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Closed Star Products and Cyclic Cohomology

ALAIN CONNES
Collège de France, 3 rue d'Ulm, F-75005 Paris, France

MOSHÉ FLATO and DANIEL STERNHEIMER
Physique Mathématique, Faculté des Sciences Mirande, Université de Bourgogne, BP 138, F-21004 Dijon Cedex, France

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Abstract. We define the notion of a closed star product. A (generalized) star product (deformation of the associative product of functions on a symplectic manifold W) is closed iff integration over W is a trace on the deformed algebra. We show that for these products the cyclic cohomology replaces the Hochschild cohomology in usual star products. We then define the character of a closed star product as the cohomology class (in the cyclic bicomplex) of a well-defined cocycle, and show that, in the case of pseudodifferential operators (standard ordering on the cotangent bundle to a compact Riemannian manifold), the character is defined and given by the Todd class, while in general it fails to satisfy the integrality condition.

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I. PRELIMINARIES

1. Let W be a symplectic manifold (a connected paracompact differentiable manifold of dimension $2l$ endowed with a closed 2-form ω, everywhere of rank $2l$; or, equivalently, with a contravariant skew-symmetric 2-tensor Λ, everywhere nondegenerate, satisfying $[\Lambda, \Lambda] = 0$ in the sense of the Schouten brackets). On the vector space of differentiable functions over W, $N = C^\infty(W)$, one defines the Poisson bracket by

$$P(f, g) = i(\Lambda)(\mathrm{d}f \wedge \mathrm{d}g), \quad f, g \in N. \tag{1}$$

This extends in a natural manner to the algebra $A = N[[\nu]]$ of formal power series in a (complex) variable ν with coefficients in N. Here, we shall call a 'star product' an associative deformation of the algebra (for the usual product) N, starting with the nontrivial Hochschild 2-cocycle P:

$$\begin{aligned} f * g &= \sum_{r=0}^{\infty} \nu^r C_r(f, g), \quad f, g \in N, \\ C_0(f, g) &= fg, \qquad C_1(f, g) = P(f, g). \end{aligned} \tag{2}$$

Here, the 2-cochains C_r are, for the time being, quite general; no hypothesis of being of the same parity as r, nor of vanishing whenever f or g are constants, nor even of bidifferentiability is made on the C_r (in contradistinction with [1]). The definition extends naturally to $f, g \in A$. We shall similarly denote by $\int f\omega^l$, for $f \in A$, the natural extension of the integral over W from N to A.

2. If $C \in C^p(N, M)$ is a p-cochain $N^p \to M$, where M is a bimodule over N (or over A, if $C: A^p \to M$), one defines the Hochschild coboundary b by

$$\begin{aligned} bC(f_0, \ldots, f_p) &= f_0 C(f_1, \ldots, f_p) - C(f_0 f_1, f_2, \ldots, f_p) + \cdots \\ &\quad + (-1)^p C(f_0, \ldots, f_{p-1} f_p) + (-1)^{p+1} C(f_0, \ldots, f_{p-1}) f_p. \end{aligned} \tag{3}$$

Then

$$(f * (g * h)) - ((f * g) * h) = \sum_{t=0}^{\infty} v^t D_t(f, g, h), \tag{4}$$

where

$$D_t = E_t - bC_t \in C^3(N, N), \tag{5}$$

$$E_t(f, g, h) = \sum_{\substack{r+s=t \\ rs \neq 0}} C_r(C_s(f, g), h) - C_r(f, C_s(g, h)) \tag{6}$$

and the bimodule is the algebra N itself. If we know $D_t = 0$ for $t \leqslant q$, we have $bE_{q+1} = 0$ and look for C_{q+1} given by $E_{q+1} = bC_{q+1}$ to show associativity at order $q + 1$: this E_{q+1} should therefore be in the zero class of the Hochschild 3-cohomology $H^3(N, N)$. In a similar fashion, the equivalence of two star products by a series $T = I + \Sigma_{r=1}^{\infty} v^r T_r$ which intertwines them is governed by the Hochschild 2-cohomology $H^2(N, N)$ at each level.

