NON-COMMUTATIVE GEOMETRY

Alain Connes

Institut des Hautes Etudes Scientifiques (IHES)

F - 91440 Bures-sur-Yvette, France

0 INTRODUCTION

For purely mathematical reasons it is necessary to consider spaces which cannot be represented as point sets and where the coordinates describing the space do not commute. In other words, spaces which are described by algebras of coordinates which are not commutative. If you consider such spaces, then it is necessary to rethink most of the notions of classical geometry and redefine them. Motivated from pure mathematics it turns out that there are very striking parallels to what is done in quantum physics. In the following lectures, I hope to discuss some of these parallels.

1 VECTORBUNDLES, DIFFERENTIALFORMS, CONNEXIONS AND THE DE-RHAM-COMPLEX

Let me start with some simple examples, where purely from mathematics one is forced to consider non-commutative spaces:

Example 1 : (Signature)

Let us consider a simply connected oriented 4k-dimensional manifold M. There is a natural symmetric form on the middle-dimensional cohomology of M, s. /2, p. 572 ff/

$$H^{2k}(M) \times H^{2k}(M) \to H^{4k}(M) \cong \mathbb{R}$$

$$(\omega_1 , \omega_2) \mapsto \omega_1 \wedge \omega_2$$

The signature of M is defined as the signature of this quadratic form (# of positive eigenvalues - # of negative eigenvalues). The signature is computable as the analytical index of the signature operator on M (see for example Bott's lecture at this school). When the manifold M is not simply

Lecture Notes by H.-W. Wiesbrock

connected but still orientable with fundamental group $\Gamma := \pi_1(M)$, you can pass to the universal covering $\tilde{M}$ over M and look at the signature. But by restricting attention to this kind of signature, you lose a lot of information which is crucial in surgery theory for example. There it is fairly obvious that one has to take into account the fundamental grop Γ and the signature has to be more involved than the signature of a quadratic form. The group Γ acts on $H^{2k}(\tilde{M})$ and $H^{2k}(\tilde{M})$ is a finite $\mathbb{C}(\Gamma)$ -module, where $\mathbb{C}(\Gamma)$ is the group ring of Γ .

$$\mathbb{C}(\Gamma) := \{ \sum_{j=1}^{n} a_j \gamma_j \,/\, a_j \in \mathbb{C}, \gamma_j \in \Gamma, n \in \mathbb{N} \}$$

with

$$\left(\sum_{j=1}^{n} a_j \gamma_j \right) \left(\sum_{l=1}^{m} a_l' \gamma_l' \right) = \sum_{\substack{j=1 \\ l=1}}^{n,m} a_j a_l' \gamma_j \gamma_l'$$

you can look at this group ring as a ring of functions defined on Γ :

$$\sum_{j=1}^{n} a_j \gamma_j \mapsto f : \begin{array}{l} \Gamma \to \mathbb{C} \\ \gamma_j \to a_j \end{array}$$

with the convolution product:

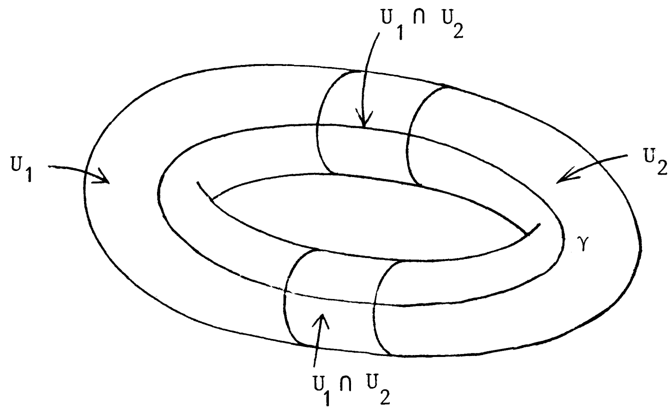
$$f * g(\gamma) := \sum_{\gamma'\gamma''=\gamma} f(\gamma') g(\gamma'')$$

If Γ is commutative, for example M a torus, $\pi_1(M) = \mathbb{Z}^4$, then by Fourier-transformation the group ring goes over to the algebra of functions on the dual of Γ , $\hat{\Gamma}$. For M a torus, $\hat{\Gamma}$ is the dual torus, with the ordinary product

$$\mathbb{C}(\Gamma) \subset C(\hat{\Gamma})$$

A finite projective module over $C(\hat{\Gamma})$ represents a vectorbundle over $\hat{\Gamma}$ (see below), and $H^{2k}(M)$ thereby defines a vectorbundle over $\hat{\Gamma}$. Now Γ acts invariantly w.r.t. the quadratic form on $H^{2k}(M)$ and you can take the (+) Eigenspace to define a subbundle over $\hat{\Gamma}$ and subtract from it the (-) Eigenspace-bundle (in K-theory of vectorbundles). The invariant you will get in this situation is no longer a real number but a (virtual) vectorbundle, the signature-bundle over the dual $\hat{\Gamma}$, in the case of a torus, over the dual torus. You can interpret this as follows.

Let $\rho \in \hat{\Gamma}$ and take the flat line bundle E_ρ over M , which is defined by the holonomy given by ρ :



The clutching map in $U_1 \cap U_2$ is given by

$$\varphi|(U_1 \cap U_2)' : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto z$$

and

$$\varphi|(U_1 \cap U_2)'' : \mathbb{C} \to \mathbb{C}$$
$$z \to \rho(\gamma) z$$

For the general construction look at the principal Γ -bundle s /17, p 42 and 160/

$$\begin{array}{ccccc} \tilde{M} & & \tilde{M} \times_\rho \mathbb{C} & & E_\rho \\ \downarrow & \Rightarrow & \downarrow & = & \downarrow \\ M & & M & & M \end{array}$$

and construct by using the homomorphism $\rho : \Gamma \to \mathbb{C}$ the associated $\mathbb{C}$ -fibre-bundle, which is in this case a flat linebundle over M .

Next take the E_ρ -twisted signature-operator on M . The kernel and co-kernel of the twisted operator depend on $\rho \in \hat{\Gamma}$ and thus give rise to the signature-bundle over $\hat{\Gamma}$.

Now assume that Γ is no longer commutative, then you cannot use Fourier transformation to get back to continuous functions on a space and talk about bundles over that space. However, it turns out that it is possible to extend the above construction. In order to do this, you have to understand that even though there is no space which is for example the dual space of a non-commutative group, you can still define vectorbundles, connections, etc. This is the program I want to talk about.

Example 2 : (Supermanifolds)

Another example which should be familiar to you is the case of supermanifolds. There is a very handy way to describe a supermanifold in terms of algebra of superfunctions on that supermanifold, which, due to the fermionic part, is not commutative but $\mathbb{Z}/_2$ graded commutative.

Example 3 : (Current Algebra)

When you are dealing with any gauge theory like QCD, the important thing is the current algebra, which is the algebra of matrix-valued functions on the manifold. This algebra, of course, is non-commutative.

The main idea which I want to persue is that in generalizing the concept of point sets the best way is to encode the information by a suitable algebra.

Example 4 : (Orbifolds)

Another simple example is an orbifold. An orbifold is not just given by a quotient space because you must remember how much folding takes place at each point of the orbifold. The natural way to encode this information is to use the following associated algebra: Let M be a manifold, π a finite group acting on M and consider the algebra

$$C(M) \rtimes \pi \qquad \text{(cross product)}$$

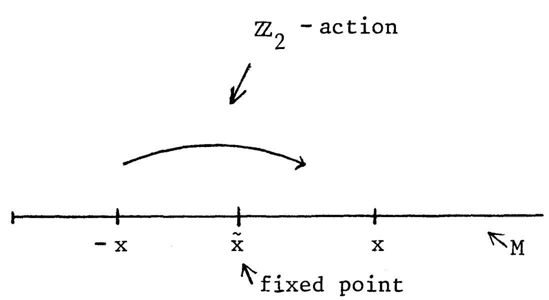
which is defined by $\mathbb{C}$ -valued functions

$$f : M \times \pi \to \mathbb{C}$$

with product

$$f * g(m,\gamma) := \sum_{\gamma' \in \pi} f(m,\gamma')\, g(\gamma' m, \gamma(\gamma')^{-1})$$

As an example, consider the 1-dim. manifold $M = \mathbb{R}$, and $\pi = \mathbb{Z}_2$ which acts on M by $x \to -x$



The cross product in this case is isomorphic to the algebra of 2x2 matrix-valued functions on M

$$x \mapsto \begin{bmatrix} f(x,+1) & f(x,-1) \\ f(-x,-1) & f(-x,+1) \end{bmatrix}$$

and it is easily shown, that restricting the domain on $\{x, -x\}$, the spectrum of the algebra you get, i.e. the set of maximal ideals of the algebra, consists of one point, if x is not a fixed point and of two points, if $x = \tilde{x}$ the fixed point. That is, looking at the spectrum associated to an orbifold, x and $-x$ are identified:

spectrum of the algebra

Example 5 : (Foliations)

The bare general case of quotient-spaces is a foliation. This situation was my original motivating example.

Now let us generalize the concept of vectorbundles, connexions et al to the non-commutative situation. To do this, we must first reformulate these concepts in the ordinary case in terms of statements about the associated (commutative) algebra:

Let X be a manifold, E a complex vectorbundle over X and denote by $\mathcal{E} = C(X,E)$ the linear space of all continuous sections of this fibre bundle. $\mathcal{E}$ is a module over $C(X)$, the algebra of $\mathbb{C}$ -valued continuous functions on X . The fibre at a point $\tilde{x}$ is obtained by

$$\mathcal{E} \otimes_{C(X)} \mathbb{C}$$

where $C(X)$ acts on $\mathbb{C}$ by

$$fz := f(\tilde{x})z$$

So passing to the module $\mathcal{E}$, you will not lose any information. There is a nice lemma due to Swan, which characterizes the modules over $C(X)$ which come from finite-dimensional vectorbundle over X :

Lemma: (Swan)

There is a one-to-one correspondence between finite dimensional locally trivial vectorbundles over X and finite projective modules over $C(X)$ s. /8/.

