K-Theory For Discrete Groups

Paul Baum,
Department of Mathematics, Brown University,
Providence, R.I. 1917, U.S.A.
Alain Connes,
I.H.E.S., 91440 Bures-sur-Yvette, France.

Let X be a C^∞ manifold.

Suppose given

(1) A C^∞ foliation F of X

or (2) A C^∞ (right) action of a Lie group G on X,
$X \times G \to X$

or (3) A C^∞ (right) action of a discrete group
Γ on X, $X \times \Gamma \to X$.

In (2) G is a Lie group with $\pi_0(G)$ finite.
In (3) "discrete" simply means that Γ is a group topologized by the discrete topology in which each point is an open set.

Each of these three cases gives rise to a C^* algebra A

(1) $A = C^*(X, F)$, the foliation C^* algebra [10] [12]

(2) $A = C_0(X) \rtimes G$, the reduced crossed-product C^*-algebra resulting from the action of G on $C_0(X)$. As usual $C_0(X)$ denotes the abelian C^* algebra of all continuous complex-valued functions on X which vanish at infinity.

(3) $A = C_0(X) \rtimes \Gamma$, the reduced crossed-product C^*-algebra resulting from the action of Γ on $C_0(X)$.

In [4] [6] we defined for each case (and also for crossed-products twisted

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by a 2-cocycle) a geometric K theory and a map μ from the geometric K theory to the K theory of the relevant C^*-algebra:

(1) $\mu : K^i(X,F) \to K_i[C^*(X,F)]$
(2) $\mu : K^i(X,F) \to K_i[C_0(X) \rtimes G)]$
(3) $\mu : K^i(X,\Gamma) \to K_i[C_0(X) \rtimes \Gamma]$ $i = 0, 1$.

The map μ in essence assigns to a symbol the index of its underlying elliptic operator. In [4] [6] we conjectured that μ is always an isomorphism. At the present time this conjecture is still alive.

Over the long run it will, of course, be fascinating to see whether such an extremely general conjecture can endure.

In this note we shall briefly review the state of the art for the special case of a discrete group Γ operating on a point. The C^*-algebra is then $C^*_r\Gamma$, the reduced C^*-algebra of Γ .

On the positive side there is the beautiful Mayer-Vietoris exact sequence of M. Pimsner [27]. This exact sequence includes earlier results of Pimsner–Voiculescu [28], T. Natsume [26], C. Lance [23], J. Cuntz [14], and J. Anderson-W. Paschke [2]. Using his bivariant KK theory, G.G. Kasparov [19] [20] [21], has verified the conjecture for discrete subgroups of $SO_0(n,1)$. Also on the positive side is the transverse fundamental class result of [13]. We shall report further on this in [7].

On the negative side there are the recent examples of G. Skandalis [39]. These examples at least indicate that life is not as simple as one might have hoped. Also on the negative side is the utter lack of a Kunneth theorem and of any calculated examples where the discrete group Γ is infinite and has Kazhdan's property T. In addition there is the somewhat disturbing observation of M. Gromov that mathematicians have never proved a non-trivial theorem about all discrete groups.

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§1. Statement of the Conjecture

For simplicity assume that Γ is countable. (For Γ uncountable see Appendix 2 of [6].) $\mathcal{C}(\cdot,\Gamma)$ denotes the category of all proper C^∞ Γ–manifolds. Thus an object of $\mathcal{C}(\cdot,\Gamma)$ is a C^∞ manifold W with a given proper (right) C^∞ action of Γ :

(1.1) $$W \times \Gamma \to W$$

W is an ordinary C^∞ manifold. W is Hausdorff, finite dimensional, second countable, and without boundary. Each $\gamma \in \Gamma$ acts on W by a diffeomorphism. Recall that the action (1.1) is proper if and only if the map

(1.2) $$W \times \Gamma \to W \times W$$

which takes (w,γ) to $(w,w\gamma)$ is proper in the usual sense, i.e. the inverse image of any compact set in $W \times W$ is a compact set in $W \times \Gamma$. This implies that each isotropy group of the action (1.1) is finite, and that the quotient space W/Γ is a Hausdorff second countable orbifold.