3. If $f, g \in A = N[[v]]$, one has

$$f * g = \left(\sum_0^{\infty} v^k f_k\right) * \left(\sum_0^{\infty} v^j g_j\right) = \sum_{r,k,j=0}^{\infty} v^{r+k+j} C_r(f_k, g_j). \tag{7}$$

To every $f \in A$, one associates in a canonical way the linear form in A^*:

$$g \to \int g f \omega^l, \quad f, g \in A.$$

This gives a natural map $C^p(A, A) \to C^p(A, A^*)$ and, therefore, to every p-cochain C_t valued in A, one associates a p-cochain $\tilde{C}_t$ valued in A^*, where A acts on A^* by $(x\varphi y)(a) = \varphi(yax)$ for $a, x, y \in A$, $\varphi \in A^*$.

The definition of b shows that $(\widetilde{bC_t}) = b\tilde{C}_t$.

4. The operator B: $C^n(A, A^*) \to C^{n-1}(A, A^*)$ is defined (for any algebra A) by the equality $B = AB_0$, where A is the cyclic antisymmetrization operator, while

$$B_0\varphi(a_0, \ldots, a_{n-1}) = \varphi(1, a_0, \ldots, a_{n-1}) - (-1)^n \varphi(a_0, \ldots, a_{n-1}, 1).$$

If $\gamma = \tilde{C} \in C^2(A, A^*)$, one has

$$B\gamma(f, g) = \gamma(1; f, g) - \gamma(1; g, f), \tag{8}$$

where $\gamma(1; f, g) = \int_W C(f, g)\omega^l$ is a normalized cochain. One has

$$Bb = -bB \quad \text{and} \quad B^2 = 0.$$

This definition of B coincides with that of the operator B in cyclic cohomology [2]. There, one considers the bicomplex (b, B) associated with the algebra A by taking the Hochschild cohomology $H^n(A, A^*)$ and setting

$$C^{n,m} = C^{n-m}(A, A^*). \tag{9}$$

The cyclic cohomology of A, $HC^n(A)$, is that of the complex (C^n_λ, b), where $C^n_\lambda(A)$ is the subspace of the $\varphi \in C^n(A, A^*)$ such that

$$\varphi(a_1, \dots, a_n)(a_o) = (-1)^n \varphi(a_2, \dots, a_n, a_o)(a_1), \quad a_i \in A, \tag{10}$$

whence a homomorphism of complexes, I: $HC^n(A) \to H^n(A, A^*)$.

B maps H^n into $HC^{n-1}(BC^n \subset C^{n-1}_\lambda$ and similarly for the cocycles, since it anticommutes with b). The essential property of the bicomplex (b, B) is the triviality of the first spectral sequence, from which it follows, in particular [2], that at each level n, $HC^n(A)$ is given equivalently by

$$\ker b \cap \ker B / b(\ker B)$$

or by cocycles in the bicomplex involving only cochains of dimension $\leqslant n$.

II. CLOSED STAR PRODUCTS

5. In the case of the Moyal product (2) on $W = \mathbf{R}^{2l}$ (where $r!C_r = P^r$, the rth power of the bidifferential operator P), the trace of the operator $\Omega(f)$ corresponding to the symbol $f \in N$ by the Weyl mapping [1], is given (when both sides are finite) by

$$\mathrm{Tr}(\Omega(f)) = (2\pi\hbar)^{-l} \int_W f\omega^l. \tag{11}$$

For the Moyal $*$-product, one has, of course [1],

$$\int (f * g)\omega^l = \int fg\omega^l = \int (g * f)\omega^l,$$

which will motivate Definition 1.