Remark:

A module $\mathcal{E}$ is finitely projective iff there exist a module $\mathcal{E}'$, such that $E \otimes E'$ is a finitely generated free module.

There is another way of looking at projective modules: Starting with the free module $[C(X)]^n$, you can describe the finite projective module by the associated projection

$$e \in M_{n,n}(C(X))$$

$M_{n,n}(A)$ - nxn matrices with entries in A, which project down on E. Now

$$M_{n,n}(C(X)) = C(X, M_{n,n}(\mathbb{C}))$$

the continuous $M_{n,n}(\mathbb{C})$ -valued functions on X, and e is an idempotent ($e^2 = e$), so that in fact e is a continuous map from X to the Grassmannian.

Geometrically this means that you can reconstruct the vectorbundle on X by pulling back via the map e the universal vectorbundle over the Grassmannian. So for a finite projective module over $C(X)$ we naturally get a vectorbundle over X and vice versa. /3, 18/

The notion of a vectorbundle as a finite projective module can easily be carried over to the non-communtative algebras.

<u>Definition</u>:

A vectorbundle over a non-commutative space with algebra A is given by a finite projective module over A or equivalently by projections in $M_{n,n}(A)$.

The projections are not uniquely defined: Let $e \in M_{n,n}(A)$, $e^2 = e$ and $\tilde{e} \in M_{n+1,n+1}(A)$

$$\tilde{e} := \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$$

The e and $\tilde{e}$ describe the same module. It is better to think of the projective module itself.

What we are doing is linear algebra with the algebra A as the ground ring. For example: A trivial bundle over A corresponds to a free module.

With these notions one can define one of the most important cohomology invariants of a space, which is its K-theory and you can extend this invariant to more complicated spaces:
First denote

$$M_\infty(A) := \lim_{\rightarrow} M_{n,n}(A)$$

the algebraic direct limit of $M_{n,n}(A)$ under the imbedding

$$a \to \begin{pmatrix} a & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

This algebra has a natural norm, the completion is called the <u>stable algebra</u> of A and is the C*-tensor product of A and $\mathcal{K}$, $\mathcal{K}$ the algebra of compact operators. To the direct sum of vectorbundles corresponds the direct sum of modules and to the equivalence-relation for vectorbundles which defines K-theory corresponds the algebraic equivalence-relation for projections

$e \sim f :\Leftrightarrow \exists\, u$ partial isometry : $e = u^* u$ and $f = uu^*$
$e, f, u \in M_\infty(A)$, e and f projections, s. /13/

The Grothendieck group, which you get by symmetrization of the direct sum, defines <u>$K_o(A)$</u> for any C* -algebra A . (It is like $\mathbb{N} \to \mathbb{Z}$).

<u>Remark</u> :

If $A = C(X)$ you have

$$K_o(A) = K^o(X)$$

where $K^o(X)$ is the ordinary K^o -group of vectorbundle-classes over the manifold X . Notice, that upper and lower subscripts refer to the transformation properties of the objects: vectorbundles transform contra-variently w.r.t. X and modules covariantly w.r.t. the algebra A .

In K-theory there is a famous result which makes everything go, the Bott-periodicity Theorem. The first proof was given by using Morse theory in infinite dimensional spaces. However, later it was shown that the appropriate setting for this theorem is the category of Banach-algebras.

<u>Theorem</u> : (Bott-periodicity)

Let A be a Banach-algebra, $Gl_N(A)$ the general linear group of NxN-matrices with entries in A . Then, if N is large enough, you have

$$\pi_n(Gl_N(A)) = \pi_{n+2}(Gl_N(A))$$

and

$$\pi_1(Gl_N(A)) \cong K_o(A)$$

Take for example $A = \mathbb{C}$. Then one gets $(2N \geq n)$

$$\pi_{n-1}(Gl_N(\mathbb{C})) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z} & n \text{ even} \quad \text{s. /8,1/} \end{cases}$$

Let me sketch the idea of the proof:

Let $[e] \in K_0(A)$ and the projection $e \in M_{n,n}(A)$ a representant of $[e]$ and look at the loop

$$\gamma : S^1 \to Gl_N(A)$$
$$t \to \exp(2\pi i te)$$

This gives you a map from $K_0(A) \to \pi_1(Gl_N(A))$ which is in fact an isomorphism.

Now define $\underline{K_1(A) := \pi_0(Gl_N(A))}$

You can also define $K_1(A)$ by mimicking the construction of a suspension to the case of Banach algebras:

$$SA := \{f: \mathbb{R} \to A \mid f \text{ continuous}, \lim_{|x| \to \infty} \|f(x)\| = 0\}$$
$$\cong C_0(\mathbb{R}) \otimes A$$

and get

$$K_1(A) \cong K_0(SA) \quad \text{s. /13/}$$

$K_0(A)$ and $K_1(A)$ are now defined for every C*-algebra A and Bott-periodicity works for them. The periodicity shortens the ordinary long exact sequence in cohomology:
If you have an exact sequence of C*-algebras

$$0 \to J \to A \to B \to 0$$

the associated exact sequence in K-theory is

Bott-periodicity map ↘

$$K_0(J) \to K_0(A) \searrow K_0(B)$$
$$K_1(B) \nearrow K_0(J)$$
$$K_1(A) \leftarrow K_1(J) \swarrow K_0(B), \quad K_1(B) \nwarrow K_1(A)$$

↖ connecting map (index map)

Take for example $U \subset X$ a subspace of the topological space X, $J = C(U)$, $A = C(X)$ and $B = C(X-U)$. In ordinary cohomology you get the long exact sequence

$$\ldots \to H^{q-1}(X,U) \to H^q(U) \to H^q(X) \to H^q(X,U) \to H^{q+1}(U) \to H^{q+1}(X) \to \ldots$$

whereas in K-theory this is shortened to the six-term diagram. This allows a lot of computations in K-theory.

The second main tool is the Chern-character. I will give a description of the Chern-character which fits well into what I want to do later.

Given a bundle E over a manifold X, the Chern-character of the bundle, given for example by the Chern-Weil-homomorphism, is equivalently given by the pairing with an arbitrary closed current (a de-Rham current is an element of the dual space of the differential forms):

$$C \mapsto \langle ch(E), C \rangle$$

But this can be rewritten very simply by use of the projection e which describes the bundle E:
Let $e \in M_{N,N}(C(X)) = C(X, M_{n,n}(\mathbb{C}))$ be a projection associated to E.
e is a continuous map from X to the Grassmannian and E is just the pull back of the universal bundle over the Grassmannian. /18,3/ There is a natural connexion for the bundle over the Grassmannian, obtained by projecting a fibre to the nearby fibre by the orthogonal projection. Pulling back this connection via e gives a connection for E and we can compute the Chern-class by applying the Chern-Weil-homomorphism to this connection. /19/ It gives

$$\langle ch\,(E), C \rangle = \langle e \underbrace{de \ldots de}_{\text{even times}}, C \rangle$$

where $de \in \Omega^1(X) \otimes M_{N,N}(\mathbb{C})$ and $de \ldots de$ is the ordinary $\wedge$-Product for $M_{N,N}(\mathbb{C})$ - valued differential forms. To describe the pairing of the closed current C with a matrix-valued differentialform, it is enough to consider the following situation:
Let $f^0, f^1, \ldots, f^n \in C^\infty(X)$

$$\tau(f^0, f^1, \ldots, f^n) := \langle f^0 df^1 \wedge \ldots \wedge df^k, C \rangle$$

is well defined. Now tensoring the f's with $N^0, \ldots, N^k \in M_{N,N}(\mathbb{C})$ gives a formula reminiscent of Chan-Paton-Factors

$$\tilde{\tau}(f^0 \otimes N^0, \ldots, f^k \otimes N^k) := \tau(f^0, \ldots, f^k)\ \text{Trace}\ (N^0 \ldots N^k)$$

Notice that the projection e associated to the vectorbundle E is not uniquely defined but it turns out (to be proved) that the function

$$e \to \langle e\, de \ldots de, C \rangle$$

although not linear in e, depends only on the K_0-class of e!
What is the right notion of Differentialforms ?

The above notions led to K-theory, which is a cohomology-theory and algebraic topology tells you that there is a corresponding homology-theory due to Alexander-Duality and all that. If you apply this to K-theory you get K-homology. By the work of Atiyah, Brown-Douglas-Fillmore and Kasparov, you can describe K-homology in a beautiful way, provided you introduce some non-commutative spaces, the odd and even spheres Σ^{odd}, Σ^{even} given by non-commutative algebras:

Let $H = H^+ \oplus H^-$ be a $\mathbb{Z}_2$-graded separable Hilbert space.