A morphism in $\mathcal{C}(\cdot,\Gamma)$ is a C^∞ Γ-equivariant map $f : W_1 \to W_2$. f is not required to be proper. Associated to f is a homomorphism of abelian groups

(1.3) $$f_! : K_i[C_0(TW_1) \rtimes \Gamma] \to K_i[C_0(TW_2) \rtimes \Gamma]$$
$$i = 0, 1$$

In (1.3) TW_j is the tangent bundle of W_j. $C_0(TW_j) \rtimes \Gamma$ is the crossed-product C^*-algebra resulting from the evident action of Γ on TW_j. To define $f_!$ first note that TW_j is an even-dimensional almost-complex manifold. The derivative map

(1.4) $$f' : TW_1 \to TW_2$$

is, therefore, a K-oriented map and so yields

$$(f') \in KK^0(C_0(TW_1)\,,\,C_0(TW_2))\,.$$

Consider the diagram

$$\begin{array}{l} KK^0_\Gamma(C_0(TW_1),C_0(TW_2)) \to KK^0(C_0(TW_1),C_0TW_2)) \\ \downarrow \\ KK^0(C_0(TW_1) \rtimes \Gamma, C_0(TW_2) \rtimes \Gamma) \\ \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(K_*[C_0(TW_1) \rtimes \Gamma], K_*[C_0(TW_2) \rtimes \Gamma]) \end{array}$$

in which the horizontal arrow is the forgetful map and the two vertical arrows are as in Kasparov [19] [20]. Since all structures involved in defining

$(f') \in KK^0(C_0(TW_1), C_0(TW_2))$ are Γ-equivariant (f') lifts to give $[f'] \in KK^0_\Gamma(C_0(TW_1), C_0(TW_2))$. The horizontal arrow in (1.5) sends $[f']$ to (f'). The map $f_!$ of (1.3) is then obtained by applying the two vertical arrows to $[f']$.

(1.6) ***Lemma:*** $f_! : K_i[C_0(TW_1) \rtimes \Gamma] \to K_i[C_0(T(W_2) \rtimes \Gamma]$ depends only on the homotopy class (as a C^∞ Γ-equivariant map) of f.

(1.7) ***Lemma:*** Let $f : W_1 \to W_2$ and $g : W_2 \to W_3$ be morphisms in $\mathcal{C}(\cdot,\Gamma)$. Then $(gf)_! = g_! f_!$.

(1.8) ***Definiton:*** $K^i(\cdot,\Gamma) = \underset{\mathcal{C}(\cdot,\Gamma)}{\xrightarrow{\text{limit}}} K_i[C_0(TW) \rtimes \Gamma] \quad i = 0,1$

Remarks: In (1.8) the limit is taken using the $f_!$ maps of (1.3). Let $F_i(\cdot,\Gamma)$ be the free abelian groups generated by all[1] pairs (W,ξ) where W is an object of $\mathcal{C}(\cdot,\Gamma)$ and $\xi \in K_i[C_0(TW) \rtimes \Gamma]$. $R_i(\cdot,\Gamma)$ denotes the subgroups of $F_i(\cdot,\Gamma)$ generated by all elements of the form

(i) $(W,\xi+\eta) - (W,\xi) - (W,\eta)$

(ii) $(W_1,\xi) - (W_2,f_!\xi)$

Then definition (1.8) is:

(1.9) $K^i(\cdot,\Gamma) = F_i(\cdot,\Gamma) / R_i(\cdot,\Gamma) \quad i = 0, 1$

For each object W of Γ we have the homomorphism of abelian groups

(1.10) $\varphi_W : K_i[C_0(TW) \rtimes \Gamma] \to K^i(\cdot,\Gamma)$

defined by

(1.11) $\varphi_W(\xi) = (W,\xi) \qquad \xi \in K_i[C_0(TW) \rtimes \Gamma]$

For any morphism $f : W_1 \to W_2$ in $\mathcal{C}(\cdot,\Gamma)$, the diagram

[1] To avoid set-theoretic difficulties assume that W is a C^∞ sub-manifold (which is a closed subset) of some Euclidean space $\mathbb{R}^m$. This is possible by the Whitney embedding theorem.

(1.12)
$$\begin{array}{ccc} K_i[C_0(TW_1) \rtimes \Gamma] & \xrightarrow{f_!} & K_i[C_0(TW_2) \rtimes \Gamma] \\ & {\scriptstyle \varphi_{W_1}} \searrow \quad \swarrow {\scriptstyle \varphi_{W_2}} & \\ & K^i(\cdot,\Gamma) & \end{array}$$

commutes.

