Crystals and the De Rham Cohomology of Schemes

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(Notes by I. Coates and O. Jussila).

Introduction.

These notes are a rough summary of five talks given at I.H.E.S
in November and December 1966. The purpose of these talks was to
outline a possible definition of a p-adic cohomology theory, via a
generalization of the de Rham cohomology which was suggested by work
of Monsky-Washnitzer [8] and Manin [7].

The contents of the notes are by no means intended to be a
complete theory. Rather, they outline the start of a program of work which has
still not been carried out (*).

(*) For a more detailed exposition and progress in this direction, we refer
to the work of P. Berthelot, to be developped presumably in SGA 8.

1. De Rham Cohomology.

1.1. **Differentiable Manifolds.** Let X be a differentiable manifold, and $\Omega^{\cdot}_{X,\mathbb{C}}$ the complex of sheaves of differential forms on X, whose coefficients are complex valued differentiable functions on X.

Theorem 1.1. (De Rham). There is a canonical isomorphism

$$H^*(X, \mathbb{C}) \xrightarrow{\sim} H^*(\Gamma(X, \Omega^{\cdot}_{X/\mathbb{C}})),$$

where $H^*(X, \mathbb{C})$ is the classical cohomology of X with complex coefficients.

To prove this, one observes that, by Poincaré's lemma, the complex $\Omega^{\cdot}_{X/\mathbb{C}}$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$ on X, and that the sheaves $\Omega^j_{X/\mathbb{C}}$ are fine for $J \geqslant 0$, so that $H^i(X, \Omega^j_{X/\mathbb{C}}) = 0$ for $i > 0$ and $j \geqslant 0$, whence the assertion.

An analogus result holds for the complex of sheaves of differential forms on X, whose coefficients are real valued differentiable functions on X.

1.2 **Complex Analytic Manifolds.** Let now X be a complex analytic manifold, and $\Omega^{\cdot}_{X,\mathbb{C}}$ the complex of sheaves of differential forms on X, whose coefficients are analytic functions on X. Then it is no longer true, in general, that the $\Omega^j_{X/\mathbb{C}}$ are fine for $j \gg 0$. For this reason, we consider the hypercohomology $\underline{H}^*(X, \Omega^{\cdot}_{X/\mathbb{C}})$ of the complex $\Omega^{\cdot}_{X/\mathbb{C}}$. Thus we have the standard spectral sequence of hypercohomology

$$E_2^{pq} = H^p(X, H^q(\Omega^{\cdot}_{X/\mathbb{C}})) \implies \underline{H}^*(X, \Omega^{\cdot}_{X/\mathbb{C}}).$$

Now Poincaré's lemma holds for X, and thus this spectral sequence degenerates, thereby showing that there is a canonical isomorphism

$$\mathbb{H}^*(X, \mathbb{C}) \xrightarrow{\sim} \underline{\mathbb{H}}^*(X, \underline{\Omega}^{\cdot}_{X \mid \mathbb{C}}).$$

On the other hand, we have the second spectral sequence of hypercohomology

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}^p_{X/\mathbb{C}}) \Longrightarrow \underline{\mathbb{H}}^*(X, \underline{\Omega}^{\cdot}_{X/\mathbb{C}}).$$

If X is a Stein manifold, $\mathbb{H}^i(X, \underline{\Omega}^j_{X/\mathbb{C}}) = 0$ if $i > 0$ and $j \geqslant 0$, and thus this spectral sequence degenerates, yielding an isomorphism

$$\underline{\mathbb{H}}^*(X, \underline{\Omega}^{\cdot}_{X/\mathbb{C}}) \xrightarrow{\sim} \mathbb{H}^*(\Gamma(X, \underline{\Omega}^{\cdot}_{X/\mathbb{C}})).$$

1. 3. The Algebraic De Rham Cohomology. Let $f : X \longrightarrow S$ be a morphism of schemes (*), and $\underline{\Omega}^{\cdot}_{X/S}$ the complex of sheaves of relative differential forms on X/S (EGA IV 16). Recall that the differential operator d of this complex is __not__ $\underline{O}_X$-linear, but only $f^{-1}(\underline{O}_S)$-linear ($f^{-1}(.)$ denotes the inverse image in the sense of topological spaces, whilst, as usual, $f^*(.)$ denotes the inverse image in the sense of ringed spaces). We define the relative De Rham cohomology of X/S to be the hyper-cohomology

$$\mathbb{H}^*_{DR}(X/S) = \underline{\mathbb{H}}^*(X, \underline{\Omega}^{\cdot}_{X/S}).$$

We have the usual spectral sequence of hypercohomology

$$E_1^{pq} = \mathbb{H}^q(X, \Omega^p_{X/S}) \longrightarrow \mathbb{H}^*_{DR}(X/S).$$

If X is __affine__, this spectral sequence degenerates, yielding a canonical isomorphism

$$\mathbb{H}^*_{DR}(X/S) \xrightarrow{\sim} \mathbb{H}^*(\Gamma(X, \underline{\Omega}^{\cdot}_{X/S})).$$

(*) Throughout, we follow the new terminology of __schemes__ and __separated__ __schemes__, instead of __preschemes__ and __schemes__, respectively, in the old terminology.

One can also consider the relative De Rham cohomology sheaf

$$\underline{H}_{DR}^{i}(X/S) = \underline{R}^{i}f_{*}(\underline{\Omega}_{X/S}^{\cdot}).$$

We have the spectral sequence

(1) $$E_{1}^{pq} = R^{q}f_{*}(\underline{\Omega}_{X/S}^{p}) \Longrightarrow \underline{H}_{DR}^{*}(X/S).$$

If f is quasi-compact and quasi-separated, one can show that the $\underline{R}^{i}f_{*}(\underline{\Omega}_{X/S}^{\cdot})$ are quasi-coherent Modules on S by using this spectral sequence.

If, moreover, X is <u>smooth</u> on S, the De Rham cohomology sheaves commute with base change, provided we work with <u>derived categories</u> (*). More precisely, if we are given an arbitrary base change

then, since $\underline{\Omega}_{X\ S}^{\cdot}$ is flat, being locally free over X which is flat, the Kunneth formula asserts that there is a canonical isomorphism

$$Lg^{*}(\underline{R}\ f_{*}(\underline{\Omega}_{X/S}^{\cdot})) \xrightarrow{\sim} \underline{R}\ f'_{*}(p^{*}(\underline{\Omega}_{X/S}^{\cdot})),$$

whence, since always $p^{*}(\underline{\Omega}_{X/S}^{\cdot}) \xrightarrow{\sim} \underline{\Omega}_{X'/S'}^{\cdot}$,

$$Lg^{*}(\underline{R}\ f_{*}(\underline{\Omega}_{X/S}^{\cdot})) \xrightarrow{\sim} \underline{R}\ f'_{*}(\underline{\Omega}_{X'/S'}^{\cdot}),$$

which is the assertion that the De Rham cohomology commutes with base change in the sense of derived categories. Note that this does not imply, in general, that the $\underline{R}^{i}f_{*}(\underline{\Omega}_{X/S}^{\cdot})$ commute with base change. However, if S is of characteristic 0, and f is proper and smooth, then one can show by transcendental arguments that the $\underline{R}^{i}f_{*}(\underline{\Omega}_{X/S}^{\cdot})$ are <u>locally</u>

(*) We use the notation for derived categories given in [6].

free, and hence commute with base change.

If X is proper on S, and S is locally noetherian, then one can show, by using the spectral sequence (1), and the finiteness theorem for proper morphisms (EGA III 3), that the $\underline{R}^i f_*(\Omega^{\cdot}_{X/S})$ are <u>coherent</u> Modules on S. Hence, in particular, if S is the spectrum of a field k, the $H^i_{DR}(X/k)$ are finite dimensional vector spaces over k.

1.4. The Comparison Theorem. Let X be a scheme which is smooth over the complex numbers $\mathbb{C}$. Let X^{an} be the analytic manifold corresponding to X(GAGA [10]), and $\underline{\Omega}^{\cdot}_{X^{an}/\mathbb{C}}$ the complex of sheaves of analytic differential forms on X^{an}. There is the canonical homomorphism of the algebraic into the analytic De Rham cohomology

$$(*) \qquad \underline{H}^*(X,\underline{\Omega}^{\cdot}_{X/\mathbb{C}}) \longrightarrow \underline{H}^*(X^{an},\underline{\Omega}^{\cdot}_{X^{an}/\mathbb{C}}) .$$

<u>Theorem 1.2.</u> <u>The homomorphism (*) is an isomorphism.</u>

<u>Corollary.</u> <u>The algebraic De Rham cohomology $H^*_{DR}(X/\mathbb{C})$ is canonically isomorphic to $H(X^{an},\mathbb{C})$.</u>

The corollary is immediate, since we saw in 1.2 that there is a canonical isomorphism

$$\underline{H}^*(X^{an},\mathbb{C}) \xrightarrow{\sim} \underline{H}^*(X^{an},\underline{\Omega}^{\cdot}_{X^{an}/\mathbb{C}}) .$$

The proof of Theorem 1.2 is given in [5], and so we omit it.

The proof in general requires Hironaka's resolution of singularities. However, if X is proper on $\mathbb{C}$, it follows immediately from the spectral sequence

$$E^{pq}_1 = H^q(X,\Omega^p_{X/\mathbb{C}}) \Longrightarrow \underline{H}^*(X,\underline{\Omega}^{\cdot}_{X/\mathbb{C}}) ,$$

together with the analogous spectral sequence for X^{an}, since GAGA shows

that the initial terms of these spectral sequences are isomorphic.

1.5. Criticism of the De Rham Cohomology. Let X be a scheme of finite type on a field k.

a) If k is of characteristic 0, and X is smooth over k, the De Rham cohomology has all the good properties one could want, since Theorem 1.2 and Lefschetz principle show that we essentially have the classical complex cohomology.

On the other hand, if k is of characteristic $p>0$, the De Rham cohomology no longer has good properties. If X is not proper on k, the $H^i(X/k)$ need not be of finite dimension on k. For one always has

$$\Gamma(X, \underline{O}_X)^p \subset H^o_{DR}(X/k),$$

and thus $H^o_{DR}(X/k)$ is certainly not finite dimensional for an affine curve, say. Even if X is proper on k, and thus the De Rham cohomology is finite dimensional, it does not always yield the good Betti numbers. For one always has (*)

$$\dim H^1_{DR}(X/k) \geq 2 \dim \text{Pic}_{X/k} \ ,$$

and there are examples ([5] , p. 103) where one has strict inequality. At least, the Riemann-Roch theorem shows that the De Rham cohomology does give the good value for the alternating sum of the Betti numbers.

b) The De Rham cohomology seems too closely bound to the assumption of smoothness.   On the other hand, one should not restrict

(*) At least if X is geometrically normal.

oneself to the study of smooth schemes only, since the fibers of a

morphism of smooth schemes need not be smooth.

c) The classical analytic De Rham cohomology theory suggests a

number of important problems for the algebraic theory [5]. For example,

if $f : X \longrightarrow Y$ is a smooth morphism of schemes smooth over C, then

there is the topological Leray spectral sequence

$$E_2^{pq} = H^p(Y^{an}, R^q f_*^{an}(C)) \Rightarrow H^*(X^{an}, C).$$

The end of this spectral sequence has, by theorem 1.2, a purely

algebraic definition as $H_{DR}^*(X/C)$. As we shall see in 3, the initial

term can also be given a purely algebraic definition ([5] p 103).