Cyclic cohomology is designed to be the natural receptacle for the Chern character in K-homology. In [3], it was shown that K-homology classes can be represented as deformations of algebras. It is thus natural to expect that the Chern character of such classes can be computed directly from the deformation. (That is in line with the general philosophy of [1], which is to treat quantum theories in an autonomous manner with algebras of symbols endowed with a deformed composition law instead of algebras of operators on a Hilbert space.) We shall now work out the explicit formula in terms of the expansion of the star product under the following crucial hypothesis of closedness.

DEFINITION 1. A star product is *closed* iff

$$\int C_r(f, g)\omega^l = \int C_r(g, f)\omega^l, \tag{12}$$

for all $f, g \in N$ and $r \leqslant l$.

In view of (7), this is equivalent to requiring that

$$\int a_l(f, g)\omega^l = \int a_l(g, f)\omega^l \quad \forall f, g \in A = N[[\nu]],$$

where $a_l(f, g) = \Sigma_{r+j+k=l} C_r(f_k, g_j)$ is the coefficient of ν^l in $f * g$ or, in other words, to saying that the map τ

$$A \ni f = \sum \nu^k f_k \to \int f_l \omega^l = \tau(f) \in \mathbf{C}$$

is a trace on A. Note that both sides of (12) are zero for $r = 1 (C_1 = P$, the Poisson bracket).

If (12) is true for all r, we shall say that the star product is *strongly closed*. This is the case of the known major examples. A star product equivalent to a closed star product will be called *weakly closed*.

6. Examples of Closed Star Products

Integration by parts shows that, on $\mathbf{R}^{2l}$, any star product for which (for $r \geqslant 1$) the cochains C_r are local (hence, bidifferential) and vanishing on constants (i.e. without constant term), with constant coefficients $C_{\alpha\beta}$ satisfying the following partial skew-symmetry condition

$$\sum (-1)^{|\beta|} C_{\alpha\beta} = 0 \quad \text{for } |\alpha| + |\beta| \text{ odd}$$

(which generalizes the case where the $C_{\alpha\beta}$ are products of $\Lambda'_{ij}s$) is strongly closed.

In particular, the Moyal product is strongly closed; in fact, there, integration of a function over $\mathbf{R}^{2l}$ is the trace of the corresponding operator in the Weyl mapping. Similarly, all equivalent orderings [4] like the standard ordering of the usual pseudo-differential calculus ('first q, then p') or the normal ordering (used in quantum field theory), are strongly closed. The equivalence of these orderings with Moyal is given by the Fourier transform (in $\mathbf{R}^{2l}$) of the weight function of the corresponding Weyl maps (the exponential of a quadratic polynomial). The associated weak Vey products [5] (where the C_r are even or odd together with r, and only the odd ones are required to vanish on constants while a zero-order term $\lambda_r fg$ with constant λ_r can be added to the even ones) are closed.

7. COROLLARY 1. *For a closed star product one has* $B\tilde{C}_2 = 0$.

Indeed, for $r = 2$, (12) reads $\int C_2(f,g)\omega^l = \int C_2(g,f)\omega^l$, i.e. $B\tilde{C}_2 = 0$ in view of (8). This necessary condition is sufficient for $l = 2$ (W of dimension 4). One starts with $C_1 = P$ and looks for C_2 such that (cf. (5) and (6)) $bC_2 = E_2$, knowing that $bE_2 = 0$. Therefore, $b\tilde{E}_2 = 0$ and $B\tilde{E}_2 = -bB\tilde{C}_2 = 0$. For the star product to be closed at order 2, it is necessary that $b\tilde{C}_2 = \tilde{E}_2 \in \ker b \cap \ker B$. Obviously, adding another B-closed cocycle in $\tilde{C}_2$ will not change that fact. Therefore

PROPOSITION 1. *The obstruction to existence, at order* 2, *of a closed star product is given by*

$$(\ker b \cap \ker B)/b(\ker B) = HC^3(A).$$

Proceeding step by step, one starts with a star product closed at order t and looks for a star product closed at order $t+1$. One has $bC_{t+1} = E_{t+1}$, with E_{t+1} given by the C_r with $r \leqslant t$, and knows that $bE_{t+1} = 0$, and $B\tilde{C}_t = 0$. Similarly, $b\tilde{E}_{t+1} = 0$ and $B\tilde{E}_{t+1} = -bB\tilde{C}_{t+1} = 0$ in view of (12) at order $t+1$. Hence, again we need to have $\tilde{E}_{t+1} \in \ker b \cap \ker B$, and we similarly find:

PROPOSITION 2. *At each level, the obstruction to existence of closed star products is given by* $HC^3(A)$.