$K^i(\cdot,\Gamma)$ has the following universal property. Let H be an abelian group. Suppose given for each object W of $\mathfrak{C}(\cdot,\Gamma)$ a homomorphism of abelian groups

(1.13) $$\psi_W : K_i[C_0(TW) \rtimes \Gamma] \to H$$

Assume that for each morphism $f : W_1 \to W_2$ in $\mathfrak{C}(\cdot,\Gamma)$ the diagram

(1.14)
$$\begin{array}{ccc} K_i[C_0(TW_1) \rtimes \Gamma] & \xrightarrow{f_!} & K_i[C_0(TW_2) \rtimes \Gamma] \\ & {\scriptstyle \psi_{W_1}} \searrow \quad \swarrow {\scriptstyle \psi_{W_2}} & \\ & H & \end{array}$$

commutes.

Then there is a unique homomorphism of abelian groups

(1.15) $$\psi : K^i(\cdot,\Gamma) \to H$$

with $\psi_W = \psi\ \varphi_W$ for each object W of $\mathfrak{C}(\cdot,\Gamma)$.

(1.15) applies to determine a homomorphism

(1.16) $$\mu : K^i(\cdot,\Gamma) \to K_i\,[C^*_r\Gamma]\,.$$

For each object W of $\mathfrak{C}(\cdot,\Gamma)$ the Dirac operator of TW is an element of $KK^0_\Gamma(C_0(TW),\mathbb{C})$. Applying Kasparov's map

(1.17) $$KK^0_\Gamma(C_0(TW),\mathbb{C}) \to KK^0(C_0(TW) \rtimes \Gamma,\ C^0_r\Gamma)$$

we then obtain a homomorphism of abelian groups

(1.18) $$\mu_W : K^i[C_0(TW) \rtimes \Gamma] \to K_i[C^*_r\Gamma]$$

(1.19) ***Lemma:*** For each morphism $f : W_1 \to W_2$ in $\mathfrak{C}(\cdot,\Gamma)$ there is commutativity in the diagram

$$\begin{array}{ccc} K_i[C_0(TW_1) \rtimes \Gamma] & \xrightarrow{f_!} & K_i[C_0(TW_2) \rtimes \Gamma] \\ & {\scriptstyle \mu_{W_1}} \searrow \quad \swarrow {\scriptstyle \mu_{W_2}} & \\ & K_i[C^*_r\Gamma] & \end{array}$$

Proof: $[D_j] \in KK_\Gamma\,(C_0(TW_j),\mathbb{C})$ denotes the element of $KK_\Gamma(C_0(TW_j),\mathbb{C})$ given by the Dirac operator of TW_j. As above we have $[f] \in KK^0_\Gamma(C_0(TW_1),C_0(TW_2))$. The Kasparov product pairing

$$KK^0_\Gamma(C_0(TW_1),C_0(TW_2)) \otimes_{\mathbb{Z}} KK^0_\Gamma(C_0(TW_2),\mathbb{C}) \to KK_\Gamma(C_0(TW_1),\mathbb{C})$$

has $$[f] \bigotimes_{C_0(TW_2)} [D_2] = [D_1]$$ □

Due to (1.19), (1.15) applies to determine a homomorphism

$$\mu : K^i(\cdot,\Gamma) \to K_i\,[C^*_r\Gamma]$$

(1.20) ***Conjecture*** $\mathbb{Z}$. For any (discrete) group

Γ $\mu : K^i(\cdot,\Gamma) \to K_i\ [C^*_r\Gamma]$ is an isomorphism of abelian groups $(i = 0,1)$.

Remarks: From a slightly heuristic point of view, conjecture (1.20) can be viewed as asserting that any element of $K_0[C^*_r\Gamma]$ is the index of a Γ–equivariant elliptic operator on a proper C^∞ Γ-manifold. The only relations imposed on these indices to obtain $K_0[C^*_r\Gamma]$, according to the conjecture, are the "obvious" index-preserving relations on the symbols.

Conjecture (1.20) appears to be quite a strong statement. It's truth implies validity of:

(i) Novikov conjecture on homotopy invariance of higher signatures [9].

(ii) Gromov - Lawson - Rosenberg conjecture on topological obstructions to the existence of Riemannian metrics of positive scalar curvature [15] [16] [33].

(iii) Kadison - Kaplansky conjecture that for Γ torsion-free there are no projections in $C^*_r\Gamma$ other then 0 and 1 .