Then

$$C(\Sigma^{even}) := \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \in L(H) \;/\; (a_{11}-a_{22}) \text{ compact} \right\}$$

where $L(H)$ is the algebra of bounded operators on H, and

$$C(\Sigma^{odd}) := \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in L(H) \;/\; a_{12}, a_{22} \text{ compact} \right\}$$

It turns out, that

$$K_o(C(\Sigma^{even})) = \mathbb{Z} \quad ; \quad K_1(C(\Sigma^{even})) = 0$$
$$K_o(C\ \Sigma^{odd})) = 0 \quad ; \quad K_1(C(\Sigma^{odd})) = \mathbb{Z}$$

In ordinary homology-theory you look at the cell-decomposition of the topological space. The basic cycles in the theory are the spheres which are mapped into the space in question. In K-theory this rôle is played by Σ^{even} and Σ^{odd} : s. /16, 13/

<u>Theorem</u>

$$K_1(X) = \mathrm{Hom}\,[\Sigma^{odd}, X]$$
$$K_o(X) = \mathrm{Hom}\,[\Sigma^{even}, X]$$

where K_o, K_1 denote the K-homology, the so-called <u>Steenrod-K-homology</u> of the ordinary space X, and Hom the homotopy classes of maps. A continuous map from $\Sigma^{\substack{even\\odd}}$ to X is a homomorphism of the corresponding algebras

$$\mathrm{Hom}(C(X), C(\Sigma^{\substack{even\\odd}}))$$

The pairing of K-theory and K-homology is given as follows:

Let $e \in K_o(C(X)) \cong K^o(X)$, $\pi \in \mathrm{Hom}(C(X), C(\Sigma^{even}))$

You can push forward e to $\pi_*(e) \in K_o(C(\Sigma^{even}))$

generally of $\mathrm{Hom}(A, C(\Sigma^{\substack{even\\odd}}))$, A any C*-algebra: Given a homomorphism of A to $C(\Sigma^{even})$ you get a representation of $H = H^+ \oplus H^-$ as an A-module.

But due to the properties of $C(\Sigma^{even})$, the two representations of A on H^+ resp. H^- differ only by compact operators! Let me summarize what you get: /6/

a) A $\mathbb{Z}_2$-graded separable Hilbert space $H = H^+ \oplus H^-$ with grading operator

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b) A homomorphism $\varphi : A \to L(H)$ such that

$[\varphi(a),\gamma] = 0 \quad \forall a \in A$

c) An odd operator $F \in L(H)$, i.e. $[\gamma,F]_+ = 0$, such that

$F^2 = 1$

$[F,\varphi(a)]$ is compact $\forall a \in A$

In the above case F is simply given by

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Remark

You can think of the grading operator γ for example as the Γ - operator in Jaffe's lecture or the γ_5 in chiral theories.

The quadruple (H,φ,F,γ) is called a Fredholm-module over A . The odd case is slightly different. There you get

a) A Hilbertspace H and a *-representation

$\varphi : A \to L(H)$

b) $P \in L(H)$ with

$P^2 = \mathrm{Id}$

$[P,\varphi(a)]$ is compact $\forall a \in A$

s. /6, 13/

To see this, notice that restricting the $\pi \in \mathrm{Hom}(A, C(\Sigma^{odd}))$ to H^+ gives a homomorphism of A modulo compact operators (a homomorphism of A to the Calkin algebra of H^+). Define

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$P_+ := \frac{1 + P}{2}$$

Then P_+ is the projection on the (+)-Eigenspace of P .

$$\rho : A \to L(P_+(H))$$
$$a \mapsto P_+\varphi(a)P_+$$

is a homomorphism of A to $L(P_+(H))$ mod. compact operators :

$$\rho(ab)-\rho(a)\rho(b) \text{ is compact.}$$

This gives the right thing to define Differentialforms: Let (H,φ,F,γ) be a Fredholm-module of A, (in the odd case (H,φ,P) . A 1-form is obtained as an operator on H

$$da := i[F,\varphi(a)] \in L(H) \quad \text{(compact)}$$

(in the odd case)

$$da := i[P,\varphi(a)] \in L(H)$$

and the general 1-form as a linear combination of

$$\sum_i \varphi(a^j)\, db^i \qquad a^j, b^j \in A$$

To simplify notation, I will drop the φ in the following, looking at H as an A-module. A general k-form is given by

$$\sum_j a_j^0\, da_j^1 \ldots da_j^k \in L(H) \qquad a_j^e \in A$$

The space of k-forms will be denoted by Ω^k . For any algebra A and Fredholm module over A this gives a graded differential algebra. The next notion we need is that of integration of forms:

We are dealing with algebras and the natural generalization of integration for algebras is the concept of trace. Here we have in the even case an additional structure, the $\mathbb{Z}_2$-grading given by γ and we define

$$\int \omega := \mathrm{Trace}_S(\omega) := \mathrm{Trace}\ (\gamma\omega) \quad \text{(supertrace)}$$

in the even case and

$$\int \omega := \mathrm{Trace}\ (\omega)$$

in the odd case where $\omega \in \Omega^k$, whenever the right sides are defined. To be able to use the concept of integration, we refine the definition of a Fredholm-module:

A Fredholm-module (H,φ,F,γ) (in the odd case (H,φ,P)) is called <u>p-summable</u> or <u>of dimension p</u> $p \in R^+$, if and only if

$$[\varphi(a),F] \in L^P(H) \quad (\text{resp. } [\varphi(a),P] \in L^P(H))$$

where $L^P(H)$ is the p.th Schatten class of H .

Remark:

Let $T \in L(H)$ compact, $\{\mu_n(T)\}_{n\in\mathbb{N}}$ the Eingenvalues of $|T|$ counted with multiplicities. Then

$$T \in L^P(H) \; :\leftrightarrow \sum_n |\mu_n(T)|^P < \infty$$

The right side defines, by taking the p.th root of the sum, a norm $\| \; \|_P$ on $L^P(H)$ and there is a Hölder inequality

$$\|T_1 \, T_2\| \leq \|T_1\|_{P_1} \|T_2\|_{P_2}$$

if $T_1 \in L^{P_1}(H), T_2 \in L^{P_2}(H)$, S./10/

In particular, for a p-summable Fredholm-module we get

$$\Omega^k \in L^1(H) \qquad k \geq p$$

and the integral of p-forms is well-defined. We get a differential graded algebra Ω^* with differential

$$d : \Omega^k \to \Omega^{k+1}$$

$$\omega \mapsto d\omega := i[F,\omega] = i(F\omega - (-1)^k \omega F) \qquad \text{(graded commutator)}$$

and an integral

$$\int : \Omega^* \to \mathbb{C}$$

with

$$\int \omega = \operatorname{trace}(\gamma\omega) \qquad \omega \in \Omega^k \qquad k \geq P$$

$$\int \omega = 0 \qquad\qquad k < P$$

Such a triple $(\Omega^*, d, \int)$ is also called a cycle for A , s/ below.

In the case of commutative algebra $C(M)$, M a manifold, the definition of dimension for Fredholm-modules corresponds to the ordinary definition of the dimension of M .

Example 6:

The fundamental class of a 2n-dimensional compact Spin^c -manifold M in ordinary homology gives naturally a $2n+\varepsilon$ -summable Fredholm-module over $C(M)$ and thereby a cycle of $C(M)$ in K-homology:
Let $S = S^+ + S^-$ be the spinor bundle over M and $H = L^2(M,S)$ the Hilbert-space of L^2 -spinors. The grading is given by γ_5 and the algebra of continuous functions on M acts on H . Denote $\not{D}$ the Dirac-operator and

$$F := \operatorname{sign} \not{D}$$

F splits H into the positive and negative energy eigenspaces of $\not{D}$.
F is a pseudodifferential operator of order 0 and the commutator of F with any continuous function $a \in C(M)$ is a pseudodifferential operator of order -1 ,

$$[F,a] \approx (1+\Delta)^{-1/2} \quad \text{(in strength)}$$

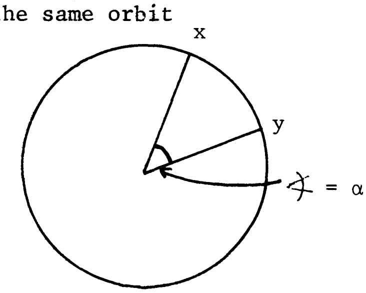
where Δ is the Laplacian on M. It is an easy exercice to check that (H,F,γ_5) is a p-summable Fredholm-module with $p > \dim M$ because a pseudo-differential operator of order $-p$ is trace class for $p > \dim M$. The dimension of the manifold which is the dimension of the K homology class given by the fundamental class of M, gives the lower limit of p.

Lecture II

In his talk, Harvey considered orbifolds, which are quotients of manifolds by a point-symmetry group. Let me start with a simple example of a space which is described by non-commutative algebra rather than as a point set and which may serve as background for the aformentioned construction of Harvey.

Example 7: (irrational rotation algebra)

Let $M = \mathbb{R}^1$ and the point symmetry given by $\mathbb{Z}+\alpha\mathbb{Z}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ irrationa
The "quotient-space" $\mathbb{R}/\mathbb{Z}+\alpha\mathbb{Z}$ doesn't have meaningful topology:
First divide $\mathbb{R}^1$ by $\mathbb{Z}$ and you get a torus $T = S^1$. Then you have to divide by $\mathbb{Z}$ acting on T by rotation by the angle α, that is, you want to identify x and y on the same orbit



If you think of this as a quotient in ordinary language, the open sets are either empty or the entire circle because the orbits are dense on S^1.
However there is a natural way to assign an algebra to this space:
Consider the q-numbers $q(x,y)$, $x,y \in \mathbb{R}$ with $q(x,y) = 0$ if $x \neq y$ $\bmod(\mathbb{Z}+\alpha\mathbb{Z})$, that is, as matrices over $\mathbb{R}$ indexed by x,y with $x = y$ $\bmod(\mathbb{Z}+\alpha\mathbb{Z})$. The multiplication is the usual matrix multiplication

$$q_1 \cdot q_2 (x,z) := \sum_y q_1(x,y) q_2(y,z)$$

Notice that the summation can be reduced to a countable set, the orbit space of x under $(\mathbb{Z}+\alpha\mathbb{Z})$. The spectrum of the subalgebra defined over one orbit consists of one point. So, for $\alpha \in \mathbb{Q}$, the spectrum of the algebra is exactly $S^1/\alpha\mathbb{Z}$. If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the algebra still makes sense and is a natural algebra associated to the above situation. There is a presentation o this algebra as the algebra generated by two matrices U and V :

$$U(x,y) = 0 \qquad x \neq y \qquad (x,y \in S^1)$$
$$U(x,x) = e^{2\pi i x}$$

and

$$V(x,y) = 1 \qquad y = e^{2\pi i \alpha}x$$
$$V(x,y) = 0 \qquad \text{otherwise}$$

and the commutation rule

$$VU = e^{2\pi i \alpha}UV$$

The smooth elements of this algebra are given by

$$\sum_{n,m} a_{nm}U^nV^m$$

with rapid decay of a_{nm} . Denote by A_α the norm closure C* algebra. /12,Th. 1.1/ This algebra is simple. Powers asked for a simple C*-algebra without idempotent. The algebra A_α was a good candidate and Powers tried to show that this algebra has no idempotent and amazingly enough he proved, together with M. Rieffel, in fact, that this algebra contains a projection, which can be described in the simple form

