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Compact metric spaces, Fredholm modules, and hyperfiniteness

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Abstract. We show that the existence of a finitely summable unbounded Fredholm module (h, D) on a C^* algebra A implies the existence of a trace state on A and that no such module exists on the C^* algebra $C^*_{\text{red}}(\Gamma)$ of a non amenable discrete group. Both for the needs of non commutative differential geometry and of analysis in infinite dimension we are led to the better notion of the θ-summable Fredholm module.

1. *Introduction*

In [**13**] Henry Dye proved that any two hyperfinite measure-preserving actions of countable groups are orbit equivalent, and moreover that groups of polynomial growth only act in a hyperfinite manner.

By a spectacular result of M. Gromov [**15**] groups of polynomial growth are almost nilpotent. Moreover by [**4, 23, 5**] any action of a solvable group or of an arbitrary amenable group is hyperfinite. This settled the problem of determining which groups act in a hyperfinite manner: the amenable ones, but it also showed that as far as measure theory or von Neumann algebras are concerned no distinction occurs between the nilpotent (polynomial growth) case and the solvable (exponential growth) case.

In this paper we shall work at the C^* algebra level, i.e. we are dealing with non commutative analogues of compact spaces, such as the duals of discrete groups. We shall develop for such spaces the analogue of the notion of *metric* on a compact space, examples include ordinary compact Riemannian manifolds as well as the duals of discrete groups Γ on which a word length is specified. We then show that the existence of a metric of *polynomial growth* implies strong results on the C^* algebra, such as the existence of a non trivial positive trace and the hyperfiniteness of the von Neumann algebra generated by certain representations. Since the definition of a metric is closely related to the *unbounded Fredholm modules* of [**1, 7**] we shall apply the above results to rule out the existence of finitely summable unbounded Fredholm modules on the group C^* algebras $C^*(\Gamma)$ for non amenable discrete groups. We also show by an example that groups with property T may have non trivial ordinary (bounded) finitely summable Fredholm modules, but we do not know what happens for lattices in semi-simple Lie groups of rank ≥ 2. For commutative C^* algebras $C(X)$, where X is a compact space, the polynomial growth condition is related to the finiteness of the dimension of X. We check using [17] that in the rigorous Wess Zumino model of A. Jaffe and his collaborators [17], the super charge operator yields an unbounded Fredholm module for the commutative C^* algebras of functions of time 0 boson fields. This module is not finitely summable due to the infinite dimension of X.

Thus we see that both because of the highly non commutative algebras which occur say in group theory, and because of the needs of infinite dimensional analysis, it is necessary to weaken the condition of finite summability used in [7]. This is what we do at the end of this paper, we show that the unbounded operator D in all the above cases satisfies the weaker assumption of θ-summability: $\exp - tD^2$ is of trace class for any $t > 0$. The next step is to extend the construction of the character of [7] to this new class but we shall not deal with this in the present paper.

2. *The non commutative analogue of a metric on a compact space*

Let A be a unital C^* algebra. Let us recall (cf. [1, 7]) that an unbounded Fredholm module $(\mathcal{H}, D)$ over A is given by:

(1) A Hilbert space $\mathcal{H}$ which is a left A-module (i.e. a * representation π of A in $\mathcal{H}$ is given).

(2) An unbounded self adjoint operator D in $\mathcal{H}$ such that:

(α) $\{a \in A; [D, a] \text{ is bounded}\}$ is norm dense in A.

(β) $(1+D^2)^{-1}$ is a compact operator.

Condition (2)(β) means that D has compact resolvent. The set of $a \in A$ satisfying (2)(α) is always a * subalgebra $\mathcal{A}$ of A.

The prototype of such an unbounded Fredholm module is the following: A is the C^* algebra of continuous complex valued functions on a compact manifold M, which we assume for convenience to be spin, $\mathcal{H}$ is the Hilbert space of L^2 spinors on M with A acting by multiplication operators. The operator D is the Dirac operator.

The key observation from which the notion of metric will be extended to the non commutative situation is the following:

PROPOSITION 1. *Let M be a compact, spin, Riemannian manifold, $A = C(M)$, $\mathcal{H} = L^2(M, S)$ and D be as above. Then the geodesic distance $d(P, Q)$ between two points P and Q of M is given by*:

$$d(P, Q) = \text{Sup}\,\{|a(P) - a(Q)|;\ a \in A,\ \|[D, a]\| \le 1\}.$$

If we think of P and Q as characters p, q of A then the above formula reads:

$$d(p, q) = \text{Sup}\,\{|p(a) - q(a)|;\ a \in A,\ \|[D, a]\| \le 1\}.$$

Proof. A continuous function $a \in C(M)$ has a bounded commutator with the Dirac operator iff it is a Lipchitz function on M, and moreover $[D, a]$ is given by the equality:

$$([D, a]\xi)(P) = \gamma((\nabla a)_p)\xi(P)\ \forall P \in M, \forall \xi \in L^2(M, S),$$

where ∇a is the gradient of a and $\gamma(X)$, $X \in T_PM$ is the Clifford multiplication by

X; $\gamma(X) \in \operatorname{End}(S_p)$. Here the section ∇a belongs to $L^\infty(M, TM)$ and:

$$\|[D, a]\| = \operatorname*{Ess\,Sup}_{P \in M} \|(\nabla a)_p\|.$$

Thus $\|[D, a]\|$ is equal to the Lipchitz norm of a for the geodesic distance:

$$\|a\|_L = \operatorname{Sup} |(a(P) - a(Q))| / d(P, Q).$$

Thus

$$\operatorname{Sup} \{|a(P) - a(Q)|; \|[D, a]\| \leq 1\} \leq d(P, Q)$$

and the converse inequality is immediate since the function a

$$a(R) = d(R, Q) \quad \text{satisfies } \|a\|_L = 1, \qquad |a(P) - a(Q)| = d(P, Q). \qquad \square$$

Remark 2. If we use the signature operator instead of the Dirac operator, Proposition 1 still works and applies now to arbitrary L^∞ Riemannian metrics on Lipchitz manifolds, using the results of [**27**].