8. Equivalence

We recall that two star products, denoted $*$ and $*'$ (with cochains denoted, respectively, by C_r and C'_r) are *equivalent* [1] if there exists a formal series (with coefficients necessarily differential operators if the C_r are bidifferential operators) with $T_o = I$ (the identity) which intertwines $*$ and $*'$:

$$T = \sum_{r=0}^{\infty} \nu^r T_r : A \to A, \tag{13}$$

$$T(f *' g) = (Tf) * (Tg). \tag{14}$$

At order t, (14) can be written as

$$bT_t = C'_t - C_t + G_t \equiv H_t, \tag{15}$$

where G_t depends only on the C'_r, C_r and T_r with $r < t$ (we set $G_1 = 0$ and take $rss' > 0$ in the sums):

$$G_t(f,g) = \sum_{r+s=t} T_s C'_r(f,g) - \sum_{s+s'=t} (T_s f)(T_{s'} g) - \\ - \sum_{r+s=t} (C_r(T_s f, g) + C_r(f, T_s g)) - \sum_{r+s+s'=t} C_r(T_s f, T_{s'} g). \tag{16}$$

One knows [1, 5] that if there is equivalence up to order t, then $bH_{t+1} = 0$, thus the obstructions to extend equivalence at each level lie in $H^2(N, N)$. If we have equivalence up to order t with a closed star product $*$, we see from (15) and (16) that $B\tilde{H}_{t+1} = 0$, thus (since $b\tilde{H}_{t+1} = 0)\tilde{H}_{t+1} \in \ker B \cap \ker b$. Since T_{t+1} will be the same if $\tilde{H}_{t+1}$ is modified by an element of $b(\ker B)$, one has:

PROPOSITION 3. *At each level, the equivalence classes of closed star products are classified by* $HC^2(A)$.

We shall, therefore, speak of *closed equivalence classes* in the category of closed star products, governed by cyclic cohomology. If we have closed equivalence up to order $t-1$, and since $bT_t(f, g)$ and all the sums in G_t, except the first, are manifestly symmetric under the exchange of f and g, a differentiable equivalence (15) at order t will be closed iff (with $rs > 0$):

$$\int_W \sum_{r+s=t} T_s C'_r(f, g)\omega^l = \int_W \sum_{r+s=t} T_s C'_r(g, f)\omega^l. \tag{17}$$

Let us now take (as in [5]) a covering of W by contractible domains (with contractible intersections) $U_\alpha (\alpha \in I)$ of Darboux (canonical) charts $(U_\alpha, \varphi_\alpha)$ and a locally finite partition of unity $\{f_\alpha\}$ subordinate to it. Then (if we denote by the same symbol, the function or operator on U_α and its image in the corresponding space $\mathbf{R}^{2l}$, and by $T^*_{\alpha s}$ the adjoint there of the differential operator T_s):

$$\begin{aligned}
&\int_W T_s C'_r(f, g)\omega^l \\
&\quad = \sum_{\alpha \in I} \int_{U_\alpha} (T_s C'_r(f, g)) f_\alpha \omega^l \\
&\quad = \sum_{\alpha \in I} \int_{\mathbf{R}^{2l}} (T_s C'_r(f, g)) f_\alpha \, \mathrm{d}^l p \wedge \mathrm{d}^l q \\
&\quad = \sum_{\alpha} \int_{\mathbf{R}^{2l}} C'_r(f, g)(T^*_s f_\alpha) \, \mathrm{d}^l p \wedge \mathrm{d}^l q \\
&\quad = \int_W C'_r(f, g)\Big(\sum_\alpha T^*_{\alpha s} f_\alpha\Big)\omega_l.
\end{aligned}$$