Thus one would like a purely algebraic definition of the spectral

sequence.

d) For X proper and smooth on k, the usual duality formalism

holds for the De Rham cohomology, the theory being analogous to that

given for the Hodge cohomology in [4]. One also has a Lefschetz

fixed point formula for the De Rham cohomology, but it only yields

the number of fixed points modulo the characteristic of k, since the

$H_{DR}^i(X/k)$ are vector spaces on k.

It would be convenient to have a more general duality formalism

of the type $f_!$, $f^!$, as developed in [6], for the De Rham cohomology.

Such a formalism would give a purely algebraic definition of the

singular homology groups of X/k.

To deal with these problems, one must certainly introduce

a more general category of coefficients than the De Rham complexes.

One of the principal aims of these notes is to propose a definition

of such a category of coefficients.

But why worry about the De Rham cohomology when one already

has the ℓ-adic cohomology ?

1.6. <u>Criticism of the ℓ-adic cohomology</u>. If X is a scheme

of finite type over an algebraically closed field k, and ℓ is any prime

number <u>distinct</u> (*) from the characteristic of k, the ℓ-adic cohomo-

logy of X is defined to be

$$H_\ell^i(X) = \varprojlim_\nu H^i(X_{et}, \mathbb{Z}/\ell^\nu \mathbb{Z}) ,$$

where X_{et} is the etale cohomology of X. Thus the $H_\ell^i(X)$ are modules

over the ring Z_ℓ of ℓ-adic integers.

If k is the field of complex numbers **C**, the comparison theorem

for the ℓ-adic cohomology [1] shows that there is a canonical

isomorphism

$$H_\ell^i(X) \xrightarrow{\sim} H^i(X^{an}, \mathbb{Z}) \otimes \mathbb{Z}_\ell$$

where $H^i(X^{an}, \mathbb{Z})$ is the classical cohomology of X with integral

coefficients. Thus, for given ℓ, the knowledge of the $H_\ell^i(X)$ is

equivalent to the knowledge of the <u>rank</u> of $H^i(X^{an}, \mathbb{Z})$ and the ℓ-

<u>primary torsion</u> subgroup of $H^i(X^{an}, Z)$. On the other hand, the full

knowledge of the structure of the $H^i(X^{an}, Z)$, including <u>all</u> torsion,

is equivalent to the knowledge of the $H_\ell^i(X)$ for all ℓ. The comparison

theorem shows immediately that the rank of $H_\ell^i(X)$ is finite and inde-

pendent of ℓ, and that the characteristic polynomial of an endomorphism

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(*) the ℓ-adic cohomology is still defined for ℓ equal to the
characteristic of k, but it no longer has too many reasonable
properties.

of $H^i_\ell(X)$ induced by a k-morphism $X \longrightarrow X$ has rational integral coefficients which are independent of ℓ.

If k is of characteristic p > 0, the situation is both intrinsically and technically not so satisfactory. When X is <u>proper</u> over k, the rank of $H^i_\ell(X)$ is <u>finite</u>, but even when X is projective and smooth over k, it is not known whether this rank is independent of ℓ (thus we certainly do not know whether the characteristic poly-nomial of an endomorphism of $H^i_\ell(X)$ induced by a k-morphism $X \longrightarrow X$ has coefficients which are independent of ℓ). If X is not proper on k, and dim(X) $\geqslant$ 2, it is not known whether the rank of $H^i_\ell(X)$ is finite. Further, the cohomological version of the deeper of Lefschetz's classi-cal theorems [13] on the hyperplane sections of a scheme X pro-jective and smooth on k has still not been proven for the ℓ-adic cohomology. In addition to our uncapacity at present to prove such basic facts for the ℓ-adic cohomology, there is the intrinsic fault that the ℓ-adic cohomology has reasonable properties only if ℓ is distinct from the characteristic of k. Thus it cannot yield information on p-torsion in the cohomology (see Tate [12]). Nor will it alone be able to show, in connection with the conjectures of Weil, that there are not powers of p in the denominators of the coefficients of the characteristic polynomial of an endomorphism of $H^i_\ell(X)$ induced by a k-morphism $X \longrightarrow X$ (*).

Thus we seek a <u>p-adic cohomology</u> to complement the ℓ-adic cohomology.

(*) Since these notes were written, it has been pointed out by Lubkin that the integrality question alluded to do fit into the so-called "standard conjectures" when using only ℓ-adic cohomology ; see Kleiman's exposé in this volume.

1.7. <u>Needed properties of a p-adic cohomology</u>. Such a cohomology theory should associate, to each scheme X of finite type over a perfect field k of characteristic p > 0, cohomology groups which are modules over an integral domain, whose quotient field is of <u>characteristic 0</u>, and which satisfy all the desirable formal properties (functorality, finite dimensionality, Poincaré duality, Kunneth formula, invariance under base change,...). This cohomology should also, most importantly, explain torsion phenomena, and in particular <u>p-torsion</u>.

The natural coefficient ring for the p-adic cohomology seems to be the ring $W(k)$ of Witt vectors of k. As Serre has pointed out, one cannot take the coefficient ring to be $\mathbb{Z}_p$ or not even $\mathbb{Q}_p$. For there exist elliptic curves X such that $\text{End}(X)$ is a <u>maximal</u> order of a quaternion algebra on $\mathbb{Z}$ and $\text{End}(X) \otimes \mathbb{Q}_p$ is a <u>field</u> (Deuring [2]). Now the existence of a reasonable cohomology of X with coefficients in $\mathbb{Q}_p$ would imply that $H_p^i(X)$ is of dimension 2 on $\mathbb{Q}_p$, and that the mapping $u \longmapsto u^1$ is a representation of the opposite algebra of $\text{End}(X)$ in $H_p^1(X)$. Thus we would have a representation of the field of quaternions $\text{End}(X) \otimes \mathbb{Q}_p$ in a vector space of dimension 2 on $\mathbb{Q}_p$, which is obviously impossible. Hence the coefficient ring for our p-adic cohomology will have to be some <u>extension</u> of $\mathbb{Q}_p$.

1.8. <u>Proposals for a p-adic Cohomology</u>. We only mention two such proposals, namely Monsky and Washnitzer's method via special affine liftings (which we discuss in n° 2), and the method using the fppf (faithfully flat and finite presentation) topology.

By analogy with the ℓ-adic cohomology, the essential idea of the fppf topology approach was to consider the cohomology of X/k, with respect to the fppf topology, with coefficient groups in the category C^ν of finite schemes of $\mathbb{Z}/p^\nu\mathbb{Z}$-modules. Examples of such schemes of modules are

$$\mathbb{Z}/p^\nu\mathbb{Z}, \quad \mu_{p^\nu} = \mathrm{Ker}(G_m \xrightarrow{p^\nu} G_m) \cdot, \quad \alpha_{p^\nu} = \mathrm{Ker}(G_a \xrightarrow{p^\nu} G_a) \quad.$$

More precisely, one hopes to prove the following conjectural theorem, with X proper on k, say :

(Conjectural Theorem). There exists a complex $L^\nu_\cdot$ in the category C^ν of sheaves of $\mathbb{Z}/p^\nu\mathbb{Z}$-modules on $\mathrm{Spec}(k)_{\mathrm{fppf}}$, with $H_i(L^\nu_\cdot) \in C^\nu$, such that for each complex $G^\cdot$ in C^ν :

$$\underline{\underline{H}}^*(X_{\mathrm{fppf}}, G^\cdot) \xrightarrow{\sim} \mathrm{Ext}^*_{C_\nu}(L^\nu_\cdot, G^\cdot) \quad.$$

The homology of X/k in dimension i would then be $(H_i(L^\nu_\cdot))_{\nu \geqslant 0}$, which is a "profinite algebraic group scheme" over k.

This apparently works well if X is of dimension 1, and yields, as it should, the Tate module $T_p(J) = (\cdot_{p^\nu} J)_{\nu \geqslant 0}$ for the first homology group of X (J being the Jacobian of X) (*). However, Artin has remarked that the cohomology of X, with respect to the fppf topology, and with coefficients in a scheme of $\mathbb{Z}/p^\nu\mathbb{Z}$-modules, vanishes in dimension $> \dim(X)+1$. Thus, if $\dim(X) > 1$, Poincaré duality cannot hold for this homology. Nevertheless, this homology should have interesting relations with the eventual p-adic cohomology. For it can still be hoped that, if X is proper, smooth, and connected, say, one can recover the "good" p-adic cohomology groups $H_p^i(X)$, for $i \leq \dim(X)+1$,

(*) and "correct" higher cohomology groups.

as suitable "Dieudonné modules" associated to the profinite groups

$(H_i(L^{\vee}))$, - and hence also all $H_p^i(X)$ if we grant Poincaré duality for

the conjectural p-adic cohomology groups $H_p^i(X)$.

2. The Cohomology of Monsky and Washnitzer.

2.1. Approach via liftings.

Suppose X_0 is a scheme on a perfect field k of characteristic

p > 0, and suppose that there exists a lifting of $X_0 \longrightarrow k$ to a scheme

X proper and smooth on S = Spec W(k) (W(k) is the ring of Witt

vectors of k). The essential idea of Monsky and Washnitzer's theory

is perhaps that the De Rham cohomology of the lifting, $H_{DR}^i(X/S)$, is

determined up to canonical isomorphism by X_0 alone, and does not

depend on the particular lifting $X \longrightarrow S$ of X_0 . Their idea was then

that the De Rham cohomology $H_{DR}^i(X/S)$ of a lifting, which are finite

dimensional vector spaces on W(k) since X is proper on S, would be the

p-adic cohomology of X_0 . This approach yields the right Betti numbers

(namely, the Betti numbers of the generic fiber of a lifting $X \longrightarrow S$),

since the comparison theorem of n° 1 shows that the De Rham cohomo-

logy of the generic fiber yields the correct Betti numbers.

A first difficulty with this approach is that no such lifting of

$X_0 \longrightarrow k$ exists, in general.

