra and $(\mathscr{H}, D, \varepsilon)$ an even θ-summable Fredholm module over A. Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be the algebras defined in Section 3 and $N \in \mathscr{L}$, $F = (DN, \varepsilon N) \in \tilde{\mathscr{L}}$ as in Proposition 3. Our aim in this section is to estimate the functional τ_{2n},

$$\tau_{2n}(a^0, \ldots, a^{2n}) = \tau(Fa^0[F, a^1]\cdots[F, a^{2n}])$$

in terms of the operator norms $\|a^0\|$ and $\|[D, a^i]\|$, $i = 1, 2, \ldots, 2n$. Given an element T of $\mathscr{L}$, we shall say that an L^1 function $f(s)$, $s \in [0, 1]$ is a majorizing function for T when the following holds for a suitable neighborhood $U = \{z \in \mathbb{C}, |z-1| \leqslant r\}$ of 1 in $\mathbb{C}$:

(a) $\forall p \in [1, \infty[, \sup_{z \in (1/p)U} (\|T(z)\|_p) \leqslant f\left(\frac{1}{p}\right)$,

(b) $T(\phi) = \int_0^\infty T(s)\phi(s)\,\mathrm{d}s$ for any $\phi \in S(\mathbb{R})$.

LEMMA 1. *Let $T_i \in \mathscr{L}$ have f_i as majorizing function, $i = 1, 2$. Then $2f_1 * f_2$ is a majorizing function for $T_1 * T_2$.*

Proof. Let $T = T_1 * T_2$. As in Lemma 3.2, one has

$$T(z) = \int_0^1 T_1(\lambda z) T_2((1-\lambda)z) z\,\mathrm{d}\lambda,$$

$$\begin{aligned} \|T(z)\|_p &\leqslant \int_0^1 \|T_1(\lambda z)\|_{p/\lambda} \|T_2((1-\lambda)z)\|_{p/(1-\lambda)} |z|\,\mathrm{d}\lambda \\ &\leqslant \int_0^1 f(\lambda/p) f((1-\lambda)/p)|z|\,\mathrm{d}\lambda \\ &\leqslant (\sup|z|) \int_0^1 f_1(\lambda/p) f_2((1-\lambda)/p) \frac{1}{p}\,\mathrm{d}\lambda \\ &\leqslant 2(f_1 * f_2)\left(\frac{1}{p}\right) \end{aligned}$$

for any $z \in (1/\mathrm{p})\,U$. □

Now let $a \in \mathrm{A}$, so that $[D, a]$ is bounded. Considering a as the element $a\delta_0$ of $\mathscr{L}$, we shall find majorizing functions for the commutators $[N, a]$ and $[DN, a]$.

LEMMA 2. *Let $U = \{z \in \mathbb{C}, |z-1| \leqslant \frac{1}{4}\}$. Then, relative to U, the commutator $[N, a]$ (resp. $[DN, a]$) has majorizing function $12\|[D, a]\|\delta^s$ (resp. $20/\sqrt{\pi s}\,\|[D, a]\|\delta^s$) where $\delta = \mathrm{trace}(\mathrm{e}^{-(1/4)D^2})$.*

Proof. Let $U_0 = \{z \in \mathbb{C}, |z-1| \leqslant \frac{1}{2}\}$. One has $N(s) = 1/\sqrt{\pi s}\,\mathrm{e}^{-sD^2}$, with majorizing function $n(s) = s^{-1/2}\,(\mathrm{trace}(\mathrm{e}^{-(1/2)D^2}))^s$ relative to U_0. One has $[N, a](s) = 1/\sqrt{\pi s}$ $[\mathrm{e}^{-sD^2}, a]$ and

$$\begin{aligned} [a, e^{-sD^2}] &= \int_0^1 \mathrm{e}^{-\lambda s D^2} [D^2, a]\, \mathrm{e}^{-(1-\lambda)sD^2} s\,\mathrm{d}\lambda \\ &= \int_0^1 (\mathrm{e}^{-\lambda s D^2} D)\,[D, a]\, \mathrm{e}^{-(1-\lambda)sD^2} s\,\mathrm{d}\lambda\, + \\ &\quad + \int_0^1 \mathrm{e}^{-\lambda s D^2} [D, a]\,(D\,\mathrm{e}^{-(1-\lambda)sD^2}) s\,\mathrm{d}\lambda. \qquad (*) \end{aligned}$$