If Γ is torsion-free, then $K^i[\cdot,\Gamma] = K_i(B\Gamma)$, the i-th K-homology (with compact supports) of the classifying space $B\Gamma$.

§2. Chern Character

Let $S(\Gamma)$ be the set of all elements in Γ of finite order. The identity element of Γ is in $S(\Gamma)$.

(2.1) $$S(\Gamma) = \{\gamma \in \Gamma \mid \gamma \text{ is of finite order } \}$$

Γ acts on $S(\Gamma)$ by conjugation. $F\Gamma$ denotes the permutation module (with coefficients $\mathbb{C}$) so obtained. Thus a typical element of $F\Gamma$ is a finite formal sum

$$\sum_{\gamma\in S(\Gamma)} \lambda_\gamma[\gamma] \qquad \lambda_\gamma \in \mathbb{C}$$

In the standard way $F\Gamma$ is a vector space over $\mathbb{C}$. The (right) action of Γ on $F\Gamma$ is

$$\left(\sum_{\gamma\in S(\Gamma)} \lambda_\gamma[\lambda]\right)\alpha = \sum_{\gamma\in S(\Gamma)} \lambda_\gamma[\alpha^{-1}\gamma\alpha] \qquad \alpha \in \Gamma \quad \lambda_\gamma \in \mathbb{C}$$

$H_j(\Gamma,F\Gamma)$ denotes the j-th homology group of Γ with coefficients $F\Gamma$. Let

Γ act on $\mathbb{C}$ trivially. Then

$$(2.2) \qquad H_j(\Gamma, F\Gamma) = \operatorname{Tor}^j_{\mathbb{C}(\Gamma)}(F\Gamma, \mathbb{C}) \qquad j = 0, 1, 2, \ldots .$$

In (2.3) $\mathbb{C}(\Gamma)$ is the group algebra of all finite formal sums

$$\sum_{\gamma \in S(\Gamma)} \lambda_\gamma [\gamma] .$$

Set
$$H_{ev}(\Gamma, F\Gamma) = \bigoplus_j H_{2j}(\Gamma, F\Gamma)$$

$$H_{odd}(\Gamma, F\Gamma) = \bigoplus_j H_{2j+1}(\Gamma, F\Gamma)$$

There is then [6] a Chern character

$$ch : K^0(\cdot, \Gamma) \to H_{ev}(\Gamma, F\Gamma)$$
$$ch : K^1(\cdot, \Gamma) \to H_{odd}(\Gamma, F\Gamma)$$

constructed by using cyclic cohomology. Its crucial property is that it becomes an isomorphism after tensoring $K^i(\cdot, \Gamma)$ with $\mathbb{C}$. Hence

$$ch : K^0(\cdot, \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \to H_{ev}(\Gamma, F\Gamma)$$

$$ch : K^1(\cdot, \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \to H_{odd}(\Gamma, F\Gamma)$$

are isomorphisms of vector spaces over $\mathbb{C}$.

Consider the diagram

$$\begin{array}{ccc} K^i(X, \Gamma) & \xrightarrow{\mu} & K_0[C^*_r \Gamma] \\ \downarrow{\scriptstyle ch} & & \downarrow \\ H^{ev}(\Gamma, F\Gamma) & \longrightarrow & K_0[C^*_r \Gamma] \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

in which the right vertical arrow is the tautological map. The lower horizontal arrow is defined by requiring commutativety in the diagram, and is denoted

$$\mu: H_{ev}(\Gamma, F\Gamma) \to K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

The same procedure determines

$$\mu: H_{odd}(\Gamma, F\Gamma) \to K_1[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

(2.4) ***Conjecture*** $\mathbb{C}$. For any group Γ

$$\mu: H_{ev}(\Gamma, F\Gamma) \to K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

and

$$\mu: H_{odd}(\Gamma, F\Gamma) \to K_1[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

are isomorphisms of vector spaces over $\mathbb{C}$.

Remarks: If (1.20) is valid for a group Γ then so is (2.4). The point of (2.4) is that it relates homological invariants of Γ to the K theory of $C_r^*\Gamma$.