If X_0/k is affine, we can at least find a lifting of X_0/k to

a formal scheme $\mathfrak{X}$ on S. One might then consider the hypercohomology

$$\underline{H}^*(\mathfrak{X}, \Omega_{\mathfrak{X}/S}^{\cdot})$$

where $\underline{\Omega}^{\cdot}_{\mathcal{X}/S} = \varprojlim \Omega_{\mathcal{X}_n/S_n}$, $S = \mathrm{Spec}(W(k)/\mathfrak{m}^{n+1})$ ($\mathfrak{m}$ being the

maximal ideal of $W(k)$), and $X_n = \mathcal{X} \times_S S_n$. Since $\mathcal{X}$ is affine, there

is a canonical isomorphism

$$\underline{H}^*(\mathcal{X}, \ \underline{\Omega}^{\cdot}_{\mathcal{X}/S}) \xrightarrow{\sim} H^*(\Gamma(\mathcal{X}, \ \underline{\Omega}^{\cdot}_{\mathcal{X}/S})) \ .$$

Now consider the particular case when $X_0 = \mathrm{Spec}\, k\,[t]$. Then $\mathcal{X} =$

$\mathrm{Spec}f\, W\,\{t\}$, where $W\{t\}$ is the ring of all formal power series

$\Sigma\, a_n t^n$, $a_n \in W(k)$, such that $v_{\mathfrak{m}}(a_n) \to \infty$ as $n \to \infty$. Thus

$$H^1(\Gamma(\mathcal{X}, \underline{\Omega}^{\cdot}_{\mathcal{X}/S})) = W\{t\}/dW\{t\},$$

where d is formal differentiation of power series. But there clearly

exist power series $\Sigma\, a_n t^n$ in $W\{t\}$ such that $\Sigma\dfrac{a_n}{n^{n+1}} t^{n+1}$ is $\underline{\text{not}}$ in

$W\{t\}$, and hence $H^1(\Gamma(\mathcal{X}, \underline{\Omega}^{\cdot}_{\mathcal{X}/S}))$ is not zero, nor even finite

dimensional when tensored with the field of fractions K of W. Thus

this cohomology is not satisfactory. However, note that this cohomology

would have been zero, if, instead of considering the ring $W\{t\}$ of

all convergent power series, we considered the ring $W^{\dagger}\{t\}$ of all

power series $\Sigma\, a_n t^n$, $a_n \in W(k)$, such that

$$v_{\mathfrak{m}}(a_n) \geqslant \rho\, n \quad \text{for n sufficiently large,}$$

for some real number $\rho > 0$. This leads us to Monsky and Washnitzer's

method of constructing their liftings.

2.2. Monsky and Washnitzer's liftings.

Monsky and Washnitzer's method was to first lift $\underline{\text{affine}}$ schemes.

To this end, they introduced a class of algebras which we shall call

M-W-algebras ("w.c.f.g. algebras" in their terminology [8]). As before, let k be a perfect field of characteristic $p > 0$, and $W = W(k)$ the ring of Witt vectors on k. Denote by $W^+\{t_1, \ldots, t_n\}$ the ring of formal power series $\sum a_{i_1 \ldots i_n} t_1^{i_1} \ldots t_n^{i_n}$, $a_{i_1 \ldots i_n} \in W(k)$, such that

$$v_{\mathfrak{M}}(a_{i_1 \ldots i_n}) \geqslant \rho \, (i_1 + \ldots + i_n) \text{ for } i_1 + \ldots + i_n$$

sufficiently large,

for some real number $\rho > 0$. Then a M-W-algebra is defined to be any quotient of such an algebra $W^+\{t_1, \ldots, t_n\}$. These M-W-algebras are similar to the building blocks used by Tate in his theory of rigid analytic spaces [11].

Then Monsky and Washnitzer prove essentially the following assertions [8] :

a) If A_0 is an algebra of finite type and smooth over k (and satisfies a further mild, and probably unnecessary condition), then there exists a M-W-algebra A, which is flat (and hence "smooth") on W, and a k-isomorphism

$$\varphi : A \underset{W}{\otimes} k \xrightarrow{\sim} A_0 \ .$$

b) If A and B are any two such flat M-W-algebras lifting A_0, they are isomorphic. More generally, if A and B are two M-W-algebras over W, A being "smooth", then any homomorphism $A_0 \longrightarrow B_0$ lifts to a homomorphism $A \longrightarrow B$.

c) If A is a M-W-algebra, one defines $\Omega^1_{A/W}$ by the universal property for W-derivations of A into separated A-modules, and then

as usual, <u>one defines</u> $\Omega^p_{A/W} = \wedge^p \Omega^1_{A/W}$, <u>thereby obtaining a complex</u>

<u>of M-W-algebras. Then, if</u> $u : A \longrightarrow B$ <u>is a homomorphism of M-W-</u>

<u>algebras smooth on W,</u> <u>the induced homomorphism</u>

$$u^* : \underset{=}{H}{}^*(\Omega^{\cdot}_{A/W}) \longrightarrow \underset{=}{H}{}^*(\Omega^{\cdot}_{B/W})$$

<u>depends only on the induced homomorphism</u> $u_o : A_o \longrightarrow B_o$.

Thus, if A_o is a variable finitely generated smooth k-algebra,

$$A_o \longmapsto H^*(\Omega^{\cdot}_{A/W})$$

is a well defined "cohomology functor" from the category of such

algebras to the category of modules on W.

By localizing the above functor with respect to the Zariski

topology, Monsky and Washnitzer construct cohomology sheaves

$\underset{=}{H}{}^*_{WM}(X_o)$ for each scheme X_o smooth on k. The global sections of these

sheaves yield satisfactory global invariants in dimension 0 and 1

for smooth schemes on k. To obtain global invariants in higher

dimensions, Monsky and Washnitzer introduced the site $X_{o_{WM}}$ of

M-W-liftings of X_o. The underlying category of this site has as

objects all pairs (U_o, A), where U_o is a Zariski open set of X_o, and

A a M-W-lifting of the coordinate ring of U_o, and its morphisms are

defined in the obvious fashion. The topology is "that of Zariski".

The complexes $\Omega^{\cdot}_{A/W}$ define a complex of sheaves $\underline{\Omega}_{X_o}$ on this site,

and one takes the hypercohomology

$$H^*_{WM}(X_o) = \underset{=}{H}{}^*(X_{o_{WM}}, \underline{\Omega}^{\cdot}_{X_o}) .$$

When X_o is affine, this can be shown (as Grothendieck understood from

Washnitzer) to be just the functor (*). On the other hand, when X_o

is proper and has a proper and smooth lifting $X \longrightarrow S$, we should have

a canonical isomorphism

$$H_{WM}^*(X_o) \xrightarrow{\sim} H_{DR}^*(X/S) \ ,$$

which would establish the invariance property of the $H_{DR}^*(X/S)$

relative to the various liftings $X \longrightarrow S$ of X_o considered in (2.1).

2.3. Remarks on Monsky and Washnitzer's method.

Their theory gives a p-adic cohomology theory, their cohomology

groups being modules over the ring W of Witt vectors of k.

According to their published work, they have only proven their

fundamental invariance assertion modulo torsion, i.e. after tensoring

by the quotient field K of W. However in a private communication,

as Grothendieck understood it, Washnitzer indicated that they have

been able to remove this restriction. It is of course of essential

importance to have a theory with coefficients in W, rather than in K.

Their method semms too closely bound to differential forms, which

practically limits its applications to smooth schemes. For this reason

perhaps, they have so far not overcome certain technical difficulties,

and have been unable so far to prove some of the usual properties for

their cohomology : for example, that their cohomology groups are of

finite rank on W, even when X_o is projective on k.

Nevertheless, they have been able to utilize their theory to give

a form of the Lefschetz fixed point theorem, using completely continuous

operators, and, by applying this to the frobenius endomorphism of a

smooth scheme over a finite field, they prove the rationality of the zeta function of this scheme (in a way similar to that of Dwork).

3. Connections on the De Rham cohomology.

For the definition of a connection and a stratification on a sheaf, see Appendix I of these notes.

3.1. Consequence of Monsky and Washnitzer's invariance result.

For simplicity, we assume that Monsky and Washnitzer have proven their invariance assertion in its more precise formulation : namely, given an X_o smooth on a field k of characteristic $p > 0$, and any two liftings h : $X \longrightarrow S$ and h' : $X' \longrightarrow S$ of X_o/k to schemes proper and smooth on S = Spec W(k), then there is a canonical (in the sense that it yields a transitive system of isomorphisms between all such liftings) isomorphism
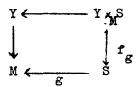
$$\underline{R}\, h_*(\underline{\Omega}^{\bullet}{}_{X/S}) \overset{\sim}{\longrightarrow} \underline{R}\, h'_*(\underline{\Omega}^{\bullet}{}_{X'/S})$$

in the derived category of $\underline{O}_S$-modules.

We now derive a consequence of this invariance assertion for an algebraic family of such liftings. Suppose

$$f : Y \longrightarrow M$$

is an algebraic family of liftings of X_o/k to schemes proper and smooth on S, in the following sense : we are given a morphism $M \longrightarrow S$ of finite type, and a proper and smooth morphism f : $Y \longrightarrow M$. Then, for each section g of M on S, passing through a fixed point t_o in the fiber of the closed point of S, the morphism f_g : $X_g \longrightarrow S$ given by base change by g

$$Y \longleftarrow Y \underset{M}{\times} S$$
$$\downarrow \qquad \uparrow \quad f_g$$
$$M \underset{g}{\longleftarrow} S$$

is a lifting of $X_0 = X_{t_0}$ to a proper and smooth scheme on S. Consider the complex of sheaves on M

$$\underline{K}^{\bullet} = \underline{R} f_*(\underline{\Omega}^{\bullet}_{Y/M}) \ .$$

Then, since f is smooth, the De Rham cohomology commutes with base change (in the derived category sense), and thus there is a <u>canonical</u> isomorphism

$$\underline{R}(f_g)_*(\underline{\Omega}^{\bullet}_{X_g/S}) \xrightarrow{\sim} Lg^*(\underline{K}^{\bullet}) \ .$$

But, by Monsky and Washnitzer's invariance result, the various

$$\underline{R}(f_g)_*(\underline{\Omega}^{\bullet}_{X_g/S})$$

are canonically isomorphic to each other for the different sections g of M on S passing through t_0. Hence, the complexes of sheaves

$$Lg^*(\underline{K}^{\bullet}) = Lg^*(\underline{R} f_*(\underline{\Omega}^{\bullet}_{Y/M})$$

on S, for the different sections g of M on S passing through t_0, are <u>canonically isomorphic</u> to each other (*).

This strongly suggests that the complex of sheaves

$$\underline{R} f_*(\underline{\Omega}^{\bullet}_{Y/M})$$

has a <u>stratification</u>, in the derived category of $\underline{O}_M$-Modules, in a neighbourhood of t_0. However, by considering families of elliptic curves on k, we shall see in (3.5) that this does not appear to be so, thereby raising a puzzle. However, we shall first give some indications in the positive direction.

(*) in the derived category, of course.