Now, by Lemma 4.1, the element T_1 of $\mathscr{L}$, $T_1(s) = D\,\mathrm{e}^{-sD^2}$, has majorizing function $t_1(s) = (s/2)^{-1/2}(\operatorname{trace}(\mathrm{e}^{-(1/4)D^2}))^s$ relative to U_0. Thus, since T_2, $T_2(s) = \mathrm{e}^{-sD^2}$ has majorizing function $t_2 = (\operatorname{trace}(\mathrm{e}^{-(1/4)D^2}))^s$ relative to U_0 we see by Lemma 1 that $[\mathrm{e}^{-sD^2}, a]$ has majorizing function

$$f(s) = 4\|[D, a]\|_\infty (t_1 * t_2)(s).$$

One has

$$(t_1 * t_2)(s) = \left(\int_0^s (u/2)^{-1/2}\, \mathrm{d}u\right)\delta^s \leqslant 3\sqrt{s}\,\delta^s$$

thus $[N, a](s)$ has majorizing function $(1/\sqrt{s})\, f(s) \leqslant 12\|[D, a]\|_\infty \delta^s$. Next $[D, a]N$ has majorizing function $\|[D, a]\| n(s)$, thus we only need to consider the term $S = D[N, a]$. Since $S(s) = (1/\sqrt{\pi s})D[\mathrm{e}^{-sD^2}, a]$ we just need to prove that $\|[D, a]\|\delta^s$ is a majorizing function for T, $T(s) = D[\mathrm{e}^{-sD^2}, a]$. The above equality (*) shows that $T = A[D, a]T_2 + T_1[D, a]T_1$, where $A(s) = D^2\,\mathrm{e}^{-sD^2}$. More precisely, A is the element of $\mathscr{L}$ whose Laplace transform is given by $\lambda \to D^2/(D^2 + \lambda) = 1 - (\lambda/(D^2 + \lambda))$ so that $A = \delta_0 + (T_2)'$. Thus

$$T = [D, a]T_2 + (T_2[D, a]T_2)' + T_1[D, a]T_1.$$

The first term has majorizing function $\|[D, a]\| t_2$, the last $\|[D, a]\| t_1 * t_1$, thus it remains to estimate $(T_2[D, a]T_2)'$. But relative to U_0, the element $T_2[D, a]T_2$ has majorizing function $\|[D, a]\| t_2 * t_2(s) = \|[D, a]\| s\delta^s$ so as in Lemma 3.1 (a), Cauchy's theorem shows that its derivative $(T_2[D, a]T_2)'$ has majorizing function $4\|[D, a]\|\delta^s$. □

LEMMA 3. *Let $(\mathscr{H}, D, \varepsilon)$ be an even θ-summable Fredholm module over A. Then for any $a^0, \ldots, a^{2n} \in$ A one has*

$$|\tau_{2n}(a^0, a^1, \ldots, a^{2n})| \leqslant \frac{(10)^{4n}}{n!}\|a^0\| \prod_1^{2n} \|[D, a^j]\| \operatorname{trace}(\mathrm{e}^{-(1/4)D^2}).$$

Proof. For each subset J of $\{1, 2, \ldots, 2n\}$ let $T_J \in \mathscr{L}$ be given by the product $B_1 \ldots B_{2n}$, where $B_j = [DN, a^j]$ for $j \notin J$ and $B_j = [N, a^j]$ for $j \in J$. The product in $\tilde{\mathscr{L}}$ of the $[F, a^j]$, $j = 1, 2, \ldots, 2n$, is given by the pair (S_0, S_1) where:

$$S_0 = \sum_{m=0}^{n} \sum_{|J| = 2m} \lambda^m T_J,$$

$$S_1 = \varepsilon \sum_{m=0}^{n-1} \sum_{|J| = 2m+1} \lambda^m T_J.$$

Now by Lemma 2, each T_J has a majorizing function given by the product

$$\prod_1^{2n} \|[D, a^j]\| \times (20)^{2n} \times \operatorname{trace}(\mathrm{e}^{-(1/4)D^2})^s \times t_J,$$

where t_J is the convolution product of $2n - |J|$ functions $1/\sqrt{\pi s}$ and $|J|$ functions equal to one. Thus, the Laplace transform of t_J is given by $\lambda \to \lambda^{-|J|}\,\lambda^{-(1/2)(2n-|J|)} =$

$\lambda^{-n-(1/2)|J|}$, so that

$$t_J(s) = \frac{1}{\Gamma(n + \frac{1}{2}|J|)} s^{n+(1/2)|J|-1}.$$

Next by definition of τ_{2n} one has

$$\tau_{2n}(a^0, \ldots, a^{2n}) = \operatorname{trace}(DNa^0S_1 + \varepsilon Na^0S_0)(1).$$

Thus we have to estimate the following terms:

(1) $\operatorname{trace}(\varepsilon\, Na^0\lambda^m T_J)(1)$ for $|J| = 2m$,

(2) $\operatorname{trace}(\varepsilon DNa^0\lambda^m T_J)(1)$ for $|J| = 2m + 1$.