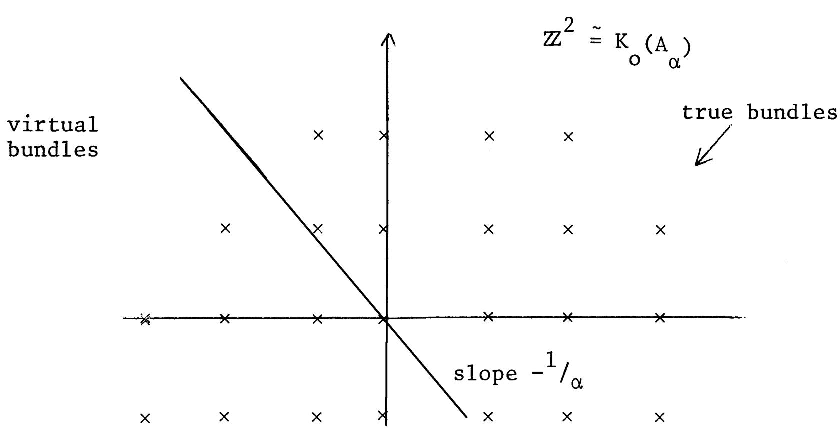
$$U^{-1}f_{-1}(V)+f_o(V)+f_1(V)U$$

with certain functions f_{-1},f_o,f_1 . It turns out that these projections together with the trivial projection 1 generate $K_o(A_\alpha)$.

Pimsner and Voiculescu calculated the $K_o(A_\alpha)$ group s. /4/

$$K_o(A_\alpha) \cong \mathbb{Z}^2 .$$

But there is more information in the K_o-group: as in the case of ordinary manifolds where K-theory contains vector bundle classes of both positive and negative virtual dimension (the latter being virtual bundles), there is also such a notion in the non-commutative case. It turns out that the true bundles are given by the points right to the linc of slope $-^1/_\alpha$:



The dimension of "vector bundles" on this space is defined as follows: Take the (if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ unique) trace

$$\tau(\sum_{n,m} a_{nm} U^n V^m) := a_{oo} = \int_{S^1} q(x,x)$$

on the algebra A_α and evaluate the trace of the idempotents associated to the vectorbundles. This defines the dimension of the bundles. You'll find that the dimension is not necessarily an integer, it can also be an irrational number. The dimension of a true bundle must be positive and this gives you the above picture.

For ordinary vectorbundles one has a well defined invariant, the Chern class of the bundle. You can mimick the computation of the Chern-Class of ordinary bundles in the above situation:

You define partial derivations w.r.t. U and V :

$$\partial_1(U) = U \qquad \partial_1(V) = 0$$
$$\partial_2(U) = 0 \qquad \partial_2(V) = V$$

∂_1, ∂_2 commute. Now write down the first Chern-Class-formula for the "vectorbundle"-idempotent e :

$$c_1(e) = \frac{1}{2\pi i}\ \tau(\ e(\partial_1 e \partial_2 e - \partial_2 e \partial_1 e))$$

where

$$e(\partial_1 e \partial_2 e - \partial_2 e \partial_1 e)$$

is the curvature. If you compute this expression for the Powers-idempotent you get an integer! /7,p.622/ But this computation gives little understanding of the reason of this integrality. It turns out, that the 1st Chern-Class defined above is the index of an operator and this "explains" why it must be an integer.

Remark:

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$, the 2x2 matrices with entries in $\mathbb{Z}$ and det = 1 . You get automorphisms of A_α by

$$U \mapsto U^a V^b$$
$$V \mapsto U^c V^d$$

Let me continue by introducing the notion of connexion for non-commutative algebras:

Usually a connexion is given by a covariant derivative

$$E \xrightarrow{\nabla} E \otimes_A \Omega^1 \qquad \text{s. /6,p.110/}$$

where E is the space of sections of the vectorbundle, Ω^1 the space of

1-forms on the base-manifold M and $A = C^\infty(M)$. This can be easily transcribed to the case of non-commutative geometry where E is replaced by a finite projective module over A and Ω^1 by the above defined 1-forms. Let us take the simplest example, that is $E \cong A$, the trivial bundle. In this case a connexion is given by

$$d : A \to A \otimes_A \Omega^1 = \Omega^1$$

and the space of connexions is an affine space over $\mathrm{Hom}_A(A,\Omega^1) \cong \Omega^1$ and we identify

$$\begin{aligned} \Omega^1 &\to \{\text{connexion}\} \\ \omega &\mapsto d + \omega \end{aligned}$$

The curvature is given by

$$R = d\omega + \omega^2 .$$

Note, that although the bundle is trivial and the algebra might be commutative, we have to include the ω^2 -term. These are operators on the Hilbert space and in general they don't commute.

I want to give some meaning to this by exploiting the simplest example:

Example 8: (Sato Grassmannian)

Take $A = C^\infty(S^1), H = L^2(S^1,\mathbb{C})$. $\{z^n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of H and define

$$F = \mathrm{sign}\,(i\frac{d}{dz})$$

or in terms of eigenfunctions

$$F(z^n) = \mathrm{sign}\,(n)\, z^n .$$

Notice, that we are in the odd case of K-homology. Let us take the trivial bundle $E = A$ and find out what are the flat connexions. In Example 6 we argued, that the 1-forms are Hilbert-Schmidt-operators. So let us take the completion $\overline{\Omega^1}$ of Ω w.r.t. this norm.

Lemma:

The space of connexions on this trivial bundle is isomorphic to the space of Hilbert-Schmidt-operators on H, i.e.

$$\overline{\Omega^1} \cong L^2(H)$$

Proof:

$L^2(H) \cong L^2(S^1\times S^1)$ by mapping the Hilbert-Schmidt-operator to the associated kernel. Assume $k \in C^\infty(S^1 \times S^1)$, with

$$k(z,\rho) = \sum_{n,m} a_{nm} z^n \rho^m$$

its Fourier expansion. Let us map the Hilbert-Schmidt-operator B with kernel

$$(z,\rho) \to z^n \rho^m$$

to Ω^1 : Let $C \in C^\infty(S^1)$

$$C(z) = \sum_n c_n z^n$$

the Fourier expansion of C and

$$a_n : S^1 \to \mathbb{C}$$
$$z \to z^n$$

Then

$$(a_n [F,a_m] C)(z) = \sum_\ell (\text{sign}(\ell) - \text{sign}(\ell + m))\, c_\ell\, z^{\ell+m+n}$$

and

$$BC = (a_{n+1}[F,a_{-m+1}] - a_n[F,a_{-m}])\, C$$

$$B = a_{n+1}\, da_{-m+1} - a_n\, d\, a_{-m}$$

What does it mean for a connexion to be flat? $d+w$ is a flat connexion iff

$$R = dw + w^2 = 0 \quad \Leftrightarrow \quad wF + wF + w^2 = 0$$

(we drop the i). Since $F^2 = 1$ this is equivalent to

$$(F+w)^2 = \text{Id} .$$

There is a little Lemma which describes the Grassmannian of Sato in terms of projection operators.

<u>Lemma:</u>

A closed subspace $W \subset H$ is in the Grassmannian, $W \in$ Grass, if and only if

$$I_W + (-I_{W^\perp}) - F = 2\, I_W - I - F$$

is Hilbert-Schmidt. (I_W denotes the orthogonal projection on W , similarly I_W)

Proof:

$$F = I_{H^+} - I_{H^-} \quad \text{and}$$

$$2\, I_W - I - F = 2\,(I_{H^+} + I_{H^-})\,(I_W) - I - F$$
$$= 2\,(I_{H^+} I_W - I_{H^+}) + 2\, I_{H^-} I_W$$

By definition $W \in$ Grass $\Leftrightarrow$ $I_{H^+} I_W : W \to H^+$ is Fredholm

and $I_{H^-} I_W : W \to H^-$ is Hilbert-Schmidt.

The statement is now obvious /20/. From these Lemmas it easily follows that

Lemma:

There is a one-to-one correspondence between flat connexions of the trivial bundle on S^1 and the Grassmannian of Sato

Proof.

"$\Leftrightarrow$" $d+w$ flat $\Leftrightarrow (F+w)^2 = I$

Let W be the 1-Eigenspace of $F+w$, then

$$w = F+w-F = I_W - I_{W\perp} - F$$

is Hilbert-Schmidt and $W \in$ Grass.

"$\Leftrightarrow$" choose $w = I_W - I_{W\perp} - F$

Carrying these ideas a little bit further, one can describe the Grassmannian as the set of critical points of the action

$$S(w) = \int (wdw + {}^2/_3\, w^3) \qquad w \in \Omega^1$$

where $\int$ denotes the integral (trace), d the derivation in the above Fredholm-module. This is an action of Chern-Simons-type. It gives a Morse function $S : \overline{\Omega^1} \to \mathbb{C}$. One cannot use conventional Morse-theory, the Hessian is a Dirac-type-operator with infinite + and - spectrum. However one can define relative Morse-indices by spectral flow methods. /14, 15/ The indices of S splits the Grassmannian into connected components corresponding to the index of the Fredholm operator $I_{H+} I_W$, s. for example Alvarez-Gaumé's talk.

