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The trace of certain operators

by

A. GROTHENDIECK (Paris)

This paper is author's answer to R. Sikorski's letter containing the following problem and conjecture:

Let X be a Banach space, $\mathfrak{A}$ — the Banach space of all bounded endomorphisms in X, $\mathfrak{A}_0$ — the set of all finitely dimensional bounded endomorphisms in X, and $\mathfrak{A}_1$ — the closure of $\mathfrak{A}_0$ in $\mathfrak{A}$. Let $T_i \in \mathfrak{A}$ be such that the functional

$$(*) \qquad \operatorname{trace} T_i K \quad (K \in \mathfrak{A}_0)$$

is continuous on $\mathfrak{A}_0$, $i = 1, 2, 3$. Denote by F_i the unique continuous extension of $(*)$ over $\mathfrak{A}_1$. Since T_2T_3, T_3T_1, T_1T_2 are nuclear, they belong to $\mathfrak{A}_1$. Consequently the numbers

$$(**) \qquad F_1(T_2T_3), \quad F_2(T_3T_1), \quad F_3(T_1T_2)$$

are well defined.

Problem: are the numbers $(**)$ equal?

Conjecture: yes, they are.

(1) Let E, F be Banach spaces, $u \in E'\hat{\otimes}F$, we say that u is *special* if for any Banach space M, and linear (continuous) maps $A: E' \to M'$, $B: F \to M$, the trace of the element $v = A\hat{\otimes}B(u)$ in $M'\hat{\otimes}M$ is equal to the sum of the proper values of the corresponding operator $\tilde{v}$ in M (or the operator ${}^t\tilde{v}$ in M', which amounts to the same), the set of these proper values (each counted with the right multiplicity) being summable. As ${}^t\tilde{v} = A\,{}^t\tilde{u}\,{}^tB$, it follows at once that u is known when we know the corresponding operator $\tilde{u}: E \to F$ (take $M = F$, $B =$ the identity). Such an operator $\tilde{u}$ will also be called *special*. If $u': E \to F$ is a special operator, it defines a well determined element u'^* in $E'\hat{\otimes}F$, and if $E = F$, then the trace of u'^* denoted also by $\operatorname{Tr} u'$, is equal to the sum of proper values of u'.

If $u: E \to F$ is a special operator, so is ${}^tu: F' \to E'$ and we have $({}^tu)^* = {}^t(u^*)$ in the obvious way; and for any operators $A: M \to E$, $B: F \to N$, BuA is special and $(BuA)^* = B(u^*)A$ (where operations on a kernel u^* have the obvious meaning). From this, and a well-known and immediate property of kernels, follows

$$\operatorname{Tr} uv = \operatorname{Tr} vu$$

if $u: E \to F$, $v: F \to E$, one of the operators u, v being special, so that the traces are well-defined. As a corollary, we have formulas

$$\mathrm{Tr}\, uvw = \mathrm{Tr}\, vwu = \mathrm{Tr}\, wuv$$

if $u: E \to F$, $v: F \to G$, $w: G \to E$, one of the three operators being special.

It is not certain, if $u, v: E \to F$ and $u+v$ are special operators, whether $(u+v)^* = u^* + v^*$, or what amounts to the same, whether (in case $E = F$) $\mathrm{Tr}(u+v) = \mathrm{Tr}\, u + \mathrm{Tr}\, v$. We will see a special case where things are nice.

(2) PROPOSITION. *Let E, F be Banach spaces, let $S(E, F)$ be the set of operators $E \to F$ which can be factored into $E \xrightarrow{u} H \xrightarrow{v} F$, H a Hilbert space, v integral (= nuclear, here). This is a vector subspace of $L(E, F)$, its elements are special operators, and the mapping $u \to u^*$ from $S(E, F)$ into $E' \hat{\otimes} F$ is linear.*

The proof is very easy, and rests essentially on the following remarks: a) The proper values of a nuclear operator in Hilbert space are summable, the sum is the trace of the operator; b) Let F, H be Banach spaces, $\alpha: P \to H$, $\beta: H \to P$ operators such that $\alpha\beta$ is compact. Then $\alpha\beta$ and $\beta\alpha$ have the same set of proper values. From this follows: c) if in (b) α comes from an element $\alpha' \in P' \hat{\otimes} H$, and if H is a Hilbert space, then $\mathrm{Tr}\, \beta\alpha'$ is equal to the sum of the proper values of $\beta\alpha$ (these being summable). From c) follows that the operator vu of proposition is special, and $(vu)^* = v^* u$ (the latter formula being also contained in (1)). Now linearity follows very easily.

COROLLARY 1. *The same assertions hold for the space $S'(E, F)$ of operators vu, where u is now assumed to be nuclear.*

An operator $E \to F$ is called of *Hilbert type* or *type* (H) if it can be factored through a Hilbert space, and of type (H′) if its compositum with any operator of Hilbert type (on left or right, it amounts to the same) is integral (or nuclear, it is the same). These notions are studied in my paper [1], where I prove that type (H′) is equivalent with the possibility of factoring through $E \to C \to L \to F''$ (where C is the space of continuous functions on some compact space, L the dual of such a space). We do not need this fact here, but merely remark that an integral operator is both of type (H′) and of type (H) (in fact, type (H′) implies type (H)). The operators of type (H), or type (H′), from E into F form a Banach space with a natural norm on it, which could be taken into account below.

COROLLARY 2. *A compositum vu ($u: E \to F$, $v: F \to G$) where one of the operators is of type* (H), *the other of type* (H′), *is special. Moreover if $E = G$, then $\mathrm{Tr}\, uv = \mathrm{Tr}\, vu$.*

In fact, we get that $\|(vu)^*\| \leqslant N_{H'}(v) N_H(u)$ resp $\leqslant N_H(v)\, N_{H'}(u)$ (1).

(1) For the definition of $N_H, N_{H'}$ see [1], p. 40.

The assertion on the traces, once factorisation through Hilbert space is written for one of the operators, follows at once from (1).

As a consequence, if we have operators $u: E \to F$, $v: F \to G$, $w: G \to E$ one of which is of type (H), another of type (H′), then we have

$$\mathrm{Tr}\, wvu = \mathrm{Tr}\, vuw = \mathrm{Tr}\, uwv.$$

Now suppose in corollary 2 that v is *integral*, and u is a limit (for the norm N_H) of operators of finite rank. A fortiori u is a limit in the usual norm of operators of finite rank, therefore by continuity $v \square u$ is defined as an element of $E' \hat{\otimes} G$, so that for $G = E$, the trace of the latter be the natural scalar product of u and v. We see by continuity that we must have

$$v \square u = (vu)^*$$

and hence, if $E = G$, $\mathrm{Tr}\, v_\square u = \mathrm{Tr}(vu)$. Therefore, the number $F_1(T_2T_3)$ in your letter can be also and more conveniently be written $\mathrm{Tr}\, T_1T_2T_3$. The affirmative answer to your question is now obvious.

Remark. If the spaces E and F satisfy the approximation property, so that the trace of a nuclear operator in either space is defined, things are much simpler still: it is easy to prove then that if $u: E \to F$ is integral and $v: E \to F$ is weakly compact, so that uv and vu are nuclear, then $\mathrm{Tr}\, uv = \mathrm{Tr}\, vu$.

References

[1] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bolletin da Sociedade de Matemática de São Paulo 8 (1956), p. 1-79.

Voir aussi P. Cartier, *Classes de formes bilinéaires sur les espaces de Banach*, Séminaire Bourbaki, février 1961, exposé 211 (ajouté pendant la correction des épreuves).

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