Let $L = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$ be a subset of $S(\Gamma)$ such that any element in $S(\Gamma)$ is conjugate to one and only one of the γ_i . $Z(\gamma_i)$ denotes the centralizer of γ_i in Γ . Let $H_j(Z(\gamma_i), \mathbb{C})$ be the j-th homology of $Z(\gamma_i)$ with coefficients $\mathbb{C}$ and trivial action of $Z(\gamma_i)$ on $\mathbb{C}$. Then for each $j = 0, 1, 2, \dots$

(2.5) $$H_j(\Gamma, F\Gamma) = \bigotimes_i H_j(Z(\gamma_i), \mathbb{C})$$

(2.5) is useful in calculating $H_*(\Gamma, F\Gamma)$ in examples.

§3. Finite Groups and Abelian Groups

As noted by J. Rosenberg, operator algebraists tend to think that any statement involving only finite or abelian groups is trivial. This is surely debatable, but let us check (1.20) and (2.4) for such groups.

(3.1) ***Lemma:*** Conjecture $\mathbb{C}$ is valid for any finite group.

Proof: Let Γ be a finite group.

$K_1[C^*_r\Gamma] = 0$ and $H_j(\Gamma, F\Gamma) = 0$ for $j > 0$, so (2.4) is valid in the odd case. $K_0[C^*_r\Gamma] = R(\Gamma)$, the representation ring of Γ. Let $C\ell(\Gamma)$ be the $\mathbb{C}$-vector-space of all complex-valued functions on Γ which are constant on each conjugacy class.

(3.2) $$C\ell(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid f(\gamma) = f(\alpha^{-1}\gamma\alpha) \text{ for all } \gamma, \alpha \in \Gamma\}$$

Let $\chi : R(\Gamma) \to C\ell(\Gamma)$ be the map which assigns to a representation its character.

$H^0(\Gamma, F\Gamma) = C\ell(\Gamma)$ and $\chi : R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \to C\ell(\Gamma)$ is the inverse to

$\mu : C\ell(\Gamma) \to R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$. □

(3.3) ***Lemma.*** Conjecture $\mathbb{Z}$ is valid for any finite group.

Proof: Let Γ be a finite group. Any (continuous) action of a finite group is proper. Let Γ act on a point. This is a final object in $\mathcal{C}(\cdot, \Gamma)$ and it is immediate that $K^i(\cdot, \Gamma) = K_i[C^*_r\Gamma]$. □

(3.4) ***Lemma.*** Conjecture $\mathbb{C}$ is valid for any abelian group.

Proof: Let Γ be an abelian group. $\hat{\Gamma}$ denotes the Pontryagin dual of Γ. Forget the group structure on $\hat{\Gamma}$, and view $\hat{\Gamma}$ as a compact Hausdorff topological space. Then $C^*_r\Gamma = C(\hat{\Gamma})$ and $K_i[C^*_r\Gamma] = K^i(\hat{\Gamma})$ the K-theory of $\hat{\Gamma}$ as defined topologically by Atiyah and Hirzebruch [1]. Moreover $H_j(\Gamma, F\Gamma) = \check{H}^j(\hat{\Gamma}; \mathbb{C})$ where $\check{H}^j(\hat{\Gamma}; \mathbb{C})$ is the j-th Čech cohomology group of $\hat{\Gamma}$ with coefficients the complex numbers $\mathbb{C}$ The topological Chern character

$$\mathrm{ch} : K^0(\hat{\Gamma}) \to \bigoplus_{j\in\mathbb{N}} H^{2j}(\hat{\Gamma};\mathbb{C})$$

$$\mathrm{ch} : K^1(\hat{\Gamma}) \to \bigoplus_{j\in\mathbb{N}} H^{2j+1}(\hat{\Gamma};\mathbb{C})$$

provides the inverse map to

$$\mu : H_{ev}(\Gamma, F\Gamma) \to K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

$$\mu : H_{odd}(\Gamma, F\Gamma) \to K_0[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$ □

(3.5) ***Lemma.*** Conjecture $\mathbb{Z}$ is valid for any abelian group.

Proof: First one checks that conjecture $\mathbb{Z}$ is valid for any finitely generated abelian group. Next, any abelian group is the direct limit of its finitely generated subgroups. Since both $K_*[C_r^*\Gamma]$ and $K^*(\cdot,\Gamma)$ commute with direct limits this completes the proof. □

§4. Evidence For The Conjecture

(4.1) ***Theorem.*** Let G be a connected simply-connected solvable Lie group. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for any discrete subgroup Γ of G.