The next proposition shows that an unbounded Fredholm module $(\mathcal{H}, D)$ over a commutative C^* algebra $A = C(X)$ defines an ordinary metric on the spectrum X of A. Note that its proof does not use condition 2β above.

PROPOSITION 3. *Let $A = C(X)$ be a commutative C^* algebra and $(\mathcal{H}, D)$ an unbounded Fredholm module over A. For any $P, Q \in X = Sp(A)$ let*

$$d(P, Q) = \operatorname{Sup} \{|a(P) - a(Q)|; a \in A, \|[D, a]\| \leq 1\}.$$

(1) *One has $d(P, R) \leq d(P, Q) + d(Q, R)$, $d(P, Q) = d(Q, P)$ for any $P, Q, R \in X$,*
(2) $d(P, Q) = 0 \Rightarrow P = Q$,
(3) *If $\{a; \|[D, a]\| \leq 1\}/\mathbb{C}1$ is bounded, then $d(P, Q) < \infty$ for any $P, Q \in X$.*

Proof. (1) is obvious. (2) follows from the density in A of the subspace $\{a \in A; [D, a]$ bounded$\}$. (3) is obvious. $\square$

The above proposition has an immediate generalisation to the non commutative case:

PROPOSITION 4. *Let A be a C^* algebra and $(\mathcal{H}, D)$ an unbounded Fredholm module over A such that: $(*)\{a \in A; \|[D, a]\| \leq 1\}/\mathbb{C}1$ is bounded. Then the following defines a metric on the state space of A:*

$$d(\varphi, \psi) = \operatorname{Sup} \{|\varphi(a) - \psi(a)|; a \in A, \|[D, a]\| \leq 1\}.$$

The proof is the same as for Proposition 3.

We shall now pass to the case of group C^* algebras $A = C^*_{\text{red}}(\Gamma)$ where Γ is a discrete group. Here A is the C^* algebra generated by the left regular representation λ of Γ in the Hilbert space $\mathcal{H} = l^2(\Gamma)$. We let L be a length function on Γ ([**15**]), i.e. a map $L: \Gamma \to \mathbb{R}_+$ such that:

(1) $L(gh) \leq L(g) + L(h) \quad \forall g, h \in \Gamma$,
(2) $L(g^{-1}) = L(g) \quad \forall g \in \Gamma$,
(3) $L(1) = 0$.

The word length with respect to a system of generators is the prototype of such a length function.

LEMMA 5. *Let Γ be a discrete group, and L a length function on Γ. Let D be the operator of multiplication by L on $l^2(\Gamma)=\mathcal{H}$*

(1) *$(\mathcal{H}, D)$ satisfies conditions (1) and (2)(α).*

(2) *If $L(g)\to\infty$ when $g\to\infty$ in Γ then (2)(β) is satisfied so that $(\mathcal{H}, D)$ is an unbounded Fredholm module over $A=C^*_{\mathrm{red}}(\Gamma)$.*

(3) *For any $g\in\Gamma$ one has $\|[D,\lambda(g)]\|=L(g)$.*

Proof. (1) Condition (1) is obvious; we just need to show that $\mathbb{C}\Gamma$, the group ring of Γ is contained in $\{a\in A;\ \|[D,a]\|<\infty\}$. Thus it is enough to prove (3). The conjugate $\lambda(g)D\lambda(g)^{-1}$ is given by the multiplication by L^g, $L^g(k)=L(g^{-1}k)$ and the conclusion follows from the equality:

$$\operatorname{Sup}_k |L(g^{-1}k)-L(k)|=L(g).$$

Finally (2) is obvious. □

An unbounded Fredholm module $(\mathcal{H}, D)$ over A is *finitely summable* (cf. [7] p. 68) iff for some $p<\infty$ one has:

$$\operatorname{Trace}((1+D^2)^{-p/2})<\infty.$$

For instance the Dirac (or signature) operator on a Riemannian spin (resp. oriented) manifold yields as in Proposition 1 a Fredholm module over $A=C(M)$ which is finitely summable (any $p>\operatorname{Dim}(M)$ works).

PROPOSITION 6. *Let Γ be a finitely generated discrete group, L, $\mathcal{H}$ and $D=D_L$ as in Lemma 5.*

(1) *If Γ has polynomial growth then taking for L the word length the module $(\mathcal{H}, D_L)$ is finitely summable.*

(2) *If $(\mathcal{H}, D_L)$ is finitely summable for some L then Γ has polynomial growth.*

Proof.

(1) Let L_0 be the word length and $B_k=\{g\in\Gamma,\ L_0(g)\le k\}$. There exists by hypothesis constants c, r such that the cardinality $|B_k|\le c(1+k)^r$. It follows that

$$\operatorname{Trace}((1+D^2)^{-p/2})=\sum(1+L_0^2(g))^{-p/2}<\infty \quad \text{for any } p>r+1.$$

(2) There exists a constant λ such that $L\le\lambda L_0$, thus the finiteness of $\sum(1+L^2(g))^{-p/2}$ implies that of

$$\sum(1+L_0^2(g))^{-p/2}=\sum(|B_k|-|B_{k-1}|)(1+k^2)^{-p/2}.$$

Thus there exists $c<\infty$ with $|B_k|<c(1+k)^{p+1}$. □

The conclusion 6(2) is rather weak since it only takes care of Fredholm modules of the form $(\mathcal{H}, D_L)$, thus it does not exclude the existence of finitely summable unbounded Fredholm modules on $A=C^*_{\mathrm{red}}(\Gamma)$, when Γ does not have polynomial growth. We shall reach this conclusion (Theorem 19 below) for certain groups, only later in section V.

Remark 7.

(1) Let $(\mathcal{H}, D)$ be an unbounded Fredholm module over a C^* algebra A and let $\mathcal{U}=\{u$ unitary in A; $[D,u]$ is bounded$\}$. Then $\mathcal{U}$ is a norm dense subgroup of

the unitary group of A and the following defines a length function on $\mathcal{U}$:

$$(l(u)) = \|[D, u]\| \quad \forall u \in \mathcal{U}.$$

(2) Let X be a finite simplicial complex. Using an embedding of X in $\mathbb{R}^N$ and a Lipschitz retraction of a neighborhood V (of X in $\mathbb{R}^N$) on X, together with the construction of Proposition 1 for the double of V one gets an unbounded finitely summable Fredholm module $(\mathcal{H}, D)$ on $A = C(X)$ verifying condition (3) of Proposition 3.