Condition (17) can then be written (with $f, g \in N$ arbitrary)

$$\int_W \sum_{r+s=t} (C'_r(f, g) - C'_r(g, f))\Big(\sum T^*_{\alpha s} f_\alpha\Big)\omega^l = 0. \tag{17'}$$

The idea is to show (17′) by induction ((17) is empty for $t = 1$) on a given symplectic manifold for an appropriate family of T_r's. Obviously, on $\mathbf{R}^{2l}$, if the T_r have constant coefficients, then we have closedness at order t. *On* $\mathbf{R}^{2l}$, *equivalence by differential operators with constant coefficients is a closed equivalence.* This is the case of the above-mentioned equivalence between orderings.

9 (a). *Existence of closed star products.* In the existence proof [5] of star products (in the sense of [1]), it is shown that there is a global star product M on W, the restriction of which to each chart U_α is a star product $T_\alpha M_\alpha$ equivalent to Moyal. But the equivalence operators T_α are only shown to exist, and it is not obvious from the construction that they are closed. However, they can probably be made so, and we can thus formulate:

PROBLEM. Show that on every symplectic manifold there exist closed star products.

(b) *Example of a nonclosed star product.* A simple example can be given by taking a current C defined by $\langle C, f_0\, df_1 \wedge df_2\rangle$, for which $dC \neq 0$. This will give a cochain C_2 with $B\tilde{C}_2 \neq 0$.

III. CHARACTER OF A CLOSED STAR PRODUCT

10. The Cyclic Cohomology Bicomplex

The cyclic cohomology can be defined using a bicomplex (b, B), where b is the Hochschild differential (of degree 1) and B is the differential previously introduced, of degree -1 and anticommuting with b, the cochain spaces being $C^{n,m} = C^{n-m}(A, A^*)$. The latter can be viewed as the space of $(n-m)+1$ linear forms on A (and defined as $\{0\}$ for $n-m<0$), in which case, b is given by, for $\psi \in C^n(A, A^*)$ and $a_i \in A$

$$(b\psi)(a_0, \ldots, a_{n+1})$$
$$\sum_{j=0}^{n} (-1)^j \psi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1}\psi(a_{n+1}a_0, \ldots, a_n) \quad (18)$$

and $B\psi = AB_o\psi$ with

$$(B_o\psi)(a_0, \ldots, a_{n-1}) = \psi(1, a_0, \ldots, a_{n-1}) - (-1)^n \psi(a_0, \ldots, a_{n-1}, 1), \quad (19)$$

$$(A\psi)(a_0, \ldots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} \psi(a_j, a_{j+1}, \ldots, a_{n-1}, a_0, \ldots, a_{j-1}). \quad (20)$$

By tensoring with the trivial algebra $\mathbf{C}$, one defines [2] the map $S\colon HC^n(A) \to HC^{n+2}(A)$, and the injection of $C^n_\lambda(A)$ in $C^n(A, A^*)$ defines the map I, whence the exact sequence

$$HC^n(A) \xrightarrow{I} H^n(A, A^*) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A)$$
$$\xrightarrow{I} H^{n+1}(A, A^*) \xrightarrow{B} HC^n(A).$$

Here

$$bC^{n,m} \subset C^{n+1,m} \quad \text{and} \quad BC^{n,m} \subset C^{n,m+1}.$$

11. Let us consider the (identity) map $f \to f$ as a linear map from the algebra $A_o = N[[v]]$ endowed with the commutative product of functions to the algebra A_v identical to A_o as a vector space but endowed with a star product. It is a 'quasi-homomorphism', and the expression (which measures its deviation from being a homomorphism):

$$\theta(f_1, f_2) = f_1 * f_2 - f_1 f_2 = v \sum_{r=1}^{\infty} v^{r-1} C_r(f_1, f_2)$$

is also a Hochschild 2-cocycle in $C^2(N, N)$ for $f_1, f_2 \in N$.