Proof: The theorem is a corollary of the Thom isomorphism [11] for crossed-products by $\mathbb{R}$. See also [32]. □

Notation. $SO(n,1)$ denotes the Lorentz group. $SO(n,1) \subset SL(n+1,\mathbb{R})$ is the subgroup of $SL(n+1,\mathbb{R})$ which preserves the form $x_1^2 + \dots + x_n^2 - x_{n+1}^2$.

(4.2) ***Theorem:*** (G.G. Kasparov [21]) Let G be a connected Lie group such that G is locally isomorphic to

$$H \times SO(n_1,1) \times SO(n_2,1) \times \ldots \times SO(n_\ell,1)$$

where H is a compact Lie group and $n_1, n_2, \ldots, n_\ell$ is any ℓ-tuple of positive integers. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for any discrete subgroup Γ of G.

Proof: See [20] [21]. □

The next theorem is a consequence of the Mayer-Victoris exact sequence of M. Pimsner [27].

(4.3) ***Theorem:*** Let Γ act on a tree without inversion. Assume that conjecture $\mathbb{Z}$ and $\mathbb{C}$ are valid for the stabilizer group of each vertex and each edge. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ.

Proof: The six-term sequence constructed by M. Pimsner also exists for $K^*(\cdot,\Gamma)$ and $H_*(\Gamma, F\Gamma)$. The theorem then follows by the five lemma. □

M. Pimsner's remarkable result [27] contains previous results of Pimsner-Voiculescu [28], J. Cuntz [14], J. Anderson- W. Paschke [2], C. Lance [23], and T. Natsume [26].

In particular, three corollaries of (4.3) are:

(4.4) ***Corollary:*** Let Γ be a subgroup of Γ_1 and Γ_2. Set $\tilde{\Gamma} = \Gamma_1 \underset{\Gamma}{*} \Gamma_2$, the free product of Γ_1 and Γ_2 with amalgamation along the common subgroup Γ. Assume the conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ, Γ_1, and Γ_2. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for $\tilde{\Gamma}$.

(4.5) ***Corollary.*** Let Γ be a subgroup of Γ_1 and let $\theta\colon \Gamma \to \Gamma_1$ be an injective homomorphism of Γ into Γ_1. Let Γ be the HNN extension determined by Γ, Γ_1, θ. Assume that conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ and Γ_1. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ.

(4.6) ***Corollary.*** (Pimsner-Voiculescu [28]) Let $\theta_1, \theta_2, \ldots, \theta_n$ be automorphisms of Γ. Denote the free group on n generators by F_n and let F_n act on Γ via $\theta_1, \theta_2, \ldots, \theta_n$. Form the semi-direct product $\Gamma \rtimes F_n$. Assume that conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for $\Gamma \rtimes F_n$.

Remark: According to [38], Corollaries (4.4) and (4.5) together are equivalent to (4.3).

Another point in favor of the conjecture is the direct limit property:

(4.7) ***Proposition:*** Let $\{\Gamma_i\}_{i\in I}$ be subgroups of Γ such that with respect to inclusions $\Gamma_i \subset \Gamma_j, \Gamma = \xrightarrow{\text{limit}} \Gamma_i$. Assume that conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for each Γ_i. Then conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for Γ.

Proof: As observed in the proof of lemma (3.5)

$$K_*[C_r^*\Gamma] = \xrightarrow{\text{limit}} K_*[C_r^*\Gamma_i]$$

and

$$K^*(\cdot,\Gamma) = \xrightarrow{\text{limit}} K^*(\cdot,\Gamma_i) \qquad \square$$

For the injectivity of $\mu : K^*(\cdot,\Gamma) \to K_*[C_r^*\Gamma]$ Kasparov [20] has proven that μ is injective whenever Γ is a discrete subgroup of a connected Lie group. If Γ is any discrete group then the transverse fundamental class result of [13] yields a quite general partial injectivity result for

$$\mu : H_*(B\Gamma;\mathbb{C}) \to K_*[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

This will be reported on in [7].