3. *Existence of a trace on C^* algebras with a finitely summable unbounded Fredholm module*

Our aim in this section is to obtain the following important necessary condition on a C^* algebra A for the existence of a finitely summable unbounded Fredholm module $(\mathcal{H}, D)$ on A:

THEOREM 8. *Let A be a unital C^* algebra and $(\mathcal{H}, D)$ an unbounded Fredholm module over A such that for some $p < \infty$ $(1 + D^2)^{-p/2}$ is of trace class. Then there exists a positive trace τ on A such that $\tau(1) = 1$.*

Proof. We shall construct a family $T(\varepsilon)$ of elements of $\mathcal{L}^1(\mathcal{H})$ such that $T(\varepsilon) > 0$, Trace $T(\varepsilon) = 1$ and that for any $a \in A$ one has $\|[a, T(\varepsilon)]\|_1 \to 0$ when $\varepsilon \to 0$. Let then φ_ε be the state on $\mathcal{L}(\mathcal{H})$ given by $\varphi_\varepsilon(x) = \text{Trace}\,(T(\varepsilon)x)\ \forall x \in \mathcal{L}(\mathcal{H})$. For any weak limit φ of the φ_ε's, the C^* algebra A is in the centralizer of the (non normal) state φ so that φ/A is a tracial state. Thus we just need to prove the following lemma:

LEMMA 9. *Let $k \geq p/2$ be an integer and $T(\varepsilon) = A(\varepsilon)/\text{Trace}\,(A(\varepsilon))$, $A(\varepsilon) = (1 + \varepsilon D^2)^{-k}$. Then $T(\varepsilon) \in \mathcal{L}^1(\mathcal{H})$, $T(\varepsilon) > 0$, $\text{Trace}\,(T(\varepsilon)) = 1$, $\|[a, T(\varepsilon)]\|_1 \to 0$ when $\varepsilon \to 0$, for any $a \in A$.*

Proof. The first assertions are clear, so we have to estimate $\|[a, T(\varepsilon)]\|_1$. We may assume that $a \in \mathcal{A} = \{a \in A, [D, a] \text{ bounded}\}$. One has

$$[a, A(\varepsilon)] = \sum_{l=0}^{k-1} (1 + \varepsilon D^2)^{-l}[a, (1 + \varepsilon D^2)^{-1}](1 + \varepsilon D^2)^{-k+l+1}, \tag{1}$$

$$\begin{aligned}[a, (1 + \varepsilon D^2)^{-1}] &= (1 + \varepsilon D^2)^{-1}\varepsilon D[D, a](1 + \varepsilon D^2)^{-1} \\ &\quad + (1 + \varepsilon D^2)^{-1}[D, a]\varepsilon D(1 + \varepsilon D^2)^{-1} \\ &= \varepsilon^{1/2} N_\varepsilon [D, a](1 + \varepsilon D^2)^{-1} + \varepsilon^{1/2}(1 + \varepsilon D^2)^{-1}[D, a]N_\varepsilon,\end{aligned} \tag{2}$$

where $N_\varepsilon = (1 + \varepsilon D^2)^{-1}\varepsilon^{1/2} D$ is bounded uniformly: $\|N_\varepsilon\| \leq \frac{1}{2}$. Thus, since $[D, a]$ is bounded, we see that $[a, T(\varepsilon)]$ is a finite sum of terms of the form:

$$\varepsilon^{1/2}(1 + \varepsilon D^2)^{-r} P_\varepsilon (1 + \varepsilon D^2)^{-s},$$

where Sup $\|P_\varepsilon\| < C$, and $r + s = k$, $r \geq 0$, $s \geq 0$. We just have to check that

$$\|(1 + \varepsilon D^2)^{-r} P_\varepsilon (1 + \varepsilon D^2)^{-s}\|_1 \leq C\ \text{Trace}\,((1 + \varepsilon D^2)^{-k}).$$

For $r = 0$ or $s = 0$ this is clear. Otherwise one has:

$$\|(1 + \varepsilon D^2)^{-r}\|_{k/r} = (\text{Trace}\,(1 + \varepsilon D^2)^{-k})^{r/k},$$

$$\|P_\varepsilon(1 + \varepsilon D^2)^{-s}\|_{k/s} \leq C(\text{Trace}\,(1 + \varepsilon D^2)^{-k})^{s/k}.$$

Thus the result follows from the Holder inequality. □

Remark 10.

(a) With the above notations, if the C^* algebra A has a unique tracial state τ (cf. [3] for examples), then the restrictions $\psi_\varepsilon = \varphi_\varepsilon / A$ of φ_ε ($\varphi_\varepsilon(x) = \text{Trace}\,(T(\varepsilon)x)\ \forall x \in \mathscr{L}(\mathscr{H})$) converge weakly to τ when $\varepsilon \to 0$.

(b) Lemma 9 shows that under the hypothesis of Theorem 8 there exists a (non normal) state φ on $\mathscr{L}(\mathscr{H})$ whose centralizer contains A.

As a corollary of Theorem 8, we see that on a C^* algebra without traces, such as the Cuntz algebra $\mathcal{O}_n$, one cannot have finitely summable unbounded Fredholm modules. The theorem is however inefficient for group C^* algebras $A = C^*_{\text{red}}(\Gamma)$ since all such C^* algebras have a (faithful) trace. We shall improve it in the next sections.

4. *Hyperfiniteness of the weak closure of a C^* algebra in a finitely summable unbounded Fredholm module*

Let A be a unital C^* algebra and $(\mathscr{H}, D)$ a finitely summable unbounded Fredholm module. Let k be such that

$$(1+D^2)^{-k} \in \mathscr{L}^1 \quad \text{and} \quad T(\varepsilon) = (1+\varepsilon D^2)^{-k} / \text{Trace}\,((1+\varepsilon D^2)^{-k})$$

as in § 3.