When $g = \Sigma_{r=0}^{\infty} v^r g_r \in A_v$, we define $\tau(g) = \int_W g_l \omega^l (\dim W = 2l)$. If the star product is *closed*, we define for $f_j \in N$ (j integer):

$$\varphi_{2k}(f_0, f_1, \ldots, f_{2k}) = \tau(f_0 * \theta(f_1, f_2) * \cdots * \theta(f_{2k-1}, f_{2k})). \tag{21}$$

Note that, by construction, $\varphi_0 = 0$ and, more generally, $\varphi_{2k} = 0$ for $0 \leqslant 2k < l$ and $k > l$. The only relevant components are thus φ_{2k} for $l \leqslant 2k \leqslant 2l$. From (8) and the fact that $b\theta = 0$, we get:

PROPOSITION 4. *Formula* (21) *defines the components of a cocycle in the* (b, B) *bicomplex.*

DEFINITION 2. The cohomology class of φ in the (b, B) bicomplex is called the *character* of the closed star product.

12. Character of a Closed Star Product on a Four-Dimensional Manifold

In this case we need only to take a C_2 such that

$$B\tilde{C}_2 = 0, \qquad bC_2 = E_2, \tag{22}$$

where

$$E_2(f_1, f_2, f_3) = P(P(f_1, f_2), f_3) - P(f_1, P(f_2, f_3)).$$

We get $\varphi_0(f_0) = 0$,

$$\varphi_2(f_0, f_1, f_2) = \int f_0 C_2(f_1, f_2)\omega^2, \quad \text{i.e. } \varphi_2 = \tilde{C}_2,$$

$$\begin{aligned} &\varphi_4(f_0, f_1, f_2, f_3, f_4) \\ &\quad = \text{coefficient of } v^2 \text{ in } \int f_0 \theta(f_1, f_2)\theta(f_3, f_4)\omega^2 \\ &\quad = \int f_0 P(f_1, f_2) P(f_3, f_4)\omega^2. \end{aligned} \tag{23}$$

Since $b\varphi_2 = b\tilde{C}_2 = \tilde{E}_2$, we have

$$b\varphi_2(f_0, f_1, f_2, f_3)$$
$$= \int f_0(P(P(f_1, f_2), f_3) - P(f_1, P(f_2, f_3)))\omega^2$$
$$= \int (P(f_3, f_0)P(f_1, f_2) - P(f_0, f_1)P(f_2, f_3))\omega^2$$
$$= -\tfrac{1}{2}\int \sum_{\text{cyclic}} (P(f_0, f_1)P(f_2, f_3))\omega^2 = -\tfrac{1}{2}B\varphi_4(f_0, f_1, f_2, f_3). \tag{24}$$

But $HC^2(N) = Z_2(W, \mathbf{C}) \oplus \mathbf{C}$, where $Z_2(W, \mathbf{C})$ is the space of closed two-dimensional currents on W and $\mathbf{C} = H_o(W, \mathbf{C})$. A change of C_2 with any element of HC^2 will thus affect the corresponding class in HC^4 and, therefore, the class of the character φ (which has no reason be be an integer, as required in the geometric quantization approach).

13. General Case: The Component φ_{2l}

One finds (as above, from (1)):

$$\varphi_{2l}(f_0, f_1, \ldots, f_{2l})$$
$$= \int f_0 P(f_1, f_2) \cdots P(f_{2l-1}, f_{2l})\omega^l$$
$$= \int f_0 \, \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2l-1} \wedge \mathrm{d}f_{2l}. \tag{25}$$

14. The Pseudodifferential Calculus

PROPOSITION 5. *If $W = T^*M$, where M is a compact Riemannian manifold, then the star product given by the composition of symbols in the usual pseudodifferential calculus is a closed star product.*