Now Na^0T_J has majorizing function given by

$$\|a^0\| \prod_1^{2n} \|[D, a^j]\| (20)^{2n} \operatorname{trace}(\mathrm{e}^{-(1/4)D^2})^s(n * t_J)$$

relative to $U = \{z \in \mathbb{C}, |z - 1| \leqslant \frac{1}{4}\}$, so that by the Cauchy formula the m th derivative at the point $1 \in \mathbb{C}$ of $f(z) = \operatorname{trace}(\varepsilon Na^0T_J)(z)$ is smaller than

$$C4^m m!\,(n * t_J)(1),$$

$$C = \|a^0\| \prod_1^{2n} \|[D, a^j]\| (20)^{2n} \operatorname{trace}(\mathrm{e}^{-(1/4)D^2}).$$

One has

$$(n * t_J)(1) = \sqrt{\pi}\, \frac{1}{\Gamma(n + \frac{1}{2}|J| + \frac{1}{2})} = \sqrt{\pi}\, \frac{1}{\Gamma(n + m + \frac{1}{2})}.$$

As

$$m!\, \frac{\sqrt{\pi}}{\Gamma(n + m + \frac{1}{2})} \leqslant \frac{\sqrt{\pi}}{(n-1)!}$$

we get that each term of the form (1) is majorized by $4^nC\sqrt{\pi}/(n-1)!$

Next $DN \in \mathscr{L}$ is by the proof of Lemma 4.2(b) of the form $D(D^2 + 1)^{-1/2}\delta_0 + 2/\pi(1 - \lambda)T$, where T has majorizing function $t(s) = (4 - \log s)\delta^s$. Thus,

$$DNa^0T_J = D(D^2 + 1)^{-1/2}a^0T_J + \frac{2}{\pi}(1 - \lambda)Ta^0T_J,$$

where the first term has majorizing function Ct_J, and Ta^0T_J has majorizing function $Ct * t_J$. Now one has

$$\begin{aligned}&\operatorname{trace}(\varepsilon DNa^0\lambda^m T_J)(1)\\ &\quad= \left(\left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \operatorname{trace}(\varepsilon D(D^2 + 1)^{-1/2}a^0T_J)(s) + \right.\\ &\qquad\left. + \frac{2}{\pi}\left(1 + \frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \operatorname{trace}(\varepsilon Ta^0T_J)(s)\right)_{s=1},\end{aligned}$$

where by the above reasoning, each term is majorized by

$$4^m m!Ct_J(1) + \frac{2}{\pi}4^m m!C(t*t_J)(1) + \frac{2}{\pi}4^{m+1}(m+1)!C(t*t_J)(1)$$

$$\leqslant C4^{(n+1)}(m+1)!\left(1+2\int_0^1 (4-\log s)\,\mathrm{d}s\right)t_J(1)$$

$$\leqslant C4^{n+1}(m+1)!\,12t_J(1) \leqslant 12 \times 4^{n+1} \times C \times$$

$$\times (m+1)!\,\frac{1}{\Gamma(n+\frac{1}{2}(2m+1))} \qquad \square$$

6. Formulae for the Character Ch($\mathscr{H}$, D, ε).

The technical part of the preceding three sections does obscure the algebraic aspects of the formula defining the character, ch($\mathscr{H}$, D, ε), of a θ-summable Fredholm module. Using the inverse Laplace transform, we shall now prove a formula showing that at a formal level (i.e. permuting the trace with an integral which is in general not possible) one can think of ch($\mathscr{H}$, D, ε) as the integral with respect to the Gaussian measure $\mathrm{e}^{-m^2}\,\mathrm{d}m/\sqrt{\pi}$, of the cocycles constructed from the action of A in $\mathscr{H}$ and the operators of square 1 given by $F(m) = (D + m\varepsilon)(D^2 + m^2)^{-1/2}$ taken for imaginary m.