THEOREM 11. *Assume that there exists a faithful normal state τ on A'', the weak closure of A in $\mathscr{L}(\mathscr{H})$, such that*:

$$(*)\ \text{Trace}\,(T(\varepsilon)a) \underset{\varepsilon \to 0}{\to} \tau(a) \quad \forall a \in A''.$$

Then A'' is a hyperfinite, finite von Neumann algebra.

Proof. By Lemma 9 the centralizer of τ contains A and hence A''. Let φ be any weak limit of the states $\varphi_\varepsilon = \text{Tr}\,(T(\varepsilon).)$ on $\mathscr{L}(\mathscr{H})$. In general φ is not normal, but our hypothesis implies that $\varphi(a) = \tau(a)\ \forall a \in A''$. It follows that the centraliser of φ contains A''. Indeed, let $z \in A''$, then for any $\varepsilon > 0$ there exists $y \in A$ such that $\tau((y-z)^*(y-z)) \le \varepsilon^2$ (and $\|y\| \le \|z\|$) ([12]) thus in the GNS construction of the pair $(\mathscr{L}(\mathscr{H}), \varphi)$ one has:

$$\|(y-z)\xi_\varphi\| \le \varepsilon, \ \|(y^* - z^*)\xi_\varphi\| \le \varepsilon.$$

It follows that for any $x \in \mathscr{L}(\mathscr{H})$ one has:

$$|\psi(zx - xz)| = |\langle (z-y)x\xi_\varphi, \xi_\varphi\rangle + \langle x(z-y)\xi_\varphi, \xi_\varphi\rangle| \le 2\|x\|\varepsilon.$$

Thus $A'' \subset \mathscr{L}(\mathscr{H})_\varphi$ and the state φ is a hypertrace ([8]) on the finite von Neumann algebra A'' so that by [8], A'' is hyperfinite. □

Before we proceed and use this theorem to rule out the existence of finitely summable unbounded Fredholm modules on certain group C^* algebras we shall show that the assumption of Theorem 11 is verified for Riemannian manifolds.

PROPOSITION 12. *Let M be a compact, spin, Riemannian manifold, $A = C(M)$ and $\mathscr{H}$, D as in Proposition 1. Then for any $a \in A'' = L^\infty(M)$, and $k > \frac{1}{2} \dim M$, one has*:

$$\lim_{\varepsilon \to 0} \text{Trace}\,(T(\varepsilon)a) = \left(\int_M a\, dv\right) \Big/ \text{Vol}\,(M),$$

where $T(\varepsilon)=(1+\varepsilon D^2)^{-k}/\text{Trace}\,((1+\varepsilon D^2)^{-k})$ *and* dv *is the Riemannian volume element.*

Proof. Let $n=\dim M$. Replacing ε by ε^2 we see that εD and hence $(1+\varepsilon^2 D^2)^{-k}$ are asymptotic differential operators with symbols respectively $\gamma(\xi)$ and $(1+\|\xi\|^2)^{-k}$. It follows that Trace $((1+\varepsilon^2 D^2)^{-k})$ is of the order of ε^{-n} when $\varepsilon\to 0$ and that the diagonal values of the kernel $k_\varepsilon(x,y)$ defined by $(1+\varepsilon^2D^2)^{-k}$ are measures $N_\varepsilon=f_\varepsilon\, dv$ where

$$\operatorname{Lim} \varepsilon^n f_\varepsilon(x)=C\int_{T_x^*}(1+\|\xi\|^{-k}\,d\xi),\quad (\text{cf. [14]}).$$

As another example we shall check:

Proposition 13. *Let* $\theta\in[0,1]$ *and* A_θ *the irrational rotation* C^* *algebra* $\delta_1\delta_2$ *the natural derivations of* A_θ([**6**, **9**]) *and* $\partial=\delta_1+i\delta_2$, D *the operator* $\left[\begin{smallmatrix}0 & \partial\\ \partial^* & 0\end{smallmatrix}\right]$ *acting in the Hilbert space* $\mathcal{H}=L^2(A_\theta,\tau)\oplus L^2(A_\theta,\tau)$ *where* τ *is the canonical trace. Then for any* $k>1$ *one has*:

$$\text{Trace}\,(T(\varepsilon)a)=\tau(a)\quad \forall a\in A_\theta'',\forall\varepsilon>0.$$

Proof. The trace τ on A_θ is the unique state invariant under the action of the 2-torus T^2 generated by δ_1 and δ_2. Since this action lifts to $\mathcal{H}$ and commutes with D each state of the form:

$$\varphi(a)=\text{Trace}\,(T(\varepsilon)a)$$

is invariant under the action of T^2 and hence equal to τ. □

5. *Group* C^* *algebras with no absolutely continuous, finitely summable, unbounded Fredholm module*

We let τ be a tracial state on a unital C^* algebra A and $L^2(A,\tau)$ the completion of A for the inner product $\langle x,y\rangle=\tau(xy^*)$. It is in a natural manner *a* bimodule over A.

Lemma 14. *The following two conditions on* A, τ *are equivalent*:

(1) *For any* $\varepsilon>0$, *there exists elements* $a_1,\dots,a_n$ *of* A *with*:

$$\xi\in L^2,\ \|\xi\|=1\,\|a_j\xi-\xi a_j\|\le 1\quad \forall j=1,\dots,n\Rightarrow\|\xi-\langle\xi,1\rangle 1\|<\varepsilon.$$

(2) *For any* $\alpha>0$, *there exists elements* $a_1,\dots,a_n\in A$ *such that for any state* φ *on* A *absolutely continuous with respect to* τ *one has*:

$$\|[a_j,\varphi]\|\le 1\quad\forall j=1,\dots,n\Rightarrow\|\varphi-\tau\|<\alpha.$$

Proof. (1)⇒(2) Any state on A which is absolutely continuous with respect to τ is represented uniquely in the form $\varphi(a)=\langle a\xi,\xi\rangle$ for a unit vector $\xi=\varphi^{1/2}\in L^2(A,\tau)^+$ (cf. [**12**]), moreover by the Powers Stormer inequality ([**24**]) one has

$$\|\varphi^{1/2}-\psi^{1/2}\|_2\le\|\varphi-\psi\|^{1/2}\le\|\varphi^{1/2}-\psi^2\|_2^{1/2}\|\varphi^{1/2}+\psi^{1/2}\|_2^{1/2}.$$

Let the a_j of (1) be of the form λu_j where the u_j's are unitaries, then for any j,

$$\|[\lambda^2u_j,\varphi]\|\le 1\Rightarrow\|u_j\varphi u_j^{-1}-\varphi\|\le\lambda^{-2}\Rightarrow\|[a_j,\varphi^{1/2}]\|_2\le 1$$

and we get the conclusion.