In [7], Widom defined what he calls 'a linear function' $L \in C^\infty(T^*M \times M)$, which is a covariant and global way to realize the equivalent of the linear function $L(q_0; p, q) = p \cdot (q - q_0)$ on $\mathbf{R}^{2l}$, with which the symbol $\sigma_A(q, p)$ of a compactly supported pseudodifferential operator A on $C^\infty(\mathbf{R}^l)$ can be defined by

$$\sigma_A(q_0, p) = A\, \mathrm{e}^{i(q - q_0)\cdot p}\big|_{q = q_0} \tag{26}$$

in such a way that the symbol of a product AB of two pseudodifferential operators A and B is given by

$$\sigma_{AB}(q, p) = \sum_{r=0}^{\infty} \frac{i^{-r}}{r!} \frac{\partial^r \sigma_A}{\partial p^r} \frac{\partial^r \sigma_B}{\partial q^r} \equiv \sigma_A * \sigma_B. \tag{27}$$

For $\sigma_A = a(q), \sigma_B = b(p)$, one gets $\sigma_{AB} = \sigma_A \sigma_B$ and, therefore, the star product given by (27) (with parameter $\nu = 1$) is what is usually called 'standard ordering'. It is equivalent to Moyal and can be directly seen to be closed on $\mathbf{R}^{2l}$. The linear function L of Widom (together with a technical function $\psi \in C^\infty(M \times M)$ equal to 1 in the diagonal and with support 'very close' to it) permits the definition

$$\sigma_A(v) = A\psi(\pi(v), x)\, \mathrm{e}^{iL(v,x)}\big|_{x = \pi(v)}, \tag{28}$$

where $T^*M \ni v$ projects onto $\pi(v) \in M$ under $\pi: T^*M \to M$, and $x \in M$, for the symbol of a pseudodifferential operator A. This gives a global formula ((3–5) in [7]) for the symbol of a product, which generalizes (27), using 'covariant' derivatives of L and of σ_B and 'contravariant' derivatives (derivatives along the fibers of T^*M) of the symbols. Different choices of ψ and L will give symbols differing by an element of the intersection $S^{-\infty}$ of all symbol spaces, and equivalent (in our sense) star products on T^*M. Indeed from the definition (28) of σ_A, one gets the equivalence operator (proposition (3.3) of [7]) giving the symbol σ'_A expressed with a connection ∇' associated with a linear of function L' in terms of contravariant derivatives of σ_A and of (∇-covariant derivatives of) L'. This operator intertwines the two corresponding star products. Moreover, on any local chart with canonical coordinates p and q, the restriction of the global expression (28) can be written as the 'canonical' one (26), which gives the closed star product (27) (on the flat space).

We then take a finite covering of M with overlapping coordinate charts, and a tiling of M with a finite number of 'tiles', the boundary of each being inside two charts (not the same for each part of the boundary). The same argument as in the flat case reduces integration over T^*M to integration on the 'cylinders', images in T^*M of these boundaries. These various integrals (of the cochains $C_r(f, g) - C_r(g, f)$) will cancel each other, and the star product thus obtained is strongly closed.

Remark. The pseudodifferential calculus gives a direct proof (anterior, by the way, to all general existence proofs) of the existence of a (closed) star product on T^*M, M compact Riemannian. Its global formulation realizes a gluing together of orderings giving the standard one in canonical charts.

15. In the case of the Moyal product on $W = \mathbf{R}^{2l}$, the trace of an operator $\Omega(f)$ corresponding to a given function f on W in the Weyl mapping is given, when both sides are defined, by [1]

$$\operatorname{Tr} \Omega(f) = (2\pi\hbar)^{-l} \int f(x)\, \mathrm{d}^{2l}x \tag{11}$$