THEOREM 1. *Let A be a unital Banach algebra and $(\mathscr{H}, D, \varepsilon)$ an even θ-summable Fredholm module over A.*

(1) *For any $a^0, \ldots, a^{2n} \in A$, the even part of the following operator is of trace class and independent of $\alpha > 0$*

$$T(a^0, \ldots, a^{2n})$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} F(im+\alpha)a^0[F(im+\alpha), a^1]\cdots[F(im+\alpha), a^{2n}]\,\mathrm{e}^{(im+\alpha)^2}\,\mathrm{d}m$$

(2) *The functionals*

$$\phi_{2n}, \phi_{2n}(a^0, \ldots, a^{2n}) = c_n \operatorname{Trace}(T(a^0, \ldots, a^{2n})) \quad c_n = (n-\tfrac{1}{2})\cdots\tfrac{3}{2}\tfrac{1}{2}.$$

define an entire even cyclic cocycle on A, equal to Ch($\mathscr{H}$, D, ε).

Proof. The operator $F(z) = (D + z\varepsilon)(D^2 + z^2)^{-1/2}$ is well defined for $\operatorname{Re} z > 0$, and its norm is majorized by

$$2\frac{|z|}{|\operatorname{Im}(z^2)|^{1/2}}$$

since for any real x one has

$$\frac{x^2}{|x^2+z^2|} \leqslant \frac{|z|^2}{|\mathrm{Im}\, z^2|} \quad \text{and} \quad \frac{|z^2|}{|x^2+z^2|} \leqslant \frac{|z|^2}{|\mathrm{Im}\, z^2|}.$$

Thus, the norm of $F(\mathrm{im} + \alpha)$ is for large m of the order of $(|m|/\alpha)^{1/2}$ which shows that the integral defining T is convergent in norm. Since $F(z)$ is an analytic operator valued function of z, $\mathrm{Re}\, z > 0$, it follows that the above integral is independent of $\alpha > 0$. The Laplace transform L is a homomorphism of the algebra $\mathscr{L}$ of Section 3 in the algebra of analytic operator-valued functions of a complex parameter λ, $\mathrm{Re}\, \lambda > 0$, we thus get two homomorphisms ρ_+ and ρ_- of $\tilde{\mathscr{L}}$ given by

$$\rho_\pm((X_0, X_1)) = L(X_0) \pm \lambda^{1/2} L(X_1),$$

where $\lambda^{1/2}$ is the branch of the square root equal to 1 for $\lambda = 1$. The image $\rho_\pm(F)$ of the element F of Proposition 4.3 is given by

$$\rho_\pm(F)(\lambda) = (D \pm \lambda^{1/2}\varepsilon)(D^2 + \lambda)^{-1/2}.$$

Thus

$$\rho_+(F)(\lambda) = F(\lambda^{1/2}), \rho_-(F)(\lambda) = -\varepsilon F(\lambda^{1/2})\varepsilon.$$

The image $\rho_\pm(\pi(a))$ of the element $(a\delta_0, 0)$, $a \in A$ is given by $\rho_\pm((a\delta_0, 0))(\lambda) = a$. The inverse Laplace transform L^{-1} applies to any element of $L(\mathscr{L})$ and gives for any $X = (X_0, X_1) \in \tilde{\mathscr{L}}$

$$X_1(1) = \frac{1}{2i\pi} \lim_{\xi \to \infty} \int_{-i\xi}^{i\xi} \tfrac{1}{2}(\rho_+(X) - \rho_-(X))(\lambda)\, \mathrm{e}^{\lambda} \lambda^{-1/2}\, \mathrm{d}\lambda.$$

Now, given $a^0, \ldots, a^{2n} \in A$, let

$X = Fa^0[F, a^1] \cdots [F, a^{2n}] \in \tilde{\mathscr{L}}$. One has $\rho_+(X)(\lambda) = S(\lambda^{1/2})$ where, for $\mathrm{Re}\, z > 0$,

$$S(z) = F(z)a^0[F(z), a^1] \cdots [F(z), a^{2n}].$$

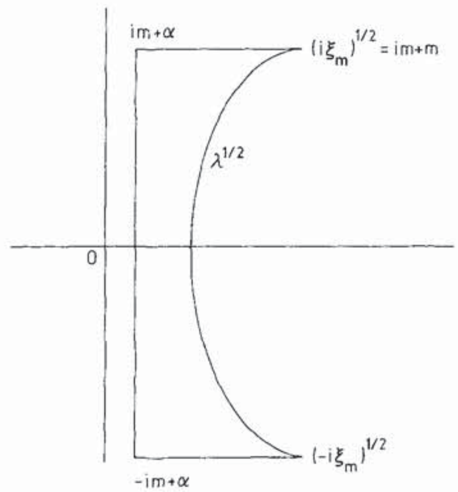
Also $\rho_-(X)(\lambda) = -\varepsilon S(\lambda^{1/2})\varepsilon$ so that $\frac{1}{2}(\rho_+(X) - \rho_-(x))$ is the even part $S_{\mathrm{ev}}(\lambda^{1/2})$ of the operator $S(\lambda^{1/2})$.