3.2. <u>The transcendental connection</u>. Suppose that S is a scheme of finite type over the complex field $\mathbb{C}$. Let $f : X \longrightarrow S$ be a proper and smooth scheme above S. Then, by the Comparison Theorem for algebraic and analytic De Rham cohomology, the coherent analytic sheaf on S^{an} defined by the coherent algebraic sheaf $\underline{R}^i f_*(\underline{\Omega}^\bullet_{X/S})$ on S is canonically isomorphic to the sheaf

$$R^i f_*^{an}(\mathbb{Z}_{X^{an}}) \otimes_{\mathbb{Z}_{S^{an}}} \mathcal{O}_{S^{an}}$$

(where $\mathbb{Z}_{X^{an}}$ is the constant sheaf of integers on X^{an}). Now there is a canonical absolute integrable connection on this analytic sheaf, namely the canonical connection on the tensor product, characterized by the condition that its horizontal sections are those of the subsheaf

$$R^i f_*^{an}(\mathbb{C}_{X^{an}}) = R^i f_*^{an}(\mathbb{Z}_{X^{an}}) \otimes_{\mathbb{Z}_{S^{an}}} \mathbb{C}_{S^{an}} \quad .$$

It will turn out that this transcendental connection comes from a connection on the algebraic De Rham cohomology sheaf $\underline{R}^i f_*(\underline{\Omega}^\bullet_{X/S})$. Moreover, this latter connection can be defined by purely <u>algebraic</u> means.

3.3. <u>The algebraic connection of Gauss-Manin</u>. Manin [7], generalizing an idea of Gauss, gave the following algebraic construction of this connection. Let k be a field, and K a separable extension of k. Each derivation ∂ of k extends to a derivation $\tilde{\partial}$ of K. The derivation $\tilde{\partial}$ operates in a natural fashion on the De Rham complex $\Omega^\bullet_{K/k}$, and so on the De Rham cohomology $H^i_{DR}(K/k)$, and Manin shows that the operation of $\tilde{\partial}$ on $H^i_{DR}(K/k)$ depends only on ∂ and not on the extension $\tilde{\partial}$ of ∂. Thus the derivations of k operate on the

$H^i_{DR}(K/k)$ in such a fashion (see Appendix I) as to define an absolute integrable connection on the $H^i_{DR}(K/k)$.

If now X is a smooth model of the function field K, Manin shows that the connection on $H^1_{DR}(K/k)$ induces one on $H^1_{DR}(X/k)$, by the canonical injection $H^1_{DR}(X/k) \hookrightarrow H^1_{DR}(K/k)$.

We can put Manin's birational construction in a more general setting. Let us first recall Cartan's homotopy formula.

Let $f : X \longrightarrow S$ be a morphism of schemes, and let ∂ be an $f^{-1}(\underline{O}_S)$-derivation of $\underline{O}_X$, i.e. $\partial \in \operatorname{Hom}_{\underline{O}_X} (\underline{\Omega}^1_{X/S}, \underline{O}_X)$. Then there is the $\underline{O}_X$-homomorphism, $i(\partial) : \underline{\Omega}^p_{X/S} \longrightarrow \underline{\Omega}^{p-1}_{X/S}$, called inner product,   /

defined by

$$i(\partial)(dx_1 \wedge \ldots \wedge dx_p) = \sum_{i=1}^{p}(-1)^{i+1} \langle dx_i, \partial \rangle dx_1 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots \wedge dx_p.$$

Further, ∂ induces an $f^{-1}(\underline{O}_S)$-homomorphism

$$\Theta(\partial) : \underline{\Omega}^\bullet_{X/S} \longrightarrow \underline{\Omega}^\bullet_{X/S} \text{ ,}$$

which is defined on $\underline{\Omega}^0_{X/S} = \underline{O}_X$ by $\Theta(\partial)(x) = \langle \partial, dx \rangle$, and then extended to $\underline{\Omega}^\bullet_{X/S}$ by imposing that it commutes with d, and that its action on all products (interior, exterior, and scalar) is given by the classical formula for the derivative of a product. Cartan's homotopy formula is then

$$\Theta(\partial) = i(\partial)d + di(\partial)$$

(the proof is immediate since both sides commute with d and agree on $\underline{\Omega}^0_{X/S}$). Thus the endomorphism $\Theta(\partial)$ of $\underline{\Omega}^\bullet_{X/S}$ induced by the $f^{-1}(\underline{O}_S)$-derivation ∂ of $\underline{O}_X$ is homotopic to zero.

Now suppose that f is formally smooth and affine. Let ∂ be a derivation of $\underline{O}_S$. Then one can locally on S lift ∂ to a derivation of O_X. For it suffices to show that,

if $A \longrightarrow B$ is a formally smooth morphism of rings, then we can extend a derivation of A to a derivation of B. But this follows immediately from the standard exact sequence (EGA O_{IV} 20).

$$0 \longrightarrow Der_A(B,B) \longrightarrow Der_Z(B,B) \longrightarrow Der_Z(A,B) \longrightarrow Exalcom_A(B,B) \longrightarrow \ldots$$

and the fact that $Exalcom_A(B,B) = 0$, since $A \longrightarrow B$ is formally smooth. Further, Cartan's homotopy formula immediately shows that the action of the local lifting on the De Rham complex depends up to homotopy only on the derivation ∂ , and <u>not</u> on the local lifting. Thus it follows that one can make the derivations of $\underline{O}_S$ operate on the De Rham cohomology sheaf $\underline{H}^i_{DR}(X/S)$, and one hopes therefore to define an absolute integrable connection on it. Of course, the difficulty with this method, when f is not affine, is that one cannot in general lift the derivations of $\underline{O}_S$ locally on S to derivations of $\underline{O}_X$.

3.4. <u>Construction of the absolute canonical connection on the De Rham cohomology</u>. By using again what is essentially Cartan's homotopy formula, we now construct an absolute connection, in the sense of <u>derived categories</u>, on the De Rham cohomology $\underline{R}f_*(\underline{\Omega}^\bullet_{X/S})$ of an arbitrary <u>smooth</u> morphism f : $X \longrightarrow S$, and even a slightly stronger structure, as we shall see. In the case when the $\underline{R}^i f_*(\underline{\Omega}^\bullet_{X/S})$ are <u>locally free</u> (and therefore commute with base change), our proof also shows that there is an absolute connection on the $\underline{R}^i f_*(\underline{\Omega}^\bullet_{X/S})$.

To give such a connection is equivalent to constructing, for each diagram

where $h : S \hookrightarrow S'$ is a closed immersion defined by an ideal I of

square zero, q_1, q_2 are two retractions of h, and $f' : X' \longrightarrow S'$,

$f'' : X'' \longrightarrow S'$ are smooth liftings of $f : X \longrightarrow S$ given by base change

by q_1, q_2, an isomorphism

$$\underline{R} \; f'_* (\underline{\Omega}^{\cdot}_{X'/S'}) \xrightarrow{\sim} \underline{R} \; f''_* (\underline{\Omega}^{\cdot}_{X''/S'})$$

(satisfying the natural condition of transitivity for a third

$q_3 : S' \longrightarrow S$). For, since f is smooth and hence commutes with base

change in the sense of derived categories, this means that we have

given a canonical isomorphism

$$\underline{L} \; q_1^* \; (\underline{R} \; f_* (\underline{\Omega}^{\cdot}_{X/S})) \xrightarrow{\sim} \underline{L} \; q_2^* (\underline{R} \; f_* (\underline{\Omega}^{\cdot}_{X/S})),$$

which is precisely the definition of a connection in the sense of

derived categories. Our construction does not use the two retractions

q_1, q_2, and is valid for any two smooth liftings X', X'' of $f : X \longrightarrow S$.

Let $\underline{G}$ denote the sheaf of germs of S'-automorphisms of X' which

induce the identity on X, and $\underline{P}$ the sheaf of germs of S'-isomorphisms

from X' onto X'', which induce the identity on X. Then there is a

canonical isomorphism [SGA 1 III]

$$\underline{G} \xrightarrow{\sim} \underline{\text{Hom}}_{\underline{O}_X} (\Omega^1_{X/S}, I \, \underline{O}_{X'}) = \underline{\text{Der}}_S(\underline{O}_X) \underset{X}{\otimes} I \, \underline{O}_{X'} \; .$$

Further, $\underline{P}$ is a right "torsor" (=principal homogenous sheaf) under

$\underline{G}$. Note also that there are canonical isomorphisms of $\underline{O}_X$-modules

$$\underline{\Omega}^P_{X/S} \xrightarrow{\sim} \underline{\Omega}^P_{X'/S'} \otimes_{\underline{O}_{X'}} \underline{O}_X \; , \quad I \, \underline{O}_{X'} \xrightarrow{\sim} I \, \underline{O}_{X''} \; ,$$

and we shall identify these canonically isomorphic Modules.

We have the following morphisms. Firstly, there is the canonical

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morphism

$$u : \underline{P} \longrightarrow \underline{\mathrm{Hom}}_{S'}^{(o)}(\underline{\Omega}^{\cdot}_{X'/S'} , \underline{\Omega}^{\cdot}_{X''/S'}) \quad (*)$$

induced by the canonical morphism $\underline{P} \longrightarrow \underline{\mathrm{Isom}}(\Omega^{\cdot}_{X''/S'} , \underline{\Omega}^{\cdot}_{X''/S'})$

given by transport of structure. Secondly, there is the canonical

morphism of interior product

$$i : \underline{G} \longrightarrow \underline{\mathrm{Hom}}_{S'}^{(-1)}(\underline{\Omega}^{\cdot}_{X'/S'} , \underline{\Omega}^{\cdot}_{X''/S'}) ,$$

which is defined by

$$i(\partial)(f'dx'_1 \wedge \ldots \wedge d'x'_n) = pf' \sum_{j=1}^{n} (-1)^{j+1} \langle \partial , dx'_j \otimes 1 \rangle p_* \ldots$$
$$\ldots (d'x'_1 \wedge \ldots \wedge \widehat{d'x'_j} \wedge \ldots d'x'_n)$$

where $\partial \in \Gamma(U, \underline{G})$, $f' \in \Gamma(U, \underline{O}_{X'})$, and $p \in \Gamma(U, \underline{P})$. This definition

does not depend on the particular section $p \in \Gamma(U, P)$. Finally,

there is the canonical morphism

$$\theta : \underline{G} \longrightarrow \underline{\mathrm{Hom}}_{S'}^{(o)}(\Omega^{\cdot}_{X'/S'}, \underline{\Omega}^{\cdot}_{X''/S'})$$

which is defined on $\underline{\Omega}^{o}_{X'/S'} = \underline{O}_{X'}$ by $\theta(\partial)(f') = \langle \partial , d'f' \otimes 1 \rangle$, and

then extended to $\underline{\Omega}^{\cdot}_{X'/S'}$ by imposing that it commutes with d', d'', and

that its action on all products is given by the classical formula for

the derivative of a product. Then

$$(*) \qquad \theta(\partial) = i(\partial)d' + d'' i(\partial).$$

Further, we have the relations

$$(**) \quad \left\{ \begin{array}{l} u(p\partial) = u(p) + \theta(\partial) \\[2mm] \theta(\partial + \partial') = \theta(\partial) + \theta(\partial') . \end{array} \right.$$

Now we can express the fact that $\underline{P}$ is a principal homogenous

space under $\underline{G}$ by giving an extension of $\underline{G}$ by $\mathbb{Z}$ (the constant sheaf

of integers)

$$0 \longrightarrow \underline{G} \xrightarrow{e} \underline{\overline{G}} \xrightarrow{j} \mathbb{Z} \longrightarrow 0$$