§5. Negative Indications

Let G be a Lie group with $\pi_0 G$ finite. Let Γ be a discrete torsion-free subgroup of G. Denote the maximal compact subgroup of G by K and consider the C^∞ manifold $K\backslash G/\Gamma$. Let $T\Gamma$ be the tangent bundle of this manifold

(5.1) $$T\Gamma = T(K\backslash G/\Gamma)$$

Then

(5.2) $$K^i(\cdot,\Gamma) = K^i(T\Gamma)$$

In (5.2) $K^i(T\Gamma)$ is the usual Atiyah-Hirzebruch K-theory (with compact supports) of $T\Gamma$. Equivalently, (5.3) $$K^i(\cdot,\Gamma) = K_i[C_0(T\Gamma)]$$

In this context, consider the case when $G = Sp(n,1)$, $n \geq 2$, and $\Gamma \subset Sp(n,1)$ is torsion-free and of finite co-volume. G. Skandalis [39] proves that for such a Γ, $C^*_r\Gamma$ is not K-nuclear. In particular $C^*_r\Gamma$ cannot be KK-equivalent to an abelian C^*-algebra so

$$\mu : K_i[C_0(T\Gamma)] \to K_i[C^*_r\Gamma]$$

cannot be a KK-equivalence of $C^*_r\Gamma$ and $C_0(T\Gamma)$.

This result of G. Skandalis came as something of a shock. The "dual Dirac" element of Kasparov-Miscenko [20] [25] in $KK(C^*_r\Gamma, C_0(T\Gamma))$ gives a map

(5.4) $$\beta : K_i[C^*_r\Gamma] \to K_i[C_0(T\Gamma)]$$

It was commonly believed [8] that at the KK level μ and β were inverses of each other. According to G. Skandalis this is not the case. But as homomorphisms of K-theory, μ and β may still be inverses of each other. Clearly something delicate and intereting is happening in these Skandalis examples which involves the difference between

$$K_*[C^*_{max}\Gamma] \text{ and } K_*[C^*_r\Gamma].$$

For a general discrete group Γ consider the standard map $C^*_{max}\Gamma \to C^*_r\Gamma$. Γ is said to be K-amenable if this map is a KK-equivalence. Any infinite group Γ having Kazhdan's [22] property T is not K-amenable. A severe weakness in the current evidence for conjectures $\mathbb{Z}$ and $\mathbb{C}$ is that we have no example of an infinite group Γ having property T for which the conjecture has been verified. Perhaps the first example that comes to mind is $SL(3,\mathbb{Z})$.

J. Anderson has observed that $SL(3,\mathbb{Z})$ is a property T group and that conjecture $\mathbb{C}$ asserts

$$K_0[C_r^*SL(3,\mathbb{Z})] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^8$$

$$K_1[C_r^*SL(3,\mathbb{Z})] \otimes_{\mathbb{Z}} \mathbb{C} = 0$$

Another weakness in the evidence for conjectures $\mathbb{Z}$ and $\mathbb{C}$ is the lack of a Künneth theorem for $K_*[C_r^*(\Gamma_1 \times \Gamma_2)]$. Here $\Gamma_1 \times \Gamma_2$ is the Cartesian product of Γ_1 and Γ_2

(5.5) $$\Gamma_1 \times \Gamma_2 = \{(\gamma_1, \gamma_2) | \gamma_i \in \Gamma_i\}$$

Set $K_i^{\mathbb{C}}(\Gamma) = K_i[C_r^*\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$

Then conjecture $\mathbb{C}$ implies

(5.6) $$K_0^{\mathbb{C}}(\Gamma_1 \times \Gamma_2) = K_0^{\mathbb{C}}(\Gamma_1) \otimes_{\mathbb{C}} K_0^{\mathbb{C}}(\Gamma_2) \oplus K_1^{\mathbb{C}}(\Gamma_1) \otimes_{\mathbb{C}} K_1^{\mathbb{C}}(\Gamma_2)$$

(5.7) $$K_1^{\mathbb{C}}(\Gamma_1 \times \Gamma_2) = K_1^{\mathbb{C}}(\Gamma_1) \otimes_{\mathbb{C}} K_0^{\mathbb{C}}(\Gamma_2) \oplus K_0^{\mathbb{C}}(\Gamma_1) \otimes_{\mathbb{C}} K_1^{\mathbb{C}}(\Gamma_2)$$

At the present time there is neither a proof nor a counter-example for (5.6) and (5.7).

Using topological methods S. Cappell [9] has proved that if the Novikov conjecture on homotopy invariance of higher signatures is valid for Γ_1 and Γ_2, then it is valid for $\Gamma_1 \times \Gamma_2$. This provides some indication that (5.6) and (5.7) may be true. If $C_r^*\Gamma_1$ is KK-equivalent to an abelian C^*-algebra (e.g. Γ_1 finite, abelian, or a discrete subgroup of $SO_0(n,1)$) then (5.6) and (5.7) are implied by the Künneth theorem of J. Rosenberg and C. Schochet [34] [35].