The gauge-transformations on the trivial bundle are U(1)-valued maps U on S^1, the unitary elements in $C(S^1)$, and they act on $w \in \Omega^1$ by

$$w \mapsto UwU^{-1} + UdU^{-1} .$$

The action S is obviously gauge invariant. Furthermore $\text{Diff}^+(S^1)$, the orientation preserving diffeomorphisms on S^1 acts unitarily on H by

$$(f.\varphi)(z) := \varphi(f^{-1}(z))\,|(f^{-1})'(z)|^{1/2} \qquad f \in \text{Diff}^+ S^1$$

and this action commutes with F modulo Hilbert-Schmidt-operators. /20,p.91/ Thereby it acts on the Fredholm-module and on $\overline{\Omega^1}$ invariantly on S. The central extension of $\text{Diff}^+(S^1)$ leads to the action of the Virasoro-algebra on $\overline{\Omega^1}$ and on the Grassmannian.

To end today's talk, I will give one more example which might explain to you, why I call the above differential forms "first quantized forms".

Example 9 : (Zitterbewegung)

Let $A = C_0^\infty(\mathbb{R}^3)$ the algebra of smooth functions on $\mathbb{R}^3$ with compac support. The natural module, in this case, s. example 6, is given by

$$H = L^2(\mathbb{R}^3, \mathbb{C}^2) \qquad D = \not{\partial} = i\,\sigma_j \frac{\partial}{\partial x_j}$$

where σ_i are the Pauli-matrices, D the Dirac-operator. The underlying manifold is not compact, D does not have discrete spectrum and we have to take care of the zeromodes in this case!

Let us first consider an operator $\tilde{D}$ which is invertible $a\tilde{D}^{-1} \in L^P(H)$ with bounded commutator $[a,\tilde{D}] \in L(H)$ for $a \in A$. /6,p.68/ It is an easy exercice to show that

$$a \mapsto \begin{pmatrix} a & o \\ o & \tilde{D}^{-1} a \tilde{D} \end{pmatrix}$$

is a homorphism

$$A \to C(\Sigma^{even})$$

The Dirac operator D is not invertible but a simple way to build an invertible operator from D which preserves most of the features like the degree etc. is to use,

$$D_m := D \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + m \,.\, I \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\tilde{H} = H \otimes \mathbb{C}^2 \,.$$

Then define

$$F := \operatorname{sign}(D_m) \,.$$

The Fredholm-module class is independent of $m \neq 0$ by homotopy-invariance.

D_m is the Dirac-Hamiltonian of a free electron moving in $\mathbb{R}^3$ with mas m . If you compute the differentiation w.r.t. F , for example of the coordinate functions x_j , you will obtain the Zitterbewegung.

Lecture III

Cyclic Cohomology

It is quite difficult to handle all Fredholm-modules of a given algebra A . Instead of doing this we weaken the data and pass to the character of a p-summable Fredholm-module and analyse them.

Like the character of group representations the character of a p-summable Fredholm-module, which is an algebra representation, is a $\mathbb{C}$ - valued function s. /6/

$\tau : A \times \ldots \times A \to \mathbb{C}$

$(a^o, \ldots, a^n) \mapsto \tau(a^o, \ldots a^n) := \int a^o\, da^1 \ldots da^n$

Remark:

By using the simple formula

$\frac{1}{2} \operatorname{tr} F(FX + XF) = \operatorname{tr} X$

whenever both sides make sense, it is easily seen that for a p-summable Fredholm-module τ is already defined for $n > p-1$.

τ captures exactly what we need to know about Fredholm-modules to do a lot of computations. The functionals τ have two very important properties.

Proposition:

a) $\tau(a^1, a^2, \ldots, a^n, a^o) = (-1)^n \tau(a^o, a^1, \ldots a^n)$ (cyclicity)

b) $$(b\tau)(a^o, \ldots, a^n, a^{n+1}) := \sum_{j=o}^{n} (-1)^j \tau(a^o, \ldots a^{j-1}, a^j a^{j+1}, a^{j+2}, \ldots, a^{n+1}) + (-1)^{n+1} \tau(a^{n+1} a^o, a^1, \ldots, a^n)$$

$$= 0 \quad \text{(cocycle-condition)}$$

b is the Hochschild Coboundary operator, $b^2 = 0$.

Proof:

a) This property doesn't involve the algebraic structure, but stems from the definition of τ by the trace.

b) This is a reformulation of the fact that the trace is a closed graded trace, i.e. $\int dw = 0$.

To understand what this means, let us consider the simplest case $n = 0$: b) $\Leftrightarrow \tau(a^o a^1) - \tau(a^1 a^o) = 0$ that is, τ a trace. b) is an elementary property of the trace (compare with the character of group representations) .

a) arises, as we already saw, if one wants to extend the integral of differential forms to the case of matrix-valued differential forms. Unlike ordinary differential forms, we do not have graded commutativity, but we still maintain cyclicity (using the cyclic property of the trace). Any cyclic cocycle, i.e. a multilinear functional τ satisfying a) and b) of the proposition defines an invariant of vectorbundles resp. of elements in $K_1(A)$ by the generalization of the Chern-Weil-map:

$K_o(A) \xrightarrow{\tau^*} \mathbb{C}$

$[e] \mapsto \tau(e, e, \ldots, e)$

e is a projection representing the equivalence class $[e]$ respectively

$$K_1(A) \overset{\tau_*}{\to} \mathbb{C}$$

$$[u] \mapsto \tau(u^{-1}, u, u^{-1}, \ldots)$$

u is a unitary representant in $[u]$.

Remark:

These maps are well defined, i.e. they don't depend on the special choice of the representants e and u . Moreoever one can prove, that the maps are group homomorphisms.

There is a trivial way to get cyclic cocycles by applying b to a cyclic φ . (But this gives only 0-maps, as one can show using the cyclicity property of τ .) /6,p.104/ If τ comes from an even Fredholm-module $(\mathcal{H}, \varphi, F, \gamma)$ then

$$\tau_*(e) = \text{ind } F_e^+ \in \mathbb{Z}$$

where F_e^+ is the Fredholm operator

$$F_e^+ = e(\mathcal{H}_+ \otimes \mathbb{C}^k) \to e(\mathcal{H}_- \otimes \mathbb{C}^k)$$

$$\varphi \mapsto e(F \otimes \text{Id}) e(\varphi)$$

when $e \in \text{Proj.}(M_{k,k}(A))$.

Now let me give you the definition of cyclic cohomology: The cochains of the complex are multilinear cyclic functionals on the algebra

$$C_\lambda^n(A) := \{\text{cyclic functionals of } (n+1) \text{ variables}\}$$

(λ refers to cyclicity) and the coboundary is the Hochschild coboundary b . Then the cyclic cohomology group is defined as

$$H_\lambda^n(A) := \frac{Z_\lambda^n(A)}{b(C_\lambda^{n-1}(A))}$$

where $Z_\lambda^n(A)$ is the group of cocycles in $C_\lambda^n(A)$ and $b(C_\lambda^{n-1})$ the group of coboundaries.

Let me describe b in low degrees:

$$b : C_\lambda^o \to C_\lambda^1$$

$$\tau \mapsto ((a^o, a^1) \to (b\tau)(a^o a^1) := \tau(a^o a^1) - \tau(a^1 a^o))$$

and $b\tau = 0 \Leftrightarrow \tau$ a trace, s. above

$$b : C_\lambda^1 \to C_\lambda^2$$

$$\tau \mapsto b\tau$$

with

$$b\tau(a^o,a^1,a^2) = \tau(a^o a^1,a^2) - \tau(a^o,a^1a^2) + \tau(a^2a^0,a^1) \; .$$

If A is a matrix-algebra, it is easy to show that a cyclic 1-cocycle gives a Lie-algebra extension by

$$(A\oplus 1) \times (A\oplus 1) \to (A\oplus 1)$$

$$((a,\alpha),(b,\beta)) \to [(a,\alpha),(b,\beta)] := (ab-ba,\tau(a,b)1)$$

s./11/.

In the case of a commutative algebra $A = C^\infty(M)$, M a compact manifold, and requiring continuous functionals in the cyclic cohomology we get

<u>Theorem:</u>

$$H^k_\lambda(A) = \{\text{closed De-Rham-currents of dimension } k\} \oplus$$

$$H_{k-2}(M,\mathbb{C}) \oplus H_{k-4}(M,\mathbb{C}) \oplus \dots$$

s. /6, Th. 4.6/

The terms $H_{k-2}(M,\mathbb{C}) \oplus H_{k-4}(M,\mathbb{C}) \oplus .. \oplus H_o(M,\mathbb{C})$ are required by the index theorem. There you have the various components of the $\hat{A}$-genus and they come from lower dimensional homology-classes. So it is absolutely necessary for our cohomology theory to reproduce these lower terms in order for the index to be an integer. Let me give you a picture which will make clear what I'm doing:

Let M be an ordinary manifold

$$\begin{array}{ccccc} & \text{K-theory } (M) & \xrightarrow{\text{Chern}^*} & \text{Cohomology } (M) & \\ \text{pairing} & \updownarrow & & \updownarrow & \text{pairing} \\ & \text{K-homology } (M) & \xrightarrow{\text{Chern}_*} & \text{Homology } (M) & \end{array}$$

The ordinary Chern maps K-theory into the cohomology of the space. What I will construct here is the Chern_* of K-homology (M) to the homology of M such that the pairings on both sides of the diagram coincide. This will reproduce the index-theorems.

Let me explain how to prove this theorem and how to make things work in general.