(2)$\Rightarrow$(1) Follows by a similar argument. □

Example 15. Let Γ be the free group on 2 generators, then on the reduced C^* algebra $A = C^*_{\text{red}}(\Gamma)$, the canonical trace τ satisfies the equivalent conditions of Lemma 14 (cf. [**26**]).

We refer to [**3**] for other examples of groups Γ for which $(C^*_{\text{red}}(\Gamma), \tau)$ satisfies the equivalent conditions of Lemma 14, as well as for a number of equivalent conditions on the group Γ.

THEOREM 16. *Let A, τ satisfy the equivalent conditions of Lemma* 14, *with* $\dim L^2(A, \tau) = +\infty$. *Let $\mathcal{H}$ be the left module $L^2(A, \tau)$ or any quasi-equivalent left module. Then there exists no unbounded selfadjoint operator D in $\mathcal{H}$ such that* (1) $(1+D^2)^{-1} \in \mathcal{L}^p(\mathcal{H})$ *for some finite p,* (2) $\{a \in A, [D, a]$ *bounded*$\}$ *is dense in A.*

Proof. Assuming the existence of D, let $T(\varepsilon) \in \mathcal{L}^1(\mathcal{H})$ be as in Lemma 9. Let ψ_ε be the restriction to A of $\varphi_\varepsilon = \text{Trace}\,(T(\varepsilon))$. Given $a_1, \ldots, a_n \in A$ there exists (Lemma 9) $\delta > 0$ such that

$$\varepsilon < \delta \Rightarrow \|[a_i, \psi_\varepsilon]\| \leq 1.$$

Thus by 14(2), we get $\|\psi_\varepsilon - \tau\| \to 0$ when $\varepsilon \to 0$. Since both ψ_ε and τ are continuous for the ultraweak topology in $\mathcal{H}$ ([**12**]) it follows that $\|\varphi_\varepsilon / A'' - \tau_{A''}\| \to 0$ when $\varepsilon \to 0$, where $\tau_{A''}$ is the continuous extension of τ to A''. Thus one has:

$$\text{Trace}\,(T(\varepsilon)a) \to \tau_{A''}(a) \quad \forall a \in A''$$

and Theorem 11 shows tht A'' is hyperfinite (and finite). Now since $L^2(A, \tau)$ is infinite dimensional, there exists nontrivial central sequences

$$(\xi_n)_{n \in \mathbb{N}}, \xi_n \in L^2(A, \tau), \|\xi_n\| = 1, \|a\xi_n - \xi_n a\| \to 0$$

when $n \to \infty$, $\langle \xi_n, 1 \rangle = 0\ \forall n \in \mathbb{N}$. Thus we contradict 14(1). □

THEOREM 17. *Let Γ be a discrete group which contains the free group on* 2 *generators. Let $\mathcal{H}$ be any representation of $C^*(\Gamma)$, absolutely continuous with respect to the canonical trace on $C^*(\Gamma)$. Then there exists no selfadjoint operator D in $\mathcal{H}$ such that $(\mathcal{H}, D)$ is a finitely summable unbounded Fredholm module over $A = C^*_{\text{red}}(\Gamma)$.*

Proof. Let $F_2 \subset \Gamma$ be a copy of the free group on 2 generators, by hypothesis the representation of Γ in $\mathcal{H}$ is subequivalent to the infinite direct sum of copies of the left regular representation of Γ in $l^2(\Gamma)$. Thus its restriction to F_2 is subequivalent to an infinite direct sum of copies of the regular representation of F_2. Thus, as a module over $C^*_{\text{red}}(F_2)$, $\mathcal{H}$ is quasi equivalent to the regular representation. Now let D in $\mathcal{H}$ be such that (1) $(1+D^2)^{-p}$ is of trace class for some finite p, (2) $\{a \in A = C^*_{\text{red}}(\Gamma)\colon [D, a]$ bounded$\}$ is dense in A. We cannot replace A by $C^*_{\text{red}}(F_2)$ since (2) does not necessarily hold for the latter. Let $T(\varepsilon)$, φ_ε, ψ_ε be as in Theorem 16, with τ the canonical trace on A. By Example 15 and Lemma 9, we get:

$$\|\varphi_\varepsilon / B'' - \tau / B''\| \to 0 \quad \text{when } \varepsilon \to 0$$

where $B = C^*_{\text{red}}(F_2) \subset A$ and B'' is its weak closure in $\mathcal{H}$. Thus again any weak limit φ of φ_ε on $\mathcal{L}(\mathcal{H})$ defines a hypertrace on B'' which is a contradiction since B'' is not hyperfinite [**26**]. □

Remark 18. The above theorem does not exclude the existence of finitely summable bounded Fredholm modules, $(\mathcal{H}, F)$ over $A = C^*_{\text{red}}(\Gamma)$ with Γ as above. In fact an example is given in [7] p. 55 of such a module, absolutely continuous with respect to the canonical trace, and defining a non trivial K homology class.