Thus, in order to prove Theorem 1, it is enough to show that

$$\lim_{\xi \to \infty} \int_{-i\xi}^{i\xi} S_{\mathrm{ev}}(\lambda^{1/2})\, \mathrm{e}^{\lambda} \lambda^{-1/2}\, \mathrm{d}\lambda = 2i \int_{-\infty}^{\infty} S_{\mathrm{ev}}(im + \alpha)\, \mathrm{e}^{(im+\alpha)^2}\, \mathrm{d}m.$$

Since $S_{\mathrm{ev}}(z)$ is an analytic function of z, $\mathrm{Re}\, z > 0$, it is enough to show that when $m \to \infty$, with $\xi_m = 2m^2$ one has

$$\left\| \int_{im+\alpha}^{(i\xi_m)^{1/2}} S_{\mathrm{ev}}(z)\, \mathrm{e}^{z^2}\, \mathrm{d}z \right\| \to 0.$$



One has

$$\left\| \int_{im+\alpha}^{(i\xi_m)^{1/2}} S_{\mathrm{ev}}(z)\, \mathrm{e}^{z^2}\, \mathrm{d}z \right\|$$
$$\leqslant \int_\alpha^m \|S_{\mathrm{ev}}(im+t)\|\, \mathrm{e}^{-m^2+t^2}\, \mathrm{d}t$$
$$\leqslant c \int_\alpha^m \left(\frac{|im+t|^2}{|\mathrm{Im}((im+t)^2)|}\right)^{n+(1/2)} \mathrm{e}^{-m^2+t^2}\, \mathrm{d}t$$
$$= c \int_\alpha^m \left(\frac{m^2+t^2}{2mt}\right)^{n+(1/2)} \mathrm{e}^{-m^2+t^2}\, \mathrm{d}t.$$

Since

$$\frac{m^2+t^2}{2mt} \leqslant \frac{m}{\alpha},$$

it is clear that the integral from α to $m/2$ tends to 0 when m tends to ∞. The other part

$$\int_{m/2}^m \left(\frac{m^2+t^2}{2mt}\right)^{n+(1/2)} \mathrm{e}^{-m^2+t^2}\, \mathrm{d}t = \int_{1/2}^1 f_m(u)\, \mathrm{d}u$$

where

$$f_m(u) = \left(\frac{1+u^2}{2u}\right)^{n+1/2} \mathrm{e}^{-m^2(1-u^2)} m,$$

converges to 0 by the Lebesgue-dominated convergence theorem, since

$$h(u) = \operatorname{Sup}_m (m\, \mathrm{e}^{-m^2(1-u^2)})$$

is integrable on $[\frac{1}{2}, 1]$ (it is of the order of $1/\sqrt{1-u^2}$). □

We shall now state the result analogous to Theorem 1 in the odd case. In order to obtain it from the even case applied to the $\mathbb{Z}/2$ graded algebra context, one uses the same method as in [2] part I.

We let

$$U(m) = (D + im)(D^2 + m^2)^{-1/2} \quad \text{for} \quad \operatorname{Re} m > 0.$$

For m real, it is a unitary operator with

$$U(m)^{-1} = U(m)^* = (D - im)(D^2 + m^2)^{-1/2}.$$

For $Re\, m < 0$ we define $U(m)$ by

$$U(m) = U(-m)^{-1} = (D + im)(D^2 + m^2)^{-1/2}$$

where the square root is given by

$$z^{1/2} = |z|^{1/2} \exp\left(i\frac{\theta}{2}\right) \quad \text{with} \quad z = |z| \exp i\theta, \theta \in]-\pi, \pi[.$$

As defined, even when $m_0 \notin \operatorname{Sp} D$, the operator-valued function $U(m)$ has a discontinuity near im_0, as the square root does. But the discontinuity

$$\delta U(im_0) = \lim_{\varepsilon \to 0+} U(im_0 + \varepsilon) - U(im_0 - \varepsilon)$$

only invokes the eigenvalues of $|D|$ smaller than $|m_0|$, and is equal to the finite rank operator $2E_{|m_0|}(D)U(im_0 +)$, where E_a is the characteristic function of the interval $[-a, a]$.