(*) the subscript S' denotes $f^{-1}(\underline{O}_{S'})$-homomorphisms.

together with a $\underline{G}$-isomorphism

$$\underline{P} \longrightarrow j^{-1}(1).$$

Let $\underline{H}^{\bullet} = \underline{\operatorname{Hom}}^{\bullet}_{S'}(\underline{\Omega}^{\bullet}_{X'/S'}, \underline{\Omega}^{\bullet}_{X''/S'})$, and let the complex $\underline{L}^{\bullet}$ be defined by

$$\cdots \longrightarrow 0 \longrightarrow \underline{L}^{-1} \overset{d}{\longrightarrow} \underline{L}^{0} \longrightarrow 0 \longrightarrow \cdots$$
$$\underset{\underline{G}}{\|} \overset{e}{\longrightarrow} \underset{\underline{G}}{\|}$$

Then there is a morphism

$$\varphi : \underline{L}^{\bullet} \longrightarrow \underline{H}^{\bullet}$$

defined by the conditions

$$\varphi^{(-1)} = i \ , \ \varphi^{(0)}\big|_{\underline{P}} = u \ , \ (\varphi^{0} \, d = \theta) \ .$$

In the derived category of abelian sheaves on the underlying topo-
logical space of X, there is an isomorphism

$$\mathbb{Z} \overset{\sim}{\longrightarrow} \underline{L}^{\bullet} \ ,$$

whence we obtain from φ a canonical homomorphism
in the derived category of $\mathbb{Z}$-Modules

$$\mathbb{Z} \longrightarrow \underline{H}^{\bullet} \ , \quad \text{i.e. an element } \Psi \in \mathbb{R}^{0} \, \Gamma_{X}(\underline{H}^{\bullet}) \ .$$

Composing with the canonical morphism

$$\underline{H}^{\bullet} \longrightarrow R \, \underline{\operatorname{Hom}}_{S'}(\underline{\Omega}^{\bullet}_{X'/S'} \ , \ \underline{\Omega}^{\bullet}_{X''/S'})$$

we obtain a canonical element of $\mathbb{R}^{0}\Gamma_{X}(\underline{H}^{\bullet}) \approx \operatorname{Hom}^{0}_{D(f^{-1}(\underline{O}_{S'}))}(\underline{\Omega}^{\bullet}_{X'/S'}, \underline{\Omega}^{\bullet}_{X''/S'})$.

This canonical element is seen to be an isomorphism by observing

that it leads to a transitive system of morphisms between the complexes

$\underline{\Omega}^{\bullet}_{X'/S'}$ for the different liftings X'/S' of X/S, and that it is the

identity when $X' = X''$. Applying the functor $\mathbb{R}f_{*}$, we get the desired

isomorphism $\mathbb{R}f_{*}(\underline{\Omega}^{\bullet}_{X'/S'}) \overset{\sim}{\longrightarrow} \mathbb{R}f_{*}(\underline{\Omega}^{\bullet}_{X''/S''})$, which defines our canonical

connection.

3.5. <u>The non-existence of canonical stratifications on the De Rham</u>

<u>cohomology</u>. One may hope that the <u>connection</u> just constructed can be deduced

from a stronger infinitesimal structure, namely a <u>stratification</u> (cf. Appendix).

We show that this is not so. Let k be a field of characteristic p>0, and M/S

the modular scheme of all elliptic curves defined on k. Then it is not pos-

sible to define a stratification relative to k on the De Rham cohomology sheaf

$$\underline{\underline{H}}^1_{DR} \ (M/S) \ ,$$

with "reasonable" functorial properties, namely, compatible with

morphisms of elliptic curves and with base change. For, the iterated

frobenius homomorphism $M \longrightarrow M^{(p^2)}$ induces a homomorphism

(*) $\qquad \widetilde{\pi} : \underline{\underline{H}}^1_{DR}(M/S)^{(p^2)} \longrightarrow \underline{\underline{H}}^1_{DR}(M/S) \qquad ,$

which must be compatible with the stratifications. Now the fiber M_s

of M at a point $s \in S$ has Hasse invariant zero if and only if the

$$\widetilde{\pi}_s : H^1_{DR}(M_s/K(s))^{(p^2)} \longrightarrow H^1_{DR}(M_s/K(s))$$

induced by (*) is <u>zero</u>. But the existence of stratifications

compatible with $\widetilde{\pi}$ would imply that, if (*) induces the zero endo-

morphism on the fiber at $s \in S$ of $\underline{\underline{H}}^1_{DR}(M/S)$, then it would induce the

zero endomorphism on all fibers of $\underline{\underline{H}}^1_{DR}(M/S)$ in some neighbourhood of

$s \in S$. But this is impossible, since, p being fixed, there are only

finitely many elliptic curves of Hasse invariant zero.

4. <u>The infinitesimal topos and stratifying topos</u>.

We now turn to the definition of a more general category of

coefficients for the De Rham cohomology. To this end we introduce two

ringed topos, the <u>infinitesimal topos</u> and the <u>stratifying topos</u>.

We shall see later that in fact these two topos work well only in characteristic 0, and at the end of these notes we propose the definition of two more ringed topos, the crystalline topos and connecting topos, the sheaves of modules of which seem. to define the good categories of coefficients for the generalized De Rham cohomology in arbitrary characteristic.

4.1. The infinitesimal topos. Let X be a scheme above the base S. We first define the infinitesimal site of X/S, Inf(X/S). The underlying category of Inf(X/S) has as objects all "nilpotent" S-immersions $U \hookrightarrow T$,i.e. those defined by a nilpotent ideal on T, U being any open set of X. The morphisms of this category are defined in the natural fashion. The topology on the category is generated by the pretopology defined by taking, as covering families of U $\longrightarrow$ T, those defined by Zariski open coverings (T_i) of T, to each T_i being associated the immersion $U_i \hookrightarrow T_i$, where $U_i = U \times_T T_i$.

A sheaf of sets on Inf(X/S) (or an infinitesimal sheaf on X/S) can be identified with a system of sheaves of sets $F_{(U, T)}$ on T, one for each object U$\longrightarrow$T of Inf(X/S), together with, for each morphism $(U \longrightarrow T) \longrightarrow (U' \longrightarrow T')$, a homomorphism of the inverse image of $F_{(U', T')}$ on T into $F_{(U, T)}$, such that this homomorphism is an isomorphism when T $\longrightarrow$T' is an open immersion, and that the resulting system of homomorphisms is transitive. The same description holds for sheaves of groups, rings etc. Thus, in particular, if for each object U$\longrightarrow$T of Inf(X/S), we take the structural sheaf of rings of T, we obtain a sheaf of rings on Inf(X/S) , which we denote by

$\underline{O}_{X_{inf}}$. The category of all sheaves of sets on $Inf(X/S)$ is therefore a ringed topos, which we call the infinitesimal topos of X/S, and which we denote by $(X/S)_{inf}$ or (if S is understood implicitly) by X_{inf}.

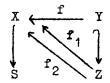Note that $Inf(X/S)$ does not have a final object, in general.

4.2. The stratifying topos. As we shall see shortly, the reason for introducing the stratifying site of X/S, $Strat(X/S)$, is that we can interpret a Module on X, fortified with a stratification relative to S, as a "special" sheaf on this site.

$Strat(X/S)$ is defined as follows. Its underlying category is the full subcategory of $Inf(X/S)$ consisting of those objects $U \longrightarrow T$ such that there exists locally a retraction $T \longrightarrow X$. The category $Strat(X/S)$ is fortified with the induced topology. Note that if X is smooth on S, then $Strat(X/S)$ and $Inf(X/S)$ are the same.

Sheaves on $Strat(X/S)$ can be described in exactly the same way as sheaves on $Inf(X/S)$. In particular, there is also a canonical sheaf of rings $\underline{O}_{X_{strat}}$ on $Strat(X/S)$, obtained by taking, for each object $U \longrightarrow T$ of $Strat(X/S)$, the structural sheaf of rings of T. Thus the category of all sheaves on $Strat(X/S)$ is a ringed topos, which we denote by $(X/S)_{strat}$ or often just X_{strat}.

We now turn to the interpretation of stratified Modules on X as "special" Modules on $Strat(X/S)$. Define an $\underline{O}_{X_{strat}}$ -Module F to be "special" if, for each morphism $(U \longrightarrow T) \longrightarrow (U' \longrightarrow T')$, the homomorphism of the inverse image (in the sense of ringed spaces) of $F_{(U',T')}$ into

$F_{(U,T)}$ is an _isomorphism_. Then the category of special $\underline{O}_{X_{strat}}$ -Modules

is equivalent to the category of Modules on X fortified with a

stratification relative to S. To see this, recall (see Appendix I)

that, if F is a Module on X fortified with a stratification relative
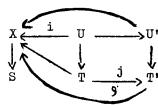
to S, then for each diagram of S-morphisms

$$X \xleftarrow{\quad f \quad} Y$$

where $Y \hookrightarrow Z$ is a nilpotent immersion, and f_1, f_2 are any two

extensions of f, there is a _canonical_ isomorphism

$$f_1^*(F) \xrightarrow{\;\sim\;} f_2^*(F)$$

which yields a transitive system of isomorphisms between the inverse

images of F by the various extensions of f. Now, if F is a Module on

X fortified with a stratification relative to S, we associate the

Module F on Strat(X/S) defined by taking, for each object

$$X \xleftarrow{\quad i \quad} U$$

of Strat(X/S), $F_{(U,T)}$ to be $g^*(F)$, where g is some retraction of T to X (which

exists locally). Note that, since F is fortified with a stratification, this

definition does not depend, up to canonical isomorphism, on the particular ex-

tension g of $i : U \hookrightarrow X$. Further, this Module F on Strat(X/S) is a "special"

Module. For if

is any morphism of Strat(X/S), then it is clear that g and $g'j$ are extensions of i, and thus, since F is stratified, there is a canonical isomorphism

$$j^* g'^*(F) \overset{\sim}{\longrightarrow} g^*(F)$$

i.e. the Module F on Strat (X/S) is "special". Conversely, if F is a "special" Module on Strat(X/S), it is clear that $F_{(X,X)}$ is a Module on X fortified with a stratification relative to S.

4.3. _The fundamental theorem._ One principal aim of these notes is to sketch a proof of the following theorem.

Theorem 4.1. _If S is of characteristic 0, and X is smooth on S, there is a canonical isomorphism_

$$H^*(X_{inf}, \underline{O}_{X_{inf}}) \overset{\sim}{\longrightarrow} H^*_{DR}(X/S) \quad .$$

The significance of this theorem is clear. It not only gives a description of the De Rham cohomology without using differential forms, but it also gives a sufficiently general context in which to study the De Rham cohomology. Further, we shall deduce that the canonical connection on the De Rham cohomology in characteristic 0 comes from a stratification which has a natural interpretation in terms of the infinitesimal site. It seems likely that the same theorem will hold in arbitrary characteristic if we replace the infinitesimal site by the crystalline site, whose definition will be given at the end of these notes. Of course, when S is of characteristic 0, the crystalline site is just the infinitesimal site.