6. *Group C^* algebras with no finitely summable unbounded Fredholm module*
Let Γ be a discrete group. Let us recall that Γ has property T of Kazhdan [**20**] iff the following holds:

For any $\varepsilon > 0$, there exists $\delta > 0$ and elements $g_1, \ldots, g_n$ of Γ such that for any representation π of Γ in a Hilbert space H:

$$\xi \in H,\ \|\xi\| = 1,\ \|\pi(g_i)\xi - \xi\| \le \delta \quad \forall i = 1, \ldots, n \Rightarrow \|\xi - P(\xi)\| \le \varepsilon,$$

where P is the projection on $H_0 = \{\xi \in H;\ \pi(g)\xi = \xi\ \forall g \in \Gamma\}$.

THEOREM 19. *Let Γ be an infinite discrete group with property T, and $A = C^*_{\text{red}}(\Gamma)$ be the reduced group C^* algebra of Γ. Then there exists no finitely summable unbounded Fredholm module over A.*

Proof. Let $(\mathcal{H}, D)$ be a finitely summable unbounded Fredholm module over A, and $T(\varepsilon) \in \mathcal{L}^1(\mathcal{H})$ be as in Lemma 9. Let $H = \mathcal{L}^2(\mathcal{H})$ be the Hilbert space of Hilbert Schmidt operators in $\mathcal{H}$ and let $\xi(\varepsilon) = T(\varepsilon)^{1/2} \in \mathcal{L}^2(\mathcal{H}) = H$. Let Γ act in H as follows:

$$\pi(g)\xi = g\xi g^{-1} \quad \forall g \in \Gamma, \quad \forall \xi \in \mathcal{L}^2(\mathcal{H}).$$

The Power-Stormer inequality [**24**] combined with Lemma 9, shows that when $\varepsilon \to 0$ one has:

$$\|\pi(g)\xi(\varepsilon) - \xi(\varepsilon)\| \to 0.$$

Since $\|\xi(\varepsilon)\| = 1$, property T shows that the representation π of Γ in H contains the trivial representation. Now one has $H = \mathcal{H} \otimes \bar{\mathcal{H}}$ and π is equivalent to $\rho \otimes \bar{\rho}$ where ρ is the original representation of Γ in $\mathcal{H}$. But since ρ defines a representation of the reduced C^* algebra $C^*_{\text{red}}(\Gamma)$, it is weakly contained in the left regular representation λ of Γ in $l^2(\Gamma)$. Thus, so is $\rho \otimes \bar{\rho}$ and it follows from the above discussion, that the trivial representation of Γ is weakly contained in the regular one, which contradicts the non amenability of Γ.† □

The next proposition shows however the existence of finitely summable bounded Fredholm modules with non trivial K homology class on the group C^* algebra $C^*_{\text{red}}(\Gamma)$ of certain groups with property T. We let G be a real semi simple Lie group of rank one, and Γ a discrete subgroup of G. We let S be the spinor bundle on the symmetric space $H = G/K$, K maximal compact subgroup of G, and $\mathcal{H} = l^2(\mathcal{O}, S)$ be the Hilbert space of l^2 sections of S on a fixed orbit of Γ. The group Γ acts unitarily in $\mathcal{H}$ and this action endows $\mathcal{H}$ with a left module structure on $A = C^*_{\text{red}}(\Gamma)$. Finally (cf. [**22, 18, 7**]) we let $F \in \mathcal{L}(\mathcal{H})$ be the operator:

$$(F\xi)(x) = \gamma\left(\frac{(x, 0)}{\|(x, 0)\|}\right)\xi(x) \quad \forall x \in \mathcal{O}, \quad \forall \xi \in \mathcal{H},$$

† As pointed out by G. Skandalis this proof applies to non amenable groups.

where $\gamma(X)$ is Clifford multiplication by X and where $(x,0)/\|(x,0)\|$ is the unit tangent vector at $x\in H$ whose geodesic contains 0.

PROPOSITION 20.

(1) *The pair $(\mathcal{H}, F)$ is a finitely summable Fredholm module over $A = C^*_{\text{red}}(\Gamma)$.*

(2) *If $G = Sp(n, 1)$ and Γ has finite covolume then Γ has property T.*

(3) *If Γ is cocompact, the class of $(\mathcal{H}, F)$ in $K^*(A)$ is not trivial.*

Proof.

(1) For any $g\in\Gamma$ the operator gFg^{-1} is given by:

$$((gFg^{-1})\xi)(x) = \gamma\left(\frac{(x, g(0))}{\|(x, g(0))\|}\right) \quad \forall x\in\mathcal{O}, \quad \forall \xi\in l^2(\mathcal{O}, S)$$

thus $T = gFg^{-1} - F$ is a diagonal operator in $l^2(\mathcal{O}, S)$, and the norm of T_x (with $(T\xi)(x) = T_x\xi(x)\ \forall\xi\in l^2$) is of the order of $\exp - d(0, x)$ (cf. [7]). As Γ is discrete the number $N(\rho)$ of elements $y\in\mathcal{O}$ with $d(0, y)\le\rho$ is bounded by a constant multiple of the volume of the ball of radius ρ in H, which in turn is of the order of $\exp(k\rho)$ for some finite k. Thus:

$$\sum \|T_x\|^p < \infty \quad \forall p > k,$$

and $(\mathcal{H}, F)$ is finitely summable.

(2) cf. [**21**].

(3) Let $z\in K_*(C^*_{\text{red}}(\Gamma))$ be the K theory class constructed from the fundamental class of $V = \Gamma\backslash H$ in K homology by the map μ of [**18**, **2**]. The results of [**18**] show that the index pairing: $K_*(A)\times K^*(A)\to\mathbb{Z}$ evaluated on $z\times(\mathcal{H}, F)$ gives the same result as the pairing: $K_*(V)\times K^*(V)\to\mathbb{Z}$ evaluated on $[V]\times\beta$ where $[V]$ is the fundamental class of V in K homology and $\beta\in K^*(V)$ the Bott element. Thus the class of $(\mathcal{H}, F)$ is non zero. □

Finally we shall relate finitely summable Fredholm modules on the group ring $A = \mathbb{C}\Gamma$ of a discrete group with group cohomology with coefficients in representations, and exploit results of [**28**] and [**11**] to exclude the existence of non trivial 2-summable Fredholm modules on $\mathbb{C}\Gamma$ when Γ has property T. (Thus in Proposition 20 the degree of summability if >2.)