THEOREM 2. *Let A be a unital Banach algebra and $(\mathscr{H}, D)$ an odd θ-summable Fredholm module over A.*

(1) *For any $a^0, \ldots, a^{2n+1} \in A$, the following operator is of trace class and independent of $\alpha > 0$:*

$$T(a^0, \ldots, a^{2n+1}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \tfrac{1}{2}(B(im + \alpha) - B(im - \alpha))\, e^{(im+\alpha)^2}\, dm$$

with

$$B(z) = a^0(a^1 - U(z)a^1U^{-1}(z)) \ldots (a^{2n+1} - U(z)a^{2n+1}U^{-1}(z)).$$

(2) *The functionals*

$$\tau_{2n+1}, \tau_{2n+1}(a^0, \ldots, a^{2n+1}) = \operatorname{Trace}(T(a^0, \ldots, a^{2n+1}))$$

define an odd entire cocycle on A.

7. The Index Formula

In this section we shall prove the following index formula:

THEOREM 1. *Let A be a unital Banach algebra and $(\mathscr{H}, D, \varepsilon)$ an even θ-summable*

Fredholm module over A. Let $(\tau_{2n})_{n\in\mathbb{N}}$ *be the character* $\tau = \mathrm{ch}(\mathscr{H}, D, \varepsilon)$. *Then for any element e of* $K_0(A)$ *one has: Index* $D_e^+ = \langle \tau, e\rangle$.

In order to prove this theorem, we can take for A the Banach algebra $A = \{a \in \mathscr{L}(\mathscr{H});\ \varepsilon a = a\varepsilon,\ \|[D,a]\| < \infty\}$ with norm given by $|||a||| = \|a\| + \|[D,a]\|$. We can, moreover, assume that e is a self-adjoint idempotent of A. We wish to use the following homotopy among operators D to go back to the easy case where $[D,e] = 0$: $D_t = D - t\delta, \delta = eD(1-e) + (1-e)De$.

One has $D_0 = D, D_1 = D - eD(1-e) - (1-e)De$ so that D_1 commutes with e. Moreover, $\delta = -e[D,e] + [D,e]e$ is bounded, so that for any t, $[D_t, e]$ is bounded. However, we need to control the trace of $\exp - (D_t)^2$, where $D_t^2 = D^2 - t(D\delta + \delta D) + t^2\delta^2$ and for this we shall prove:

LEMMA 2. (*a*) *Without changing the class of e in* $K_0(A)$ *one can assume that* $[|D|^{1/2}, e]$ *and* $[|D|^{1/2}, [D,e]]$ *are bounded.*

(*b*) *With* e *as in* (*a*) *one has*

$$\sup_{t\in[0,1]} \mathrm{Tr}(\exp - (D_t)^2) < \infty.$$

Proof. (a) Let us first show that $[|D|^{1/2}, a]$ is bounded for any $a \in A$. One has

$$|D|^{1/2} = (\sqrt{2}\pi)^{-1}\int_0^\infty \frac{D^2}{(D^2+\lambda)\lambda}\lambda^{1/4}\,\mathrm{d}\lambda$$

$$= (\sqrt{2}\pi)^{-1}\int_0^\infty \left(\frac{1}{\lambda} - \frac{1}{D^2+\lambda}\right)\lambda^{1/4}\,\mathrm{d}\lambda$$

so that

$$[|D|^{1/2}, a] = (\sqrt{2}\pi)^{-1}\int_0^\infty \frac{1}{D^2+\lambda}(D[D,a] + [D,a]D)\frac{1}{D^2+\lambda}\lambda^{1/4}\,\mathrm{d}\lambda.$$

Moreover $\|D(D^2+\lambda)^{-1}\| \leqslant (2\lambda^{1/2})^{-1}$ so that for λ large, the norm of the term under the integral is of the order of $\|[D,a]\|2\lambda^{-1/2}\lambda^{-1}\lambda^{1/4} \sim \lambda^{-5/4}$. There is no problem near $\lambda = 0$, since one may always replace D by an invertible operator D' such that $D - D'$ and $|D|^{1/2} - |D'|^{1/2}$ are both bounded.

It follows that for any $a \in A$ the map

$$t \in \mathbb{R} \to \alpha_t(a) = \exp(it|D|^{1/2})\,a\,\exp(-it|D|^{1/2}) \in \mathscr{L}(\mathscr{H}),$$

is norm continuous. Thus, if we let $B = \{a \in A; t \to \alpha_t(a) \in A \text{ of class } C^\infty\}$, we see that the closure of B in $\mathscr{L}(\mathscr{H})$ is the same as the closure of A.

We have two subalgebras $B \subset A$ of the norm closure of A in $\mathscr{L}(\mathscr{H})$, both norm dense and stable under holomorphic functional calculus so the inclusion $B \subset A$ induces an isomorphism in K-theory.