The properties of cyclic cohomology will be dictated by the situation of Fredholm-modules. There we get these characters τ_n for $n \geq p-1$. In general, one can recover from a cyclic cocycle a cycle $(\Omega,d,\int)$, which is a graded algebra with a graded derivation of degree $1, d^2 = 0$ and $\int : \Omega^n \to \mathbb{C}$ closed graded trace, n - the dimension of the cycle, and a homomorphism $\rho : A \to \Omega^o$. But this cycle need not come from a p-summable Fredholm-module. /6, p.98/

We have seen that in the case of p-summable Fredholm $\tau_n, \tau_{n+2}, ..$ $n \geq p$ all give cyclic cocycles and the question arises, is the relevant information about the module already contained in τ_n ?

Theorem :

There exists a universal map $S : H^n_\lambda(A) \to H^{n+2}_\lambda(A)$ defined by

$$(S\tau_n)(a^o,..,a^{n+2}) := 2\pi i \sum_{j=0}^{n+2} \tau_n \,(a^o da^1..da^{i-1})a^i a^{i+1}(da^{j+2}..da^{n+2})$$

(purely algebraically), s.t. in the above question

$$\tau_{n+2k} = (S)^k(\tau_n) \in H^{n+2k}_\lambda(A)$$

/6, Th. 1/

This nice result leads to another question: When is a given τ in the image of S ? Or: Given a τ , when is it possible to rewrite τ as $\tau = S^k\tau'$ with a τ' of a lower-dimensional Fredholm-module, which thereby depends on fewer variables?

Theorem 2:

τ is in the image of S , if and only if $\tau \in b(C^{n-1})$ that is, τ is a coboundary of a Hochschild cochain, not necessarily of a cyclic cochain. /6, Cor. 35/

The use of this fact is that the Hochschild cohomology is computable in principle at least, you can use all the developed techniques of homological algebra. In the case of an ordinary manifold we get

Lemma:

For $A = C^\infty(M)$ the Hochschild cohomology $H^k(A,A^*)$ is identical with the space Ω_k of de-Rham currents of dimension k/6, Lemma 45/ .

Remark:

Look at the multilinear functionals

$$\tau : A\times..\times A \to \mathbb{C}$$
$$(a^o,..,a^k) \mapsto \tau(a^o,..,a^k) \ ?$$

These maps give you a map

$$\tilde{\tau}_{a^1,..,a^k} : A \to \mathbb{C}$$
$$a^o \to \tau(a^o,a^1,..,a^k)$$

where A^* is the dual of A . The maps $\tilde{\tau}$ are the k-cochains of the Hochschild-complex, the coboundary is b , s. the Proposition above. The Hochschild cohomology is defined by

$$H^k(A,A^*) = \frac{\ker b \quad (\text{k-cochains})}{\operatorname{Im} b \quad ((\text{k}-1)\text{-cochains}}$$

The remarkable thing about this Lemma is that you get all the de-Rham-currents, the duals of the differential forms, purely algebraically, without involving exterior algebra considerations. We can get rid of the commutativity of the ground ring and recover the currents and the differential for forms by a theorem, not by definition! This will then work for non-commutative algebras where the notion of exterior algebra is not applicable. The Hochschild cohomology is the right generalization of currents in the non commutative case. In the commutative case the above identification is quite simple:

Given a de-Rham current $C \in \Omega_k$ we assign to it the multilinear form

$$\varphi(f^o, f^1, .., f^k) := < C, f^o df^1 .. df^k >$$

$f^i \in C^\infty(M)$. Notice that I never used properties of locality in the definition of the multilinear functionals defining the cochains of the Hochschild-complex. It is a corollary of the theorem that any element in the Hochschild-cohomology class is equivalent to a local one.

Our aim is the computation of the cyclic cohomology. The cyclic complex is a subcomplex of the Hochschild-complex but not a retraction. The latter would imply that the cohomology groups injected. To show you that this is not so, consider the following simple example.

Example 10:

Let $A = \mathbb{C}$. Then $H^k(\mathbb{C},\mathbb{C}) = 0$, the Hochschild complex is trivial. For the cyclic complex you get:

Let $\varphi \in H^k_\lambda(\mathbb{C})$, then by multilinearity

$$\varphi(a^o, .., a^k) = a^o .. a^k \varphi(1,1,..,1)$$

and by cyclicity $H^{2k+1}_\lambda(\mathbb{C}) = 0$ but $H^{2k}_\lambda(\mathbb{C}) = \mathbb{C}$

A generator of $H^2_\lambda(\mathbb{C})$ is given by $\sigma(1,1,1) = 1$ s./6, p.105/.
It turns out that S is just the cup-product with this generator.

Summarizing the situation we have

$$H^n_\lambda(A) \overset{S}{\to} H^{n+2}_\lambda(A) \overset{I}{\to} H^{n+2}(A,A^*)$$

where I is the imbedding. There is a universal operator B , defined purely algebraically by

$$B_\varphi = AB_o\varphi$$

where A is the cyclic antisymmetrization, the projection onto the cyclic part, and

$$(B_o\varphi)(a^o,..,a^{n-1}) = \varphi(1,a^o,..,a^{n-1})-(-1)^n \varphi(a^o,..,a^{n-1},1)$$

B_o is the contraction with 1 in the first variable and subtracting a term s.t. it will be zero for cyclic cochains and this operator B continues the above diagram to a long exact sequence

$$\ldots\to H^n_\lambda(A) \xrightarrow{S} H^{n+2}_\lambda(A) \xrightarrow{I} H^{n+2}_\lambda(A,A^*) \xrightarrow{B} H^{n+1}_\lambda(A) \xrightarrow{S} H^{n+3}_\lambda(A) \to \ldots$$

In the case $A = C^\infty(M)$, the $I.B$ operator is exactly the de-Rham boundary for currents. The main lemma in proving the exactness is

<u>Main Lemma</u>: $\dfrac{\operatorname{Im} B \cap \ker b}{b(\operatorname{Im} B)} \cong \dfrac{\ker B \cap \ker b}{b(\ker B)}$ s./6, Lemma 36/

what you get is an exact couple

$$\begin{array}{ccc} & H^*(A,A^*) & \\ {}^{B}\swarrow & & \nwarrow{}^{I} \\ H^*_\lambda(A) & \xrightarrow{S} & H^*_\lambda(A) \end{array}$$

and you can use all the nice tools of homological algebra: For any exact couple we get a spectral sequence. Let

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ k \nwarrow & & \swarrow j \\ & B & \end{array}$$

be an exact sequence of abelian groups (<u>exact couple</u>) and

$$d : B \to B \quad d = j \circ k$$

Then $d^2 = 0$. Define $B^1 := H(B) := \dfrac{\ker d}{\operatorname{Im} d}$ and $A^1 = i(A)$. Then you get a <u>derived couple</u> by defining

$$i^1 : i(A) \to i(A) \qquad\qquad j^1 : i(A) \to H(B)$$
$$(ia) \to i^1(ia) := i \circ i(a) \qquad\qquad (ia) \to j^1(ia) := [ja]$$

and

$$k^1 : H(B) \to i(A)$$
$$[b] \to k^1([b]) := kb$$ s./1, p.155/

It is not difficult to prove that the maps are well-defined and

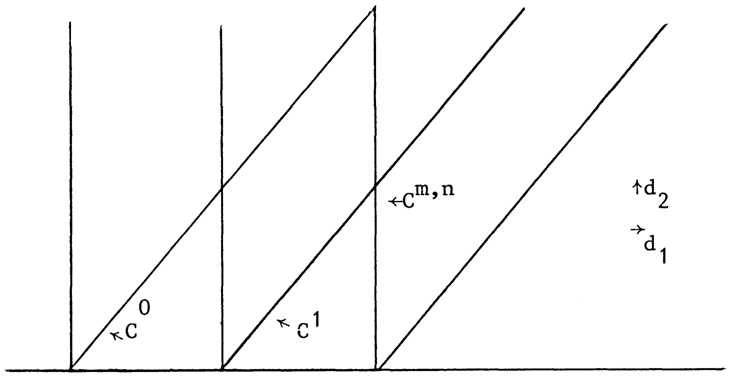
$$\begin{array}{ccc} A^1 & \xrightarrow{i^1} & A^1 \\ k^1 \nwarrow & B^1 & \swarrow j^1 \end{array}$$

is an exact couple. By this construction you get a sequence of exact couples, the spectral sequence of the exact couple. In the case of cyclic cohomology we get another spectral sequence associated to filtrations of a double complex:

We have an exact sequence of complexes

$$0 \to C_\lambda \to C \to C/_{C_\lambda} \to 0$$

C_λ - the cyclic, C the Hochschild complex. Let $C^{m,n} := C^{n-m}(A,A^*)n,m \in \mathbb{Z}$



where

$$d_1\varphi = (n-m+1)b\varphi \in C^{n+1,m} \quad , \varphi \in C^{n,m}$$

$$d_2\varphi = \frac{1}{n-m} B\varphi \quad \in C^{n,m+1} \quad , \varphi \in C^{n,m} \quad C = 0 \quad \text{for} \quad n = m)$$

s. /6, p.123/

by $B^2 = b^2 = Bb + bB = 0$ we get

$$d_1^2 = d_2^2 = d_1d_2+d_2d_1 = 0$$

when you have a double complex of this sort, you get two spectral sequences associated to the two filtrations

$$F_pC = \sum_{n\geq p} C^{n,m}$$

and

$$F^qC = \sum_{m\geq q} C^{n,m}$$

The spectral sequence of the first filtration might not be convergent but for the second we get

Lemma: $H^P(F^qC) = H^n_\lambda(A) \quad$ for $n = P-2q$

s. /6, Th. 40/

where $H^P(F^qC)$ is the cohomology group of the complex (F^qC, d_1)

$$H^P(F^qC) = \frac{\ker d_1 \Gamma(\sum_{m\geq q} C^{P,m})}{\operatorname{Im} d_n \Gamma(\sum_{m\geq q} C^{P-1,m})}$$

Furthermore the spectral sequence associated to the second filtration converges and coincides with the spectral sequence of the exact couple.