Finally, combining Theorems 4.1 and 1.2, we see that the following conjecture is true when X is smooth on $\mathbb{C}$:

Conjecture 4.2. If X is locally of finite type (not necessarily smooth) over $\mathbb{C}$, there are canonical isomorphisms

$$H^*(X_{strat}, \underline{O}_{X_{strat}}) \xrightarrow{\sim} H^*(X_{inf}, \underline{O}_{X_{inf}}) \xrightarrow{\sim} H^*(X^{an}, \mathbb{C}).$$

A canonical homomorphism

$$H^*(X_{strat}, \underline{O}_{X_{strat}}) \longrightarrow H^*_{DR}(X/\mathbb{C})$$

which, it is conjectured, is an isomorphism and hence by (1.2) yields

$H^*(X_{strat}, \underline{O}_{X_{strat}}) \xrightarrow{\sim} H^*(X^{an}, \mathbb{C})$, is defined in n° 6. A heuristic

argument in favour of this conjecture is also given in (5.5). The truth of this conjecture (which seems very plausible) would imply that the infinitesimal (or stratifying) topos allows, for schemes of finite type over a field of characteristic 0, a reasonable, purely algebraic, analogue of the classical transcendentally defined complex cohomology. (Compare with the tentative description of singular homology [5]).

5. Čech calculations.

We now consider the cohomology of the infinitesimal topos and the stratifying topos (*).

5.1. The stratifying topos. Let X be a scheme above the base S. The category Strat(X/S) does not, in general, have a final object. However, the sheaf of sets $\tilde{X}$ on Strat(X/S), represented by the object $X \xrightarrow{1_X} X$ of Strat(X/S), covers the final object of the stratifying topos. Note that this is only true for the stratifying

(*) For a general discussion of the cohomology of a topos, see (SGA 4 V).

site, and is not true, in general, for the infinitesimal site. Thus,
if F is any Module on Strat(X/S), we have the Leray spectral
sequence

$$H^*(X_{strat}, F) \Longleftarrow E_2^{pq} = H^p(\nu \longmapsto H^q(\tilde{X}^{\nu+1}, F))$$

($\tilde{X}^{\nu+1}$ denotes the product of $\tilde{X}$ with itself $\nu + 1$ times). The sheaf
$\tilde{X}^{\nu+1}$ is usually not representable for $\nu \geq 1$. But it is an inductive
limit of representable sheaves, namely

$$\tilde{X}^{\nu+1} = \varinjlim_i \widetilde{(\Delta_{X/S}^\nu(i))}$$

where $\Delta_{X/S}^\nu(i)$ is the ith infinitesimal neighbourhood of the diagonal
of $X_{(S)} = X \times_S \cdots \times_S X$ ($\nu + 1$ times) endowed with the diagonal aug-
mentation, the $\sim$ denoting the sheaf on Strat(X/S) represented by
this object.

In all that follows in n° 5, we suppose that the Module F satisfies
the following two conditions (which could be considerably relaxed) :

(1) For each object $U \hookrightarrow T$, $F_{(U,T)}$ is quasi-coherent.

(2) For each morphism

$$\begin{array}{ccc} U & \hookleftarrow & T \\ \uparrow & & \uparrow j \\ U' & \hookrightarrow & T' \end{array}$$

of Strat(X/S) such that $j : T \hookrightarrow T'$ is an immersion, we have
$F_{(U,T)} = j^*(F_{(U,T')})$.

Now for each object $U \longrightarrow T$ of Strat(X/S), there is a canonical
isomorphism

$$H^*_{strat}((U \longrightarrow T), F) \xrightarrow{\sim} H^*_{zar}(T, F_{(U,T)}) .$$

Suppose that X is <u>affine</u>, whence the $\Delta_{X/S}^\nu(i)$ are also affine.
Thus

$$H^q_{strat}((X \longrightarrow \Delta^\nu_{X/S}(i)),F) = 0 \quad \text{if } q > 0 \ .$$

Further, by Mittag–Leffler style arguments, it can be shown that

$$H^q(\widetilde{X}^{\nu+1},F) = H^q(\varinjlim_i \Delta^\nu_{X/S}(i),F) = \varinjlim_i H^q(\Delta^\nu_{X/S}(i),F),$$

whence

$$H^q(\widetilde{X}^{\nu+1},F) = 0 \quad \text{if } q > 0,$$

$$H(\widetilde{X}^{\nu+1},F) = \varprojlim_i F(\Delta^\nu_{X/S}(i)) \ .$$

Hence the Leray spectral sequence degenerates, yielding a canonical isomorphism

$$(\overset{*}{\ }) \qquad H^*(X_{strat},F) \overset{\sim}{\longrightarrow} H^*(\nu \longmapsto F(X^\nu_{/X})) \ ,$$

where $X^\nu_{/X}$ denotes the formal scheme $\varinjlim \Delta^\nu_{X/S}(i)$.

Now no longer assume that X is affine. Then we introduce, for each non-negative integer ν , the sheaf $\mathcal{F}^\nu$ on X_{zar} defined by

$$\mathcal{F}^{(\nu)} : \quad U \longmapsto F(U^\nu_{/U}) = \varprojlim_i F(\Delta^\nu_{U/S}(i)) \ ,$$

where $U^\nu_{/U}$ denotes the formal scheme $\varinjlim \Delta^\nu_{U/S}(i)$. For variable ν , the $\mathcal{F}^{(\nu)}$ form a cosimplicial sheaf $\mathcal{F}^*$ on X_{zar}. Taking a covering of X by affine open sets, and utilising the isomorphism $(\overset{*}{\ })$ in the affine case, we deduce that there is a canonical isomorphism

$$H^*(X_{strat},F) \overset{\sim}{\underset{=}{\longrightarrow}} H^*(X_{zar}, \mathcal{F}^*) \ .$$

In particular, we have then the spectral sequence

$$H^*(X_{strat},F) \Longleftarrow E^{pq}_2 = H^p(\nu \longmapsto H^q(X_{zar}, \mathcal{F}^{(\nu)})).$$

giving back the isomorphism $(\overset{*}{\ })$ when X is affine.

5.2. **The infinitesimal topos.** It is not usually true that the sheaf $\widetilde{X}$ represented by the object $X \overset{1_X}{\longrightarrow} X$ of $\mathrm{Inf}(X/S)$ covers the

final object of the infinitesimal topos. However, let us assume

that we can find an S-immersion $X \hookrightarrow Y$, with Y formally smooth on

S (for example, this is possible if X and S are affine, or if X is

quasi-projective on S and S admits an ample Module). Then consider

the sheaf on $\mathrm{Inf}(X/S)$ "represented" (we put " " since $X \hookrightarrow Y$ may not

be an object of $\mathrm{Inf}(X/S)$) by $X \longrightarrow Y$, namely

$$\tilde{Y} : (U,T) \longmapsto \mathrm{Hom}((U,T),(X,Y)) \ .$$

One can also describe this sheaf by

$$\tilde{Y} = \varinjlim_{i} \widetilde{Y(i)} \ ,$$

where $Y(i)$ denotes the ith infinitesimal neighbourhood of X in Y,

and, as usual, $\sim$ denotes the sheaf on $\mathrm{Inf}(X/S)$ represented by the

object. Since Y is formally smooth on S, $\tilde{Y}$ covers the final object of

$\mathrm{Inf}(X/S)$. Hence, if F is any Module on $\mathrm{Inf}(X/S)$, we have the Leray

spectral sequence

$$H^{*}(X_{\mathrm{inf}},F) \Longleftarrow E_{2}^{pq} = H^{p}(\nu \longmapsto H^{q}(\tilde{Y}^{\nu+1},F)) \ .$$

Assume that F satisfies the conditions (1) and (2) given in

(5.1). Then, if X is affine, the same argument as in the stratifying

case shows that this spectral sequence degenerates, yielding a canonical

isomorphism

$$(*) \qquad H^{*}(X_{\mathrm{inf}},F) \xrightarrow{\sim} H^{*}(\nu \longmapsto F(Y_{/X}^{\nu})) \ ,$$

where $Y_{/X}^{\nu}$ is the formal completion of $Y_{(S)}^{\nu} = Y \times_{S} \cdots \times_{S} Y$ ($\nu+1$ times)
along X.

If we no longer assume X affine, we introduce, for each non-

negative integer ν , the sheaf $\mathcal{F}^{(\nu)}$ on X_{zar} defined by

$$\mathcal{F}^{(\nu)} : U \longmapsto \varprojlim_i F(U \longrightarrow U_Y^{\nu}(i)) = F(Y_{/U}^{\nu}) \quad ,$$

where $Y_{/U}^{\nu}$ is the formal completion of $Y_{(S)}^{\nu}$ along U. For variable ν, we obtain a cosimplicial sheaf $\mathcal{F}^*$ on X_{zar}. Taking a covering of X by affine open sets, and utilizing the isomorphism ($\overset{*}{\cdot}$) in the affine case, we deduce that there is a canonical isomorphism

$$H^*(X_{inf},F) \xrightarrow{\sim} \underline{H}^*(X_{zar},\mathcal{F}^*) \quad .$$

Thus there is the spectral sequence

$$H^*(X_{inf},F) \Longleftarrow E_2^{pq} = H^p(\nu \longmapsto H^q(X_{zar},\mathcal{F}^*)) \quad ,$$

yielding back the isomorphism (*) when X is affine.

5.3. <u>Stratification on the De Rham cohomology in characteristic 0.</u>

As before, suppose there is an S-immersion $X \hookrightarrow Y$. Assume we are given a second scheme X_0 on S, and a nilpotent immersion

$$X_0 \hookrightarrow X .$$

Then there is the obvious functor

$$u_* : Inf(X/S) \longrightarrow Inf(X_0/S) \quad .$$

If F_0 is a sheaf on $Inf(X_0/S)$, we define its "restriction" F to $Inf(X/S)$ to be its direct image by the functor u_* , namely

$$F = u_*^* F_0 \quad : \quad (U \longrightarrow T) \longmapsto F (U_0 \hookrightarrow U \hookrightarrow T) \quad .$$

Now, for each non-negative integer ν , and each open set U of X, there is a canonical isomorphism

$$Y_{/U}^{\nu} \xrightarrow{\sim} Y_{/U_0}^{\nu} \quad ,$$

where U_0 is the open set of X_0 corresponding to U(U and U_0 are fortified with their induced structural sheaves). Hence, if $\mathcal{F}_0^*$ and $\mathcal{F}^*$ denote the cosimplicial sheaves on $X_{zar} = X_{0\,zar}$ associated with

F_0 and F, respectively, there is a canonical isomorphism

$$\mathcal{F}_0^* \xrightarrow{\sim} \mathcal{F}^* \ ,$$

whence a canonical isomorphism

$$H^*((X_0/S)_{inf}, F) \xrightarrow{\sim} H^*((X/S)_{inf}, F) \ .$$

Note that we could remove the assumption that X can be immersed in Y formally smooth on S by taking an affine open covering of X and utilizing this isomorphism in the affine case (*).