PROPOSITION 21. *Let Γ be a discrete group and $(\mathcal{H}, F)$ a p-summable Fredholm module over $A = \mathbb{C}\Gamma$.*†

(1) *The map $c\colon \Gamma\to\mathcal{L}^p(\mathcal{H})$, $c(g) = gFg^{-1} - F$ is a 1-cocycle for the action of Γ on $\mathcal{L}^p$ by conjugation.*

(2) *Let $n\in\mathbb{N}$ have the parity of $(\mathcal{H}, F)$ and satisfy $n\ge p$. Let τ be the n-dimensional character of $(\mathcal{H}, F)$, then:*

$$\tau(g^0, g^1, \ldots, g^n) = \mathrm{Tr}_\varepsilon(c^n(g^1, \ldots, g^n)) \quad \forall g^i\in\Gamma, g^0, g^1, \ldots, g^n = 1,$$

where c^n is the cup product of n copies of c; $c^n\in H^n(\Gamma, \mathcal{L}^1)$ while Tr_ε is the trace if n is odd and the super trace if n is even.

(3) *If Γ has property T and $p = 2$ then there exists a Hilbert Schmidt operator Δ such that $F+\Delta$ commutes with Γ, i.e. $(\mathcal{H}, F)$ is trivial.*

† I.e. $[F, a]\in\mathcal{L}^p(\mathcal{H})\ \forall a\in A$.

Proof.

(1) One checks that $c(g_1 g_2) = c(g_1) + g_1 c(g_2) g_1^{-1}\ \forall g_1, g_2 \in \Gamma$.

(2) The composition of operators defines a multilinear Γ-equivariant map $\mathscr{L}^p(\mathscr{H}) \otimes \cdots \otimes \mathscr{L}^p(\mathscr{H}) \to \mathscr{L}^1(\mathscr{H})$, so that the cup product c^n is well defined. One has

$$c^n(g_1, g_2, \ldots, g_n) = c(g_1) g_1 c(g_2) g_2 c(g_3) \cdots g_{n-1} c(g_n) g_{n-1}^{-1} \cdots g_1^{-1}.$$

Since $c(g)g = (gF - Fg)$ equality (2) follows easily.

(3) By [**28, 11**] the cocycle c is a coboundary, $c(g) = g\Delta g^{-1} - \Delta$. □

Remark 22. We do not know any example of non trivial finitely summable Fredholm module on $C^*_{\text{red}}(\Gamma)$ or $\mathbb{C}\Gamma$ for Γ a lattice in a Lie group of rank >1.

7. *θ-summable unbounded Fredholm modules*

The results of §§ 4 and 5 show that certain C^* algebras do not possess any finitely summable unbounded Fredholm module, and so in the sense of § 1, have no metric of polynomial growth. We shall now give a weaker condition on an unbounded Fredholm module and examples of such Fredholm modules both on the group C^* algebra of any finitely generated group, and on the infinite dimensional space occurring in constructive quantum field theory.

DEFINITION 23. *We shall say that an unbounded Fredholm module $(\mathscr{H}, D)$ over a C^* algebra A is θ-summable iff*

$$\operatorname{Trace}(e^{-tD^2}) < \infty \quad \forall t > 0.$$

This condition implies that $(1 + D^2)^{-1}$ is a compact operator, so that by [**1**] the pair $(\mathscr{H}, F)$ where $F = D(1 + D^2)^{-1/2}$ defines an element of the Kasparov group $KK(A, \mathbb{C})$.

Obviously every finitely summable $(\mathscr{H}, D)$ is also θ-summable and as in [**10**] one gets a bound: Trace $(e^{-tD^2}) = 0(t^{-p})$ when $t \to 0$. We shall now give general existence results for such modules.

PROPOSITION 24. *Let Γ be a finitely generated discrete group, and l the word length function, relative to some generating subset. Let $\mathscr{H} = l^2(\Gamma)$ as a left module over $A = C^*_{\text{red}}(\Gamma)$ and D be the multiplication by l. Then $(\mathscr{H}, D)$ is a θ-summable unbounded Fredholm module over A.*

Proof. We just have to check that $\sum_{g \in \Gamma} \exp - t l^2(g) < \infty, \forall t > 0$, which is immediate since with the notations of Proposition 6, $|B_k|$ grows at most as q^k for some finite q. □

Next we shall check that the natural Fredholm modules on the group C^* algebras $C^*(\Gamma)$ where Γ is a discrete subgroup of a semi simple Lie group G, are θ-summable. Let us recall their construction [**19**]. Let $H = G\backslash K$ be the quotient of G by a maximal compact subgroup, and endow H with its G-invariant Riemannian metric of negative curvature. Let V be the (not necessarily compact) manifold $V = \Gamma\backslash H$ and on V consider the bundle of Hilbert space $(H_a)_{a \in V}$, obtained by:

$$H_a = l^2(\pi^{-1}(a), S_a) \quad \forall a \in V,$$

where $\pi: H \to \Gamma\backslash H$ is the projection and S_a is the restriction to $\pi^{-1}\{a\}$ of the spinor bundle on H. We assume for simplicity that H is even dimensional and that the isotropy representation of K lifts to spin, so that both S and its $\mathbb{Z}/2$ grading $S = S^+ \oplus S^-$ are well defined. Now, once an origin $0 = 1/K \in H$ has been chosen, the bundle H has a natural superconnexion in the sense of [25], given by the combination $\nabla + \delta$ of the Levi Civita connexion ∇ of the spinor bundle S on H (viewed on $\Gamma\backslash H = V$ as a connexion on H) and of the dual Dirac operator δ which for each $a \in V$ defines the following odd endomorphism of H_a:

$$(\delta_a \xi)(x) = \gamma((x,0))\xi(x) \quad \forall x \in \pi^{-1}\{a\}.$$

Here $\xi(x) \in S_x$ and $\gamma((x,0))$ is the Clifford multiplication in S_x by the element $(x,0)$ of $T_x(H)$ whose exponential is equal to $0 \in H$. Thus we get a natural Dirac operator on $\Gamma\backslash H$ with coefficients in H given as an operator D in $L^2(V, S_V \otimes H)$, where S_V is the spinor bundle on V.