To prove this, one needs to use the Main Lemma again.

Let me remind you that the characters τ and $S\tau$ for Fredholm-modules contain essentially the same information and in order to get the non-redundant information, we have to "divide" the cyclic cohomology by the action of S . This is performed by defining

$$H^*(A) := \lim_{\rightarrow}(H^n_\lambda(A),S) = H^*_\lambda(A) \otimes_{\mathbb{C}_p(S)} \mathbb{C}$$

/6, p. 52/

where $\mathbb{C}_p(S)$ denotes the polynomial ring of S and the action of $p \in \mathbb{C}_p(S)$ on $\mathbb{C}$ is given by $p(S)z := P(1)z$.

As the final result you get, that the cohomology of the double complex is given by

$$H^n(C) = H^{even}(A) \qquad n \text{ even}$$

$$H^n(C) = H^{odd}(A) \qquad n \text{ odd}$$

/6, Th. 40/

Let me go back to the aim : to get some information from the Fredholm-modules, there are three steps in doing that :

1) Compute the cyclic cohomology of the algebra
2) Compute the Chern-character of the Fredholm-module
3) State the Index-Theorem.

As an example of this computation, let me return to

Example 7 : (irrational rotation algebra, II)

In the computation of the currents, i.e. the Hochschild cohomology of A_α , one gets :

If α satisfies the diophantine condition, i.e.

$$|1-e^{2\pi i n\alpha}|^{-1} \underset{n\to\infty}{\approx} \mathcal{O}(n^k)$$

for some k , then

$$H^j(A_\alpha, A^*_\alpha) = \mathbb{C}^2 \qquad j = 1,2$$
$$= 0 \qquad \text{otherwise} \quad \text{s./6, p. 133/}$$

but if α doesn't satisfy the diophantine condition H^1, H^2 are infinite dimensional non-Hausdorff spaces. Compare this result with $\alpha \in \mathbb{Q}$: in general A_α is stably isomorphic to $C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}^2$, i.e.

$A_\alpha \otimes K \cong C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}^2$ s./4 , p.145 or 12, p.417/

where K is the ideal of compact operator and the action of $\mathbb{R}^2$ on $C_o(\mathbb{R}^2)$ is given by

$$(\xi(\psi))\ (x) := e^{i2\pi\alpha\xi\wedge x}\ \psi(x-\xi)$$

with $\xi \wedge x = \xi_1 x_2 - \xi_2 x_1$.

By looking at the definition of K-theory et al one notices that two stably isomorphic algebras have the same K-groups et al.

If $\alpha \in \mathbb{Q}$, A_α is thereby stably isomorphic to the commutative algebra of smooth functions on a 2-torus and by the above theorem $H^j(A_\alpha, A_\alpha^*)$ is infinite dimensional for $j \leq 2$. But when we pass to cyclic cohomology we get

$$H^{even}(A_\alpha) \cong \mathbb{C}^2$$

$$H^{odd}\ (A_\alpha) \cong \mathbb{C}^2$$

for <u>all</u> α and as a basis

a) even case τ_o, τ_2

denote $f^j = \sum_{n,m} a^j_{n,m}\ U^n V^m \in A_\alpha$

then $\tau_o(f) = a_{o,o}$

and $\tau_2(f^o, f^1, f^2) = \tau_o(f^o(\partial_1 f^1\ \partial_2 f^2 - \partial_2 f^1\ \partial_1 f^2))$

where ∂_j are defined as before, s. Example 7 I .

b) odd case τ_1, τ_1^c

$$\tau_1(f^o f^1) = \tau_o(f^o \partial_1 f^1)$$

$$\tau_1^c(f^o f^1) = \tau_o(f^o \partial_2 f^1) \qquad \text{s./6, p.138/}$$

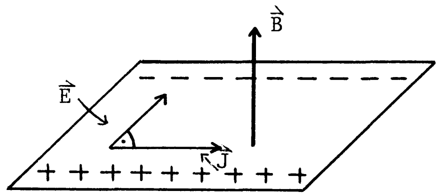
It is an easy exercise to show that these τ are cyclic cocycles.

Lecture IV

<u>The Quantum Hall Effect</u> (after Bellissard)

There are several different approaches to explaining this effect but I want to stress that of Bellissard which, I feel, explains a lot of things in a conceptually simple fashion. /4, 5/

Let me start by describing the classical Hall effect (1880) :



$\vec{B}$ - uniform magnetic field perpendicular to the plane
$\vec{J}$ - electric current along the plane

Hall (1880) observed an electric field $\vec{E}$ perpendicular to $\vec{J}$. The $\pm$ direction of E (sign) depends on the metal used, i.e. whether the electric current is carried by electrons or by holes.

In a stationary state the classical Lorentz-force is

$$Ne\vec{E} + \vec{J} \wedge \vec{B} = 0$$

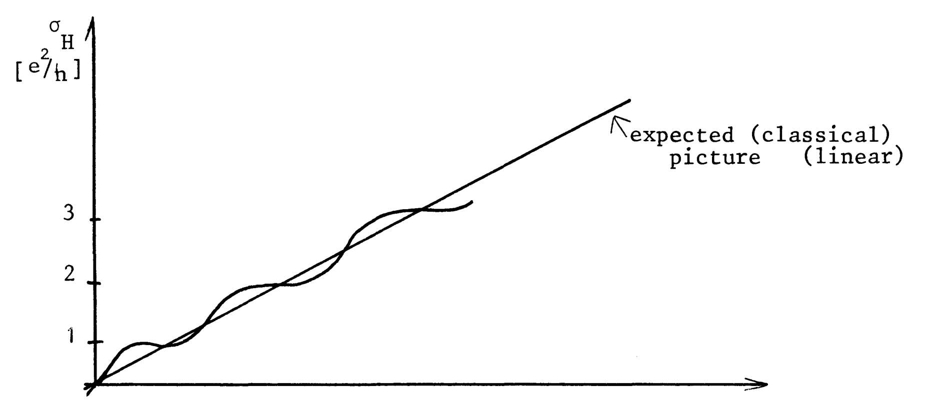
N-charge carrier density = # of charge per unit of surface, and therefore

$$\vec{J} = \frac{Ne\ \vec{B} \wedge \vec{E}}{|\vec{B}|^2} \quad , \qquad |\vec{J}| = \sigma_H |\vec{E}|$$

with $\sigma_H = \rho_H^{-1}$ the Hall conductivity

$\rho_H = Ne/|\vec{B}|$ the Hall resistivity

The Hall conductivity is classically a linear function of the charge-carrier-density. But if one measures at low temperature and in a very large magnetic field $\vec{B}$ (von Klitzing, Pepper and Dorda) one observes



The conductivity becomes stationary for some N , moreover and this is absolutely amazing, the plateaus are exactly at integer multiples of e^2/h with an accuracy better than 10^{-7} . The main surprise was not the Quantum-Hall effect itself but the extremely high accuracy of the result !

I want to give an account for this which is due to Bellissard : /4, 5/

First one has to use Quantum Physics, but we will not use Quantum Field Theory, just the picture of a free Fermi gas, that is a gas of electrons in the 2-dim plane, not interacting with each other. The expectation value of an observable T in this case is given by the thermo-dynamical average w.r.t. the Fermi-Dirac statistic :

$$\langle T\rangle_\beta = \lim_{V\to\infty} \frac{1}{|V|} \text{Trace } \chi_V[(1+e^{\beta(H_\omega-\mu)})^{-1}\, T]$$

$\beta = (KT_o)^{-1}$ - the inverse temperature

χ_V - characteristic function of $V \subset \mathbb{R}^2$

μ - chemical potential (sometimes called Fermi energy)

$$\begin{aligned} H_\omega &= \frac{1}{2m}((p_1-eA_1)^2 + (p_2-eA_2)^2) + V_\omega(x) \\ &= H_o + V_\omega \end{aligned}$$

A_1, A_2 is a suitable gauge-potential for the electromagnetic field, take for example

$$A_1 = |\vec{B}|x_2 \qquad A_2 = |\vec{B}|x_1$$

and V_ω is a potential. If we had a perfect lattice, V_ω would be a periodic potential. In general there are some impurities in the metal and the atoms of the lattice have random positions, so that V_ω depends randomly on a parameter ω , which represents the configuration of disorder.

The average is normalized in such a way that

$$\langle 1\rangle_\beta = \text{charge-carrier-density } N$$

and this fixes the chemical potential μ . To get the Hall conductivity, one has to compute the expectation value of the current. In practice, one induces a current and observes the electric potential E . For doing calculations, one reverses the order, imposes the electric field E and computes the average current in equilibrium. For a stationary state, the result will be the same :

$$\vec{J} = i\, e/\hbar\, [H_\omega, \vec{x}\,]$$

the time evolution, when the electric field E is turned on at time 0 is given by

$$\vec{J}(t) = e^{it/h(H_\omega + \vec{E}\,\vec{x})}\,\vec{J}\,e^{-it/h(H_\omega + \vec{E}\,\vec{x})}$$

the time independent part by

$$\vec{J}_{ave} = \lim_{t\to\infty} \frac{1}{t}\int_o^t < \vec{J}(t) >_\beta \, dt$$

In the case of a perfect lattice and $V_\omega = 0$ one easily computes

$$\vec{J}_o(t) = ie\,\frac{|\vec{E}|}{|\vec{B}|}\,1 + e^{it\omega_c}\,\vec{J}_o \quad , \quad \omega_c = e|\vec{B}|/m \quad \text{cyclotron frequency}$$

and

$$\vec{J}_{o,ave} = ie\,\frac{|\vec{E}|}{|\vec{B}|}\,\lim_{V\to\infty}\frac{1}{|V|}\,\mathrm{trace}(\chi_V\,(1 + e^{\beta(H_o-\mu)})^{-1}\,)$$
$$= iNe|\vec{E}|/|\vec{B}| \quad .$$

Notice that the Hall conductivity is not quantized as a function of the charge-carrier-density as is observed in experiments !