Before turning to the significance of this isomorphism, let us note that this result is <u>not</u> true if we take the stratifying site instead of the infinitesimal site. Indeed, let us assume $S = \mathrm{Spec}A$, and that X lies over $S_0 = \mathrm{Spec}(A/J)$. As every object of the stratifying site of X_0/S lies in fact over S_0, it follows that the sheaf $\underline{O}_{(X/S)_{strat}}$ is a sheaf of A_0-modules, hence its cohomology groups are A_0-modules, i.e. A-modules annihilated by J, which is generally not true for the cohomology of $\underline{O}_{(X/S)_{strat}}$. This shows that if A_0 is of char. $p > 0$, the cohomology invariants defined by $(X_0/S)_{strat}$ are also modules over a ring of char. $p > 0$, contrarily to what can be achieved using the infinitesimal topos instead.

Now let X be a smooth scheme on S. Suppose that we are given a nilpotent immersion $S_0 \hookrightarrow S$, so that

$$X_0 = X \times_S S_0 \hookleftarrow X$$

is a nilpotent immersion. If we consider X_0 as a scheme on S, instead of S_0, then the above shows that, for each Module F_0 on $\mathrm{Inf}(X_0/S)$, there is a canonical isomorphism

$$H^*((X_0/S)_{inf}, F) \xrightarrow{\sim} H^*((X/S)_{inf}, F) \ .$$

(*) We can get trivially the preceding isomorphism, without restriction on X nor F, by observing that the functor u_* is <u>exact</u> on sheaves of sets.

Thus the right hand side is independent, up to canonical isomorphism, of the particular lifting of X_o/S_o to a smooth scheme on S.

In particular, if we take $F_o = O_{X_{o_{inf}}}$, so that $F = \underline{O}_{X_{inf}}$, and assume that S is of characteristic 0, then Theorem 4.1 (which will be proven in n° 6), shows that there is a canonical isomorphism

$$H^*((X/S)_{inf}, \underline{O}_{X_{inf}}) \xrightarrow{\sim} H^*_{DR}(X/S) \quad .$$

Hence the De Rham cohomology groups $H^i_{DR}(X/S)$ can be identified with the $H^i(X_o/S)_{inf}$, $X_{o_{inf}})$, and hence are independent, up to canonical isomorphism, of the particular lifting of X_o/S_o to a smooth scheme on S, S being of characteristic 0. This does not directly imply that the De Rham cohomology sheaf $\underline{R}^i f_*(\underline{\Omega}^{\bullet}_{X/S})$ is fortified with an absolute stratification, as this sheaf does not commute with base change. However, one could develop the previous considerations for the relative De Rham cohomology in the sense of derived categories, and in this way, one could show that

$$\underline{R} \, f_*(\underline{\Omega}^{\bullet}_{X/S}) \quad ,$$

where $f : X \longrightarrow S$ is a smooth morphism and S is of characteristic 0, is fortified with an absolute stratification. In fact this would show more, namely that $\underline{R} f_*(\underline{\Omega}^{\bullet}_{X/S})$ is an absolute crystal, i.e. it extends canonically to all infinitesimal neighbourhoods of S, and not just those with retractions onto S.

5.4. Differential operators. We note that differential operators (EGA IV 16) arise naturally in the context of the infinitesimal and stratifying sites, via the cosimplicial sheaves $\mathcal{J}^*$. For example, let

$\mathcal{F}^*$ be the cosimplicial sheaf on X_{zar} associated with $\underline{O}_{X_{strat}}$. Then, for each positive integer ν, there are the $\nu + 1$ canonical homomorphisms of sheaves of rings

$$p_i^\nu \; : \; \underline{O}_X \longrightarrow \mathcal{F}(\nu)$$

(induced by the family of canonical homomorphisms $p_i^\nu(n) \; : \; \underline{O}_X \to \underline{O}_{\Delta_{X/S}^\nu}(n)$ $n = 0, 1,\ldots, i = 0,\ldots, \nu + 1$). If we choose one of these canonical homomorphisms, say p_0^ν, to give $\mathcal{F}(\nu)$ the structure of an $\underline{O}_X$-algebra, then the remaining morphisms (which are not $\underline{O}_X$-linear), or rather their truncations

$$p_i^\nu(n) \; : \; \underline{O}_X \longrightarrow \underline{O}_{\Delta_{X/S}^\nu}(n)$$

can be interpreted as differential operators from $\underline{O}_X$ to $\underline{O}_{\Delta_{X/S}^\nu}(n)$. For $\nu = 1$, this truncation is just the "universal differential operator of order n on $\underline{O}_X$" (EGA IV 16).

5.5. <u>Analogy with cochains of Čech-Alexander</u>. If X is a topological space, and A any abelian group, the sheaf of germs of cochains, of Čech-Alexander of degree ν, with values in A, is defined to be the sheaf $C^{(\nu)}(X;A)$ associated with the presheaf

$$U \longmapsto F^{\nu+1}(U;A).$$

where $F^{\nu+1}(U,A)$ is the group of functions from $U^{\nu+1}$ into A, modulo the subgroup of functions which vanish in some neighbourhood of the diagonal of $U^{\nu+1}$. For variable ν, we obtain a cosimplicial sheaf $C^*(X,A)$ on X, and, under suitable assumptions on X (*), it can be shown that the cohomology of $\Gamma C^*(X;A)$ is in fact the cohomology of X with coefficients in A.

(*) The complex C*(X;A) being clearly a resolution of the constant sheaf X, it is enough that each F^ν be a <u>sheaf</u> (as it will be necessarily flask).

It is natural to ask if a similar result holds when we take the "functions" on the formal completion $X^{(\nu)}_{/X}$ of $X^{(\nu)} = X \times_S \ldots \times_S X$ ($\nu+1$ times) along X the diagonal, X being a scheme over S. More precisely, let X be a proper scheme over the complex field $\mathbb{C}$. Then we have the spectral sequence for the stratifying cohomology

$$H^*(X_{strat}, \underline{O}_{X_{strat}}) \Longleftarrow E_2^{pq} = H^p(\nu \longmapsto H^q(X^\nu_{/X}, \underline{O}_{X^\nu_{/X}})) \ .$$

Since X is proper on $\mathbb{C}$, the $H^q(\Delta^\nu_X(i), \underline{O}_{\Delta^\nu_X(i)})$ are finite dimensional vector spaces on $\mathbb{C}$, and a Mittag-Leffler style argument then shows that

$$H^q(X^\nu_{/X}, \underline{O}_{X^\nu_{/X}}) = \varprojlim_i H^q(\Delta^\nu_X(i), \underline{O}_{\Delta^\nu_X(i)}) \ ,$$

whence

$$E_2^{pq} = \varprojlim_i H^p(\nu \longmapsto H^q(\Delta^{(\nu)}_X(i), \underline{O}_{\Delta^\nu_X(i)}) \ .$$

Now, by GAGA [10], we can interpret the $H^q(\Delta^\nu_X(i), \underline{O}_{\Delta^\nu_X}(i))$ as the cohomology of the corresponding analytic sheaf on the corresponding analytic manifold. Hence, defining the sheaf

$$\mathcal{F}^{(\nu)}_{X^{an}} : \quad U \longmapsto (U^\nu_{/U}, \underline{O}_{U^\nu_{/U}})$$

on X^{an} (where the formal completion is in the sense of analytic manifolds), there is a canonical isomorphism

$$H^*(X_{strat}, \underline{O}_{X_{strat}}) \xrightarrow{\sim} H^*(X^{an}, \mathcal{F}_X^{an}) \ .$$

We are then led to the following conjecture.

Conjecture 5.1. <u>For any analytic space</u> $\mathfrak{X}$, <u>the complex</u> $\mathcal{F}^*_{\mathfrak{X}}$ <u>is a resolution of the constant sheaf</u> $\mathbb{C}$.

There is a heuristic argument in favour of this conjecture. Define the <u>formal</u> <u>fiber</u> $\widehat{\mathcal{F}}^{(\nu)}_{\mathfrak{X},x}$ of $\mathcal{F}^{(\nu)}_{\mathfrak{X}}$ at $x \in \mathfrak{X}$ to be

$$\mathcal{F}^{(\nu)}_{\mathfrak{X},x} = \varprojlim_i \underline{O}_{\Delta^\nu_{\mathfrak{X}}(i),x} \ ,$$

where $\wedge$ denotes the completion of the local rings on the right hand side. Then the complex $\widehat{\mathcal{F}}^*_{\mathcal{X},x}$ of formal fibers at $x \in \mathcal{X}$ is a resolution of the field $\mathbb{C}$.

Note that if the (purely local) conjecture 5.1 were true, then we would know that, for a scheme X <u>proper</u> on $\mathbb{C}$, there is a canonical isomorphism

$$H^*(X_{strat}, \underline{O}_{X_{strat}}) \overset{\sim}{\longrightarrow} H^*(X^{an}, \mathbb{C}) \quad .$$

This, together with 1.2, gives some support for conjecture 4.2.

6. Comparison of the Infinitesimal and De Rham Cohomologies.

6.1. The basic Idea. Let X be a scheme above S, and F a quasi-coherent Module on X fortified with a stratification relative to S. Then, as was shown in (4.2), F defines a Module F_{strat} on $\text{Strat}(X/S)$. In (5.1), it was shown that we could associate with F_{strat} a complex of differential operators of infinite order

$$C^*(F) = \mathcal{F}^*$$

on X_{zar}, where

$$C^{(\nu)}(F) = \mathcal{F}^{(\nu)} = \varprojlim_i F_{(X, \Delta^\nu_{X/S}(i))}$$

(recall that $\Delta^\nu_{X/S}(i)$ denotes the ith infinitesimal neighbourhood of the diagonal of $X_{(S)}^\nu = X \times_S \dots \times_S X$ ($\nu+1$ times) , and that $F_{(X, \Delta^\nu_{X/S}(i))}$ is the inverse image of F by any of the $\nu + 1$ canonical projections $\Delta^\nu_X(i) \longrightarrow X$, all of these inverse images being canonically isomorphic because of the stratification on F). It was then shown that there is a canonical isomorphism

$$H^*(X_{strat}, F) \overset{\sim}{\longrightarrow} \underset{=}{H}^*(X_{zar}, C^*(F)) \quad .$$

Thus we have shown that the stratifying cohomology of a stratified

sheaf can be interpreted as the Zariski hypercohomology of a complex

of "differential operators of infinite order". In the following, we

shall show that a converse relation holds, i.e. the Zariski

hypercohomology of any complex of differential operators can be

expressed as the stratifying hypercohomology of a suitable complex

of stratified sheaves. Applying this to the De Rham complex $\Omega^{\bullet}_{X/S}$

we shall thus prove theorem 4.1.