Let N be a normal subgroup of Γ so that the sequence

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow \Gamma/N \longrightarrow 1$$

is exact. If conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for N and Γ/N are they valid for Γ ? Our understanding of this question appears to be in an extremely primitive state. For example, let B_n be the braid group on n strings [3]. As usual B_n maps to the symmetric group S_n . The kernel is the pure braid group B_n^0

$$1 \longrightarrow B_n^0 \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

V. Jones has pointed out that by using the Pimsner-Voiculescu exact sequence [28] and an inductive argument one can verify conjectures $\mathbb{Z}$ and $\mathbb{C}$ for B_n^0. By (3.1) and (3.3) conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for the symmetric group S_n . We do not know if conjectures $\mathbb{Z}$ and $\mathbb{C}$ are valid for B_n . According to conjecture $\mathbb{C}$

$$K_0[C_r^* B_n] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}$$

$$K_1[C_r^* B_n] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}$$

At the opposite extreme from discrete subgroups of Lie groups are the groups constructed by D. Kan and W. Thurston in [18]. The Kan-Thurston construction begins by selecting a "large" acyclic group Γ . For example, Γ could be the acyclic group of J. Mather [24]. This Γ is the group of all compactly supported homeomorphisms of the real line $\mathbb{R}$ onto itself. According to conjecture $\mathbb{Z}$

$$K_0[C_r^* \Gamma] = \mathbb{Z}$$

$$K_1[C_r^* \Gamma] = 0$$

Mather's acyclic Γ is torsion free. Other "large" acyclic discrete groups are studied in [17] [37] [40].

§6. The More General Conjecture

This note has been devoted to $K_*[C^*_r\Gamma]$. As pointed out in the introduction conjectures $\mathbb{Z}$ and $\mathbb{C}$ are special cases of a very general conjecture involving foliations, Lie group actions, and actions of discrete groups. For a precise formulation of the more general conjecture see [4] [6]. It is somewhat artificial to focus attention exclusively on conjectures $\mathbb{Z}$ and $\mathbb{C}$ because there is a great deal of inter-action among the various cases of the general conjecture. For example, let Γ be a discrete group acting by a C^∞ (right) action on a C^∞ manifold X. In [6] we construct a commutative diagram:

$$\begin{array}{ccc} K^i(X,\Gamma) & \xrightarrow{\mu} & K_i[C_0(X) \rtimes \Gamma] \\ \Big\downarrow ch_\Gamma & & \Big\downarrow \\ H^i(X,\Gamma) & \xrightarrow{\underline{\mu}} & K_i[C_0(X) \rtimes \Gamma] \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

in which $C_0(X) \rtimes \Gamma$ is the reduced crossed-product C^*-algebra and the right vertical map is the tautological map.

(6.1) ***Conjecture:*** $\mu : K^i(X,\Gamma) \to K_i[C_0(X) \rtimes \Gamma]$

and

$$\underline{\mu} : H^i(X,\Gamma) \to K_i[C_0(X) \rtimes \Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

are isomorphisms.

Conjecture (6.1) is hereditary in Γ, i.e. if conjecture (6.1) is valid for any C^∞ action of a discrete group Γ, then conjecture (6.1) is valid for any C^∞ action of any subgroup H of Γ. The reason is that there is a "Shapiro's lemma" [29] [30] [31] [36]

$$K^i(X,H) \cong K^i(X \times_H \Gamma, \Gamma)$$

$$K_i[C_0(X) \rtimes H] \cong K_i[C_0(X \times_H \Gamma) \rtimes \Gamma]$$

So far any discrete group Γ for which conjectures $\mathbb{Z}$ and $\mathbb{C}$ are known to be valid is also known to satisfy conjecture (6.1) for any C^∞ action of Γ.

§7. Gromov's Principle

Gromov's Principle is: "There is no statement about all discrete groups which is both non-trivial and true." If Gromov is right, then the universe of all discrete groups is too wild and too crazy for us to say anything of substance about all discrete groups. In this case the correct strategy for conjectures $\mathbb{Z}$ and $\mathbb{C}$ is to try to prove them only for certain kinds of groups, e.g. discrete subgroups of Lie groups. But is Gromov right? Time will tell.

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