(b) Let e and δ be as above, one has $D_t^2 = D^2 - t(D\delta + \delta D) + t^2\delta^2$, and since δ is bounded, it is enough to show that $\mathrm{Trace}(\exp - (D^2 - t(D\delta + \delta D)))$ is bounded. One has

$$D\delta = |D|F\delta = |D|^{1/2}[|D|^{1/2}, F\delta] + |D|^{1/2} F\delta |D|^{1/2}$$

where $D = F|D| = |D|F$ is the polar decomposition of D. By hypothesis on e, we know that $[|D|^{1/2}, \delta]$ is bounded, and F commutes with $|D|^{1/2}$ so that $[|D|^{1/2}, F\delta]$ is bounded. Thus $D\delta = |D|^{1/2}T + |D|^{1/2}T'|D|^{1/2}$, where T and T' are bounded operators. Hence

$$D\delta + \delta D = |D|^{1/2}T_0 + T_1|D|^{1/2} + |D|^{1/2}T_2|D|^{1/2},$$

where T_0, T_1, T_2 are bounded operators. Then the equality

$$\begin{aligned}&\mathrm{Trace}\,(\exp(-D^2 + t(D\delta + \delta D))\\ &= \sum_0^\infty t^n \int_{0\leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant 1} \mathrm{d}s_1 \ldots \mathrm{d}s_n \,\mathrm{Trace}\, \exp(-s_1 D^2)(D\delta + \delta D) \times\\ &\quad \times \exp(-(s_2 - s_1)D^2) \ldots (D\delta + \delta D) \exp - (1 - s_n)D^2\end{aligned}$$

together with Lemma 4.1 and the Holder inequality give a bound of the form:

$$\sum_0^\infty t^n C^n \int_{0\leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant 1} s_1^{-1/2}(s_2 - s_1)^{-1/2} \ldots (1 - s_n)^{-1/2}\, \mathrm{d}s_1 \ldots \mathrm{d}s_n$$

which is enough to ensure the convergence □

Lemma 2 shows that in order to prove Theorem 1, one can assume that e satisfies 2(a) and, hence, gives rise to a family $(D_t)_{t\in[0,1]}$ of θ-summable Fredholm modules over A, such that

(1) $\mathrm{Sup}_t \left\| \frac{\mathrm{d}}{\mathrm{d}t} D_t \right\| < \infty$,

(2) $\mathrm{Sup}_t \,\mathrm{Trace}\,(\exp - \frac{1}{4}D_t^2) < \infty$.

PROPOSITION 3. *Let D_t be a family of unbounded self-adjoint operators in $\mathscr{H}$ such that $(\mathscr{H}, D_t, \varepsilon)$ is an unbounded Fredholm module over A for any $t \in [0,1]$, and satisfying (1), (2). Then the characters of D_0 and D_1 differ by a coboundary.*

Proof. Let F_t be the element of $\tilde{\mathscr{L}}$ given by Proposition 4.3 applied to D_t. Let $N_t \in \mathscr{L}$ correspond similarly to D_t. For each n let

$$\begin{aligned}\psi^t_{2n-1}(a^0, \ldots, a^{2n-1}) &= \sum_0^{2n-1} (-1)^i \tau\Big(F_t a^0 [F_t, a^1] \ldots [F_t, a^i] \times\\ &\quad \times \left(\frac{\mathrm{d}}{\mathrm{d}t} F_t\right)[F_t, a^{i+1}] \ldots [F_t, a^{2n-1}]\Big),\end{aligned}$$

where for $i = 0$ one takes

$$\tau\left(a^0 \left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)[F_t, a^1] \ldots [F_t, a^{2n-1}]\right).$$

Since $(\mathrm{d}/\mathrm{d}t)D_t$ is bounded, the proof of Lemma 5.2 applies verbatim, replacing commutators with a by $\mathrm{d}/\mathrm{d}t$ to show that $(\mathrm{d}/\mathrm{d}t)N_t$ (resp $(\mathrm{d}/\mathrm{d}t)(D_tN_t)$) has majorizing function

$$12\left\|\frac{\mathrm{d}}{\mathrm{d}t}D_t\right\|C^s \qquad \left(\mathrm{resp}\,\frac{20}{\sqrt{\pi s}}\left\|\frac{\mathrm{d}}{\mathrm{d}t}D_t\right\|C^s\right)$$

where

$$C = \operatorname*{Sup}_t \operatorname{Trace}(\exp - \tfrac{1}{4}D_t^2).$$

It follows then as in Lemma 5.3, that

$$\|\psi^t_{2n-1}\| \leqslant \frac{(10)^{4n}}{n!} \times 2n \times \left\|\frac{\mathrm{d}}{\mathrm{d}t}D_t\right\| \times C. \tag{*}$$