If one rewrites

$$H_o = \frac{1}{2m}\,(k_1^2 + k_2^2) \qquad k_j = (p_j - eA_j)$$

with $[k_1,k_2] = i\hbar\, e/c\ 1$

one sees that H_o describes an harmonic oscillator and

$$N = \lim_{V\to\infty}\frac{1}{|V|}\,\mathrm{trace}\,\chi_V(1+e^{\beta(H_o-\mu)})^{-1} = \sum_{n>o}(1+e^{\beta(\hbar\,\omega_c(n+\frac{1}{2})-\mu)})_{-1}$$

Passing to the zero temperature limit, $\beta \to \infty$, one gets that N as a function of the chemical potential μ (the Fermi-level or energy) becomes a step function. But μ is adjusted by the normalization $<1>_\beta = N$! Moreover one always has disordered configurations, the effective potential will usually be larger than the spacing between two Landau levels, i.e. $\geq \hbar\,\omega_c$. A priori there is no evidence that the step-behavior will survive in experiment.

The Hamiltonian of the system is given by

$$H_\omega = H_o + V_\omega$$

as above. The physical system is homogeneous in space and the translation by $\alpha \in \mathbb{R}^2$ will result in a physically equivalent disordered configuration $T^\alpha\omega$. Notice that an electric-magnetic field acts on the system, the wave-functions are sections in a U(1)-bundle and the translations are given by

$$U(a)\psi(x) = e^{2\pi i|\vec{B}|e/\hbar(x\wedge a)}\psi(x-a)$$

the electromagnetic translations, defined by the gauge-connection (-potential). The homogeneity of the underlying physical situation is reflection by

$$U(a)H_\omega U(a)^* = H_{T^a\omega}$$

and H_ω and $H_{T^a\omega}$ should describe the experiment equivalently. The algebra of observables must contain the subalgebra generated by

$$\{H_{T^a\omega}\}_{a\in\mathbb{R}^2}$$

so that it will be a non-commutative algebra ! The $H_{T^a\omega}$ are unbounded operators and to generate the C^*-algebra of observables, look at the resolvents

$$A_\omega := (z-H_\omega)^{-1} \qquad z \in \mathbb{C} \setminus \mathbb{R} \ .$$

Rewriting this operator in terms of "matrices" w.r.t. the generalized eigenfunctions of the position operator

$$\langle\delta_x, A_\omega\delta_{x'}\rangle$$

and by using the translation properties we get

$$\begin{aligned}\langle\delta_x, A_\omega\delta_{x'}\rangle &= \langle\delta_o, A_{T^{-x}\omega}\delta_{(x'-x)}\rangle e^{i\pi|\vec{B}|e/\hbar(x\wedge x')} \\ &= a(T^{-x}\omega, x-x')\end{aligned}$$

where a is a function on $\Omega\times\mathbb{R}^2$, Ω the parameter set of the disordered configurations. The algebra of observables can be represented by an algebra of functions on $\Omega\times\mathbb{R}^2$, where $\mathbb{R}^2$ acts on Ω and the operator product becomes /4, p.143/

$$a*b(\omega,x) := \int_{\mathbb{R}^2} dx'\ a(\omega,x')\circ(T^{-x'}\omega, x-x')e^{i\pi|\vec{B}|e/\hbar(x\wedge x')}$$

It turns out that the algebra generated by H_ω is isomorphic to $C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}^2$, $\alpha = |\vec{B}|e/\hbar$, s. Example 7 (II). The two derivations of A in Example 7 (II) are now given by

$$\partial_j a(\omega,x) = x_j a(\omega,x)$$

as an easy calculation shows. Going back to the operator, these derivatives are just the commutator with x_j! Now, the current is given by

$$\vec{J} = 2\pi e/\hbar\ [H,\vec{x}]$$

and the function on $\Omega\times\mathbb{R}^2$ which represents $\vec{J}$ is thereby a derivation of a certain function. To compute the Hall conductivity we have to compute a certain trace involving the current $\vec{J}$.

Passing to the zero-temperature limit $B \to \infty$ leads to

$$(1+e^{B(H_\omega-\mu)})^{-1} \underset{B\to\infty}{\longrightarrow} E_\mu \qquad \text{(strongly)}$$

where E_μ is the spectral projection of H_ω on $]-\infty,\mu]$. If μ lies in a gap of the spectrum of H_ω, E_μ is a continuous function of H_ω and an element of the C^*-algebra generated by H_ω . Let us pass to the isomorphic algebra $C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}_2$:

For $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\alpha = |\vec{B}|e/\hbar$ there is a unique trace on this algebra, $\vec{J}$ is represented by a derivation of a function and denote by P_μ the image of E_μ under this isomorphism.

Bellissard proved (using previous work of Thouless et al.) that by the Kubo formula in the zero-temperature-limit one gets for the Hall conductivity:

$$\sigma_H = e^2/\hbar \ \frac{1}{2\pi i} \ \tau(P_\mu[\partial_1 P_\mu, \partial_2 P_\mu])$$

which is exactly the Chern-character of the module associated to the idempotent P_μ! This might explain the "stability" of the step-behavior for disordered configurations.

What is left to prove now is that σ_H is really quantized, i.e. that

$$\tau(P_\mu[\partial_1 P_\mu, \partial_2 P_\mu]) \in \mathbb{Z} \ .$$

This is shown by the Index Theory :

Let $H = H^+ + H^- = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the grading and by identifying $\mathbb{C} \cong \mathbb{R}^2$, $z = x_1 + ix_2$

$$F = \begin{pmatrix} 0 & \frac{z}{|z|} \\ \frac{z^*}{|z|} & 0 \end{pmatrix}$$

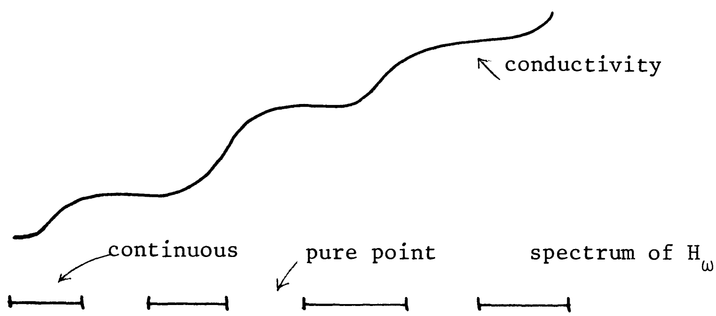
The module-structure is given by the operator-formalism, i.e.

$$[\rho_\omega(a)\psi](x) := \int_{\mathbb{R}^2} d^2y \ a(T^{-x}\omega, y-x) e^{2\pi i |\vec{B}| e/\hbar \ x \wedge y} \ \psi(y)$$

$(H, \rho_\omega, F, \varepsilon)$ is a 2-summable Fredholm-module for $C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}^2$ and

$$\begin{aligned} \tau(P_\mu[\partial_1 P_\mu, \partial_2 P_\mu]) &= \mathrm{trace}(\varepsilon P_\mu dP_\mu dP_\mu) \\ &= \mathrm{ind}(F^+_{P_\mu}) \in \mathbb{Z} \ . \end{aligned}$$

This argument works for $P_\mu \in C_o(\mathbb{R}^2) \rtimes_\alpha \mathbb{R}^2$ that is when μ is in a gap of the spectrum of H_ω . In general this is not true, the spectrum consists of a continuous part and a pure point part, the corresponding "eigenstates" to free electron states resp. localized states. The first contribute to the conductivity the latter represent bound electrons which don't contribute to the conductivity.



The observed plateus correspond exactly to the region of localized states. Bellissard showed that the proof of the Index Theorem works in a slightly more general situation that is if

$$\|\partial_1 P_\mu\|^2 + \|\partial_2 P_\mu\|^2 < \infty$$

The P_μ need not be in the algebra but need only have finite Dirichlet-integral. When you write down this finiteness condition, you can recognize it as exactly the definition of localized states. What you see is that the Index-Theorem and thereby the quantization of σ_H will hold exactly in the region of localized states and will force the conductivity to be an integer multiple of e^2/h .

Let me stop with this and finish with some remarks and open problems.

1) Can one extend the orbifold string theory to more complicated fibers with spectrum one point.

2) The work of A. Jaffe and coll. defines using $F = Q(1+Q^2)^{-1/2}$ a Fredholm module over the algebra of bosons, compute the ∞ dim. cycle assigned to it :

$$\int \omega_2\omega_1 - (-1)^{\partial_1\partial_2} \omega_2\omega_1 = \int d\omega_1 d\omega_2 \qquad \forall\ \omega_1, \omega_2$$

3) Formulate gauge theories for spaces of non integral Hausdorff dimension, using p-summable Fredholm modules + action :

$$\|dA+A^2\|^2_{H.S.}$$

4) Interpret results in totally int. systems of dim. 1. Using the flat connexions on $S^1 \equiv$ Grassmanian + Action :

+ Action $\int AdA + \frac{2}{3} A^3$.

5) Is string theory for a given target space X yielding an action function on the space of 1-summable Fredholm modules.

6) Higher Schwinger terms $(D+1 > 2)$ and regulator maps :

$$K_{alg}(A) \longrightarrow \mathbb{C}^{x}$$

$$\uparrow$$

function on Space

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