and the definition makes sense, since the Moyal star product is closed. This can be easily seen for projectors on a finite-dimensional subspace of $L^2(\mathbf{R}^l)$ and extended to general trace class operators by a limiting procedure. The passage to standard ordering, i.e. replacement of $\Omega(f)$ by $\Omega_S(f)$ given by the introduction [4] of a weight (the exponential of a second-order polynomial) in the Weyl map, is an equivalence relation (the equivalence being given the exponential of the Fourier transform of this polynomial, i.e. the identity plus positive powers of $\hbar$ times differential operators). Thus

$$\mathrm{Tr}(\Omega_S(f)) = (2\pi\hbar)^{-l}\int_W f(x)\,\mathrm{d}x + O(\hbar^{1-l}) \tag{29}$$

and the same will be true [7] in the case of the pseudodifferential calculus. Therefore, the components φ_{2k} of the character defined by (21) are the same as those defined by the corresponding pseudodifferential operators:

$$\mathrm{Tr}(S(f_0)\Theta(f_1,f_2)\Theta(f_{2k-1},f_{2k})) = \varphi_{2k}(f_0,f_1,\ldots,f_{2k}), \tag{30}$$

where f_0 is the symbol of $S(f_0)$ and

$$\Theta(f_1,f_2) = -S(f_1f_2) + S(f_1)S(f_2).$$

PROPOSITION 6. *The character φ defined with the star product of symbols of pseudodifferential operators coincides with the character defined by the trace on these operators.*

Remarks. (a) The correspondence between trace and integral over phase space is exact at all orders of $\hbar$ in the Moyal (symmetric) ordering, which also exhibits the highest symmetry (algebra of preferred observables [1]). It would, therefore, be appropriate to study more carefully this 'symmetric pseudodifferential' calculus.

(b) If we apply the character given by formula (30) to the projector e (of A_o), one gets [8] half the trace of $(1-2e)((1-2e)^2)^{-1/2}$, which has a spectrum ± 1. Its trace is thus the index of e (in $A_\hbar$), an integer (independent of $\hbar$, therefore). It is thus enough to look at the power of $\hbar$ such that its coefficient remains finite $\neq 0$ when $\hbar \to 0$, i.e. that of $\hbar^l$, which explains the choice made for the trace τ.

16. We are now in position to translate the classical theory of pseudodifferential operators in star-product language, for the above-mentioned star product and, therefore, to express the character φ in terms of geometrical invariants. In particular, we get (cf. [2]) that the coupling between φ and the idempotent $e \in K_o(T^*M)$ is equal to the Atiyah–Singer index of the corresponding operator. Thus, it is not surprising that the character φ contains the Todd genus of the manifold.

THEOREM. *When $W = T^*M$, the character $\varphi \in HC^{ev}(T^*M)$ associated with the star product of the pseudodifferential calculus (M compact Riemannian manifold) is given by $Td(T^*M)$ as current over T^*M.*

The proof follows by the standard argument [9] of invariance theory, since the whole construction is functorial and local in terms of the Riemannian structure.

It is therefore interesting to try and develop a similar calculus for other orderings, especially for the Weyl ordering, study what kind of invariants one can get from an analogous theory, and extend this scheme to general symplectic or Poisson manifolds.

Though, in the above case, the K-theory proofs of the Atiyah–Singer index theorem are, in some sense, simpler than the 'asymptotic expansion' proofs (cf. e.g. [9]), in the general case of a closed star product on a symplectic manifold, only the asymptotic statement is meaningful. So the general open index problem (which, by the above theorem, extends the Atiyah–Singer index theorem) is that of finding an explicit formula for the cyclic cohomology class for the character of a closed star product in terms of Poisson geometry.

If a closed star product (with a fixed value of ν) gives us (for suitable functions) the compact operators, then the trace we have is necessarily proportional to the (unique) trace of compact operators. It is thus integral on the K-theory, and the cyclic cocycle is proportional to an integral one: $\langle K_o, \varphi \rangle \subset \lambda \mathbf{Z}$ for some constant λ. However, from the definition of φ, it is obvious that this integrality condition need not be satisfied. We thus have examples where there are obstructions to the traditional quantization, but where the more general framework of star product (deformation) quantization can be applied.

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