If we identify $L^2(V, S_V \otimes H)$ with the Hilbert space $\mathcal{H}$ of L^2 differential forms on $H = G/K$, using the natural isomorphism between the tensor product $S_x \otimes S_x$ and the exterior algebra ΛT_x^*, for $x \in H$, both the operator D and the action of $C^*_{\text{red}}(\Gamma)$ become explicit and simple:

(1) One has $(Dw)(x) = ((d+d^*)w)(x) + X(x)\Lambda w(x) - i_{X(x)}w(x)$ where $X(x) = (x,0)$ is the tangent vector at x whose image under $\exp_x$ is $0 \in H$.

(2) The group Γ acts in $\mathcal{H}$ by left translations.

PROPOSITION 25. *The pair $(\mathcal{H}, D)$ is a θ-summable unbounded Fredholm module over $C^*_{\text{red}}(\Gamma)$.*

Proof. Classical results [4] show that D is essentially selfadjoint on the domain $C_c^\infty(H, \Lambda T^*)$ of smooth forms with compact support. A simple calculation shows that the square of D is the same as the Laplacian used in [16] and [29] for the Morse function $W(x) = (d(0,x))^2$ on H. Thus up to lower order terms D^2 is equal to $\Delta_H + W$ and it follows from the Golden–Thomson inequality that $\exp(-tD^2)$ is of trace class for any finite $t > 0$. Finally for any $g \in \Gamma$ the left translation $\lambda(g)$ by g in $\mathcal{H}$ commutes with $d + d^*$ and the inequality $\|(x,0) - (x, g^{-1}(0))\| \le d(0, g^{-1}(0))$ (which follows from the comparison theorem applied to the triangles $(0, (x,0), (x, g^{-1}(0)))$ in $T_x(H)$ and $(x, 0, g^{-1}(0))$ in H) shows that $[\lambda(g), D]$ is a bounded operator for any $g \in \Gamma$. □

The results of Kasparov [19] show that, if in the above construction one allows to take coefficients in a (finite dimensional) vector bundle E over V, one obtains enough Fredholm modules on $C^*_{\text{red}}(\Gamma)$ to show that the map $\mu: K_*(V) \to K_0(C^*_{\text{red}}(\Gamma))$ is injective. The proof of proposition 23 easily adapts to show that all these modules are θ-summable.

We shall end this section with another example of a θ-summable Fredholm module in which the lack of finite summability is no longer due as above to the non commutativity of the algebra A but to the infinite dimensionality of the space X for which $A = C(X)$. We shall also use the occasion to clarify, on an example, the link between the formalism of K homology and that of supersymmetry. The example is known as the $N=2$ Wess-Zumino quantum fields and we refer to [17] for its discussion. We just give the dictionary which allows to read the results of [17] in our language. Let $X=D'(T^1)$ be the space of complex distributions on T^1, the one dimensional torus. Any smooth $f\in C^\infty(T^1)=D(T^1)$ defines a function $\varphi(f)$. We let A be the (commutative) C^* algebra of functions on X generated by those of the form $\varphi(f)(1+|\varphi(f)|^2)^{-1/2}$ for $f\in C^\infty(S^1)$, with norm given by the sup norm and involution given by complex conjugation.† The spectrum of A is a suitable compactification of X. The Hilbert space $\mathscr{H}$ of the model is of the form $\mathscr{H}_b\otimes\mathscr{H}_f$ where the Bosonic Hilbert space $\mathscr{H}_b$ is identical with $L^2(X, d\mu)$ for the Gaussian measure $d\mu$ with covariance $G=(-(d/dx)^2+m^2)^{-1/2}(m\neq 0)$ while the Fermionic Hilbert space $\mathscr{H}_f$ is the Hilbert space of a suitable irreducible representation of the Clifford algebra on the (infinite dimensional) tangent space to X. Thus $\mathscr{H}_b\otimes\mathscr{H}_f=\mathscr{H}$ can be thought of as the space of L^2 spinors on X. The algebra A acts by multiplication operators in $\mathscr{H}$. Let Q be the supercharge operator in $\mathscr{H}$:

$$Q=\frac{1}{\sqrt{2}}\int dx\,\psi_1(x)(\pi(x)-\partial_1\varphi^*(x)-i(\partial V)(\varphi(x))+\psi_2(x)(\pi^*(x)$$
$$-\partial_1\varphi(x)-i\partial V(\varphi(x))^*)+\text{h.c.}$$

with the notations of [17, II.5]. Finally let $\Gamma=(-1)^{Nf}$ be the natural $\mathbb{Z}/2$ grading of the Fermionic Hilbert space, where Nf is the Fermion number operator. The results of [17, Proposition II.7] yield:

PROPOSITION 26. *Let $\mathscr{H}$ be the Hilbert space of the model, A act in $\mathscr{H}$ by multiplication, D be the supercharge operator Q in $\mathscr{H}$ and $\varepsilon=\Gamma$ the $\mathbb{Z}/2$ grading given by parity of the Fermion number. Then the triple $(\mathscr{H}, D, \varepsilon)$ is an even, θ-summable, unbounded Fredholm module over A.*

Proof. Cf. [17]. □

Thus in conclusion to this paper we see that both for the needs of non commutative geometry, whose aim is to understand non commutative spaces such as the dual of a discrete group, and for those of infinite dimensional analysis, in the set up of supersymmetric rigorous models of Quantum field theory, it is important to extend the construction of [7] of the Chern character, so that the character still makes sense for θ-summable Fredholm modules. It turns out that the cohomology of cochains in the (b, B) bicomplex of [7] which are no longer of finite support but satisfy a suitable growth condition is the natural candidate for infinite dimensional cohomology. It will be the subject of another paper.

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† The C^* algebra A is separable if we generate it by the $\varphi(f_n)(1+|\varphi(f_n)|^2)^{-1/2}$ where $f_n(\theta)=\exp$ in $\theta, \forall\theta\in T^1$.

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