Using the equality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)F_t + F_t\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right) = 0$$

a direct calculation in $\tilde{\mathscr{L}}$ yields

$$\begin{aligned}
&(b\psi^t_{2n-1})(a^0, \dots, a^{2n})\\
&= -\sum_{i=0}^{2n-1}\tau\left(F_t a^0[F_t, a^1]\dots[F_t, a^i]\left[\frac{\mathrm{d}}{\mathrm{d}t}F_t, a^{i+1}\right][F_t, a^{i+2}]\dots\right) + \\
&\quad + \sum_{i=0}^{2n-1}(-1)^i\tau\left(a^0[F_t, a^1]\dots[F_t, a^i]\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)[F_t, a^{i+1}]\dots\right).
\end{aligned}$$

Thus, with τ^t_{2n} the components of the character of D_t, we get

$$\begin{aligned}
&\left(b\psi^t_{2n-1} + \frac{\mathrm{d}}{\mathrm{d}t}\tau^t_{2n-1}\right)(a^0, \dots, a^{2n})\\
&= \tau\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)a_0[F_t, a^1]\dots[F_t, a^{2n}]\right) + \\
&\quad + \sum_{i=0}^{2n-1}(-1)^i\tau\left(a^0[F_t, a^1]\dots[F_t, a^i]\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)[F_t, a^{i+1}]\dots\right) =
\end{aligned}$$

One has

$$\begin{aligned}
&(-1)^i\tau\left(a^0[F_t, a^1]\dots[F_t, a^i]\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)[F_t, a^{i+1}]\dots[F_t, a^{2n}]\right)\\
&= -\tfrac{1}{2}\tau\left([F_t, a^0][F_t, a^1]\dots[F_t, a^i]F_t\frac{\mathrm{d}}{\mathrm{d}t}F_t[F_t, a^{i+1}]\dots[F_t, a^{2n}]\right)
\end{aligned}$$

and

$$\tau\left(\left(\frac{d}{dt}F_t\right)a^0[F_t,a^1]\dots[F_t,a^{2n}]\right)$$
$$= -\tfrac{1}{2}\tau\left(\left(F_t\frac{d}{dt}F_t\right)[F_t,a^0]\dots[F_t,a^{2n}]\right).$$

Thus

$$b\psi^t_{2n-1} + \frac{d}{dt}\tau^t_{2n} = -\frac{1}{2}A\theta^t_{2n},$$

where

$$\theta^t_{2n}(a^0,\dots,a^{2n}) = \tau\left(\left(F_t\frac{d}{dt}F_t\right)[F_t,a^0]\dots[F_t,a^{2n}]\right)$$

Similarly, one has

$$B_0\psi^t_{2n+1}(a^0,\dots,a^{2n})$$
$$= \sum_{i=0}^{2n+1}\tau\left([F_t,a^0]\dots[F_t,a^{i-1}]\left(F_t\frac{d}{dt}F_t\right)[F_t,a^i]\dots[F_t,a^{2n}]\right),$$

so that:

$$AB_0\psi^t_{2n+1} = (2n+2)A\theta^t_{2n}$$

Hence

$$-\frac{d}{dt}\tau^t_{2n} = b\psi^t_{2n-1} + \frac{1}{2}\frac{1}{2n+2}B\psi^t_{2n+1}$$

Equivalently with

$$\varphi^t_{2n} = 2^{-n}\times 1\times 3\times\cdots\times(2n-1)\tau^t_{2n},$$
$$\gamma^t_{2n-1} = (2n)^{-1}\times 2^{-n}\times 1\times 3\times\cdots\times(2n-1)\psi^t_{2n-1},$$

we get

$$-\frac{d}{dt}\varphi^t = (d_1+d_2)\gamma^t.$$

Since by $(*)$ $(\gamma_{2n-1})_{n\in\mathbb{N}}$ is an entire cochain the conclusion follows. □

Proof of Theorem 1. We can assume that e is a self-adjoint idempotent of A satisfying the conditions of Lemma 2(a). Then Proposition 3 and Theorem 1.8 show that we can replace D by $D_1 = D - eD(1-e) - (1-e)De$ without changing the value of $\langle ch(\mathscr{H}, D, \varepsilon),\ e\rangle$. Thus, the latter is given by Trace$(\varepsilon e\exp - D_1^2)$ which by the MacKean-Singer identity is equal to the index of the operator $e De^+$. □

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