Our method will be to construct a functor $Q^{\circ}(.)$ from the

category $Dif(X/S)$ of Modules on X, with morphisms differential

operators relative to S, to the category of Artin-Riesz pro-Modules

on X fortified with a stratification relative to S. The functor $Q^{\circ}(.)$

will be called the formalizing functor, and it can be viewed intui-

tively as "linearizing" differential operators.

In the particular case when $\mathrm{diag}_{X/S} : X \longrightarrow X \times_S X$ is nilpotent,

so that a stratification on F relative to S is a descent on F relative

to S, the functor $Q^{\circ}(.)$ can be taken to be $Q^{\circ}(F) = f^{*}f_{*}(F)$, where

$f : X \longrightarrow S$ is the structural morphism of X/S.

6.2. Definition of $Q^{\circ}(.)$. We first recall the definition of the

category of Artin-Riesz pro-objects of a category C. The objects of

this category are pro-objects (A_i) of C indexed by $\mathbb{Z}$, whilst the set

of morphisms between two such objects (A_i) and (B_i) is defined to be

$$\varinjlim_{k} \mathrm{Hom}((A_i)_k, (B_i)) \quad ,$$

where $(A_i)_k$ denotes the pro-object obtained from (A_i) by shifting k

places to the right, i.e. its ith component is A_{i+k}, (k is any integer). In other words, a morphism from (A_i) to (B_i) is given by a commutative diagram (suitable k)

$$\cdots \longrightarrow \underset{\downarrow}{\overset{A_{i+k+1}}{\uparrow}} \longrightarrow \underset{\downarrow}{\overset{A_{i+k}}{\uparrow}} \longrightarrow \underset{\downarrow}{\overset{A_{i+k+1}}{\uparrow}} \longrightarrow \cdots$$
$$\cdots \longrightarrow \overset{B_{i+1}}{} \longrightarrow \overset{B_i}{} \longrightarrow \overset{B_{i-1}}{} \longrightarrow \cdots$$

We next consider the category of Artin-Riesz pro-objects of the category of sheaves on the underlying space of X.

As before, let $\Delta_X^\nu(i)$ be the ith infinitesimal neighbourhood of the diagonal of $X_{(S)}^\nu$, and let $P^\nu(i)$ be the structural sheaf of $\Delta_X^\nu(i)$. For variable i, we obtain a pro-object P^ν of the category of sheaves on the underlying space of X, and for variable ν, the P^ν form a cosimplicial pro-object P. In particular, $P^0 = \underline{O}_X$, and for any ν, there are the $\nu+1$ canonical homomorphisms of pro-rings

$$p_j^\nu : \underline{O}_X \longrightarrow P^\nu \qquad (j = 0,\ldots,\nu) .$$

These $\nu + 1$ homomorphisms are distinct, in general, and hence define distinct structures of an $\underline{O}_X$-algebra on P^ν. The $\underline{O}_X$-algebra structure on P^ν defined by p_0^ν (resp. p_ν^ν) will be called the <u>extreme left</u> (resp. <u>extreme right</u>) structure, and will be written on the left (resp. right).

We define the cosimplicial pro-object Q^* by

$$Q^\nu = P^{\nu+1} \qquad (*).$$

For each non-negative integer ν, there is the canonical homomorphism of pro-rings $\alpha^\nu : P^\nu \longrightarrow Q^\nu$ defined by the canonical injection $\alpha : \{0, 1,\ldots, \nu\} \longrightarrow \{0,1,\ldots, \nu,\nu + 1\}$, where $\alpha(h) = h$.

(*) More precisely, as a functor on totally ordered finite sets, Q^* is défined as $I \longmapsto P(I \amalg \{a\})$, where $I \amalg \{a\}$ is deduced from I "by adding a first element a".

In other words, there is a homomorphism of cosimplicial rings

$$\alpha^* : P^* \longrightarrow Q^* .$$

Now let M be an $\underline{O}_X$-Module. For each non-negative integer ν , define

$$Q^\nu(M) = Q^\nu \otimes_{\underline{O}_X} M ,$$

where the tensor product is taken with respect to the extreme <u>right</u> structure of an $\underline{O}_X$-Module on Q^ν . Then $Q^\nu(M)$ is clearly a Q^ν-Module, hence an $\underline{O}_X$-bimodule (the left structure being given by the extreme left structure on Q^ν , and the right structure by the extreme right structure on Q^ν), and a P^ν-Module by restriction of scalars via the canonical homomorphism $\alpha^\nu : P^\nu \longrightarrow Q^\nu$. This structure of a P^ν-Module on $Q^\nu(M)$ commutes with the right structure of an $\underline{O}_X$-module on $Q^\nu(M)$.

Let N be a second $\underline{O}_X$-Module, and D : M $\longrightarrow$ N a <u>differential operator</u> from M to N, i.e. a homomorphism of sheaves of abelian groups which factors (uniquely) in the form

$$M \longrightarrow P^1(k) \otimes_{\underline{O}_X} M \longrightarrow N \qquad \text{(k some non-negative integer)},$$

where the first morphism is the obvious one, and the second morphism is an $\underline{O}_X$-Module homomorphism with respect to the left structure on $P^1(k) \otimes_{\underline{O}_X} M$. Then, since there is a unique homomorphism of $\underline{O}_X$-modules (with respect to the right structures)

$$S(i) : P^{\nu+1}(k+i) \longrightarrow P^{\nu+1}(i) \otimes_{\underline{O}_X} P^{\nu+1}(k) ,$$

such that the diagram

$$\begin{array}{ccc}
\underline{O}_X & \xrightarrow{\ p^{\nu+1}_{\nu+1}(k+i)\ } & P^{+1}(k+i) \\
{\scriptstyle p^{\nu+1}_{\nu+1}(k)}\downarrow & & \downarrow {\scriptstyle \delta(i)} \\
P^{\nu+1}(k) & \longrightarrow & P^{\nu+1}(i) \otimes_{\underline{O}_X} P^{\nu+1}(k)
\end{array}$$

is commutative, the differential operator $D : M \longrightarrow N$ induces, for each non-negative integer i, a homomorphism

$$P^{\nu+1}(i+k) \otimes_{\underline{O}_X} M \longrightarrow P^{\nu+1}(i) \otimes_{\underline{O}_X} M \quad ,$$

i.e. an Artin-Riesz homomorphism $Q^\nu(D) : Q^\nu(M) \longrightarrow Q^\nu(N)$. Further it is clear that $Q^o(D)$ is linear with respect to the extreme left structure. Thus $Q^o(.)$ is a functor from $\mathrm{Diff}(X/S)$, the category of Modules on X, with differential operators as morphisms, to the category of Artin-Riesz pro-Modules on $\underline{O}_X$.

6.3. Properties of $Q^o(.)$. The fundamental property of the functor $Q^o(.)$ is that, for each Module $M, Q^o(M)$ is fortified with a canonical stratification relative to S. We call $Q^o(M)$ the formalization of M.

Further, one can recover M from $Q^o(M)$ as the subsheaf of horizontal sections, this subsheaf being endowed with the structure induced from the extreme right structure of $Q^o(M)$. A differential operator $D : M \longrightarrow N$ is recovered from $Q^o(D)$ as the morphism induced on the subsheaf of horizontal sections.

Finally, we note that the cosimplicial pro-Module $Q^*(M)$ on P^* is just the cosimplicial pro-Module associated with the stratified pro-Module $Q^o(M)$, i.e. if $n \leq m$, $Q^n(M)$ is obtained from $Q^m(M)$ by base change with respect to any of the canonical morphisms $P^n \longrightarrow P^m$.

6.4. Interpretation of Zariski hypercohomology of a complex of differential operators as a stratifying hypercohomology. Let M^* be a

complex of differential operators on X bounded from below. Then one checks easily, by looking term by term, that

$$\varprojlim \ Q^*(M^\cdot)$$

is a resolution of $M^\cdot$, and thus there is a canonical isomorphism

$$\underline{\underline{H}}^*(X_{zar},M^\cdot) \overset{\sim}{\longrightarrow} \underline{\underline{H}}^*(X_{zar},\varprojlim \ Q^*(M^\cdot)) \ .$$

Now, since $Q\circ(M^\cdot)$ is a complex of stratified $\underline{O}_X$-pro-Modules, it defines, as described in (4.2) (where, of course, one passes to the limit after taking the inverse image), a complex of sheaves $Q\circ(M^\cdot)_{strat}$ on $\mathrm{Strat}(X/S)$. Then the argument given in (5.1) shows that there is a canonical isomorphism

$$\underline{\underline{H}}^*(X_{strat},Q\circ(M^\cdot)_{strat}) \overset{\sim}{\longrightarrow} \underline{\underline{H}}^*(X_{zar},C^*(Q\circ(M^\cdot)_{strat})) \ ,$$

where $C^*(Q\circ(M^\cdot_{strat}))$ is the Zariski complex of differential operators associated with the stratifying complex $Q\circ(M^\cdot)_{strat}$. But, by construction

$$C^*(Q\circ(M^\cdot)_{strat}) = \varprojlim \ Q^*(M^\cdot) \ ,$$

and thus there is a canonical isomorphism

$$\underline{\underline{H}}^*(X_{zar},M^\cdot) \overset{\sim}{\longrightarrow} \underline{\underline{H}}^*(X_{strat},Q\circ(M^\cdot)_{strat}) \ .$$

Hence, in particular, there is the spectral sequence

$$(^*) \qquad E_2^{pq} = H^p(X_{strat},H^q(Q\circ(M^\cdot)_{strat})) \Longrightarrow \underline{\underline{H}}^*(X_{zar},M^\cdot) \ .$$

6.5. **Proof of theorem 4.1.** Let F be a module on X, endowed with a stratification relative to S, and F_{strat} the associated Module on $\mathrm{Strat}(X/S)$. Then it is well known that the differential operators of the De Rham complex $\underline{\Omega}^\cdot_{X/S}$ extend to differential operators on

$$\underline{\Omega}^\cdot_{X/S} \otimes_{\underline{O}_X} F$$

thereby making this latter a complex of differential operators.

Thus, interpreting the Zariski hypercohomology of this complex in terms of stratifying hypercohomology, we have

$$\underline{H}^*(X_{zar}, \underline{\Omega}^{\bullet}_{X/S} \otimes_{\underline{O}_X} F) \xrightarrow{\sim} \underline{H}^*(X_{strat}, Q^{\circ}(\underline{\Omega}^{\bullet}_{X/S} \otimes_{\underline{O}_X} F)_{strat})$$

Now there is always a canonical homomorphism

$$F_{strat} \longrightarrow Q^{\circ}(\underline{\Omega}^{\bullet}_{X/S} \otimes_{\underline{O}_X} F)_{strat} \quad ,$$

and hence a canonical homomorphism

$$(**) \qquad \underline{H}^*(X_{strat}, F_{strat}) \longrightarrow \underline{H}^*(X_{zar}, \underline{\Omega}^{\bullet}_{X/S} \otimes_{O_X} F) \quad .$$

We would have liked $(**)$ to be an isomorphism, but unfortunately, this is not always true, as will be explained shortly. However, this is so if we assume that S is of <u>characteristic</u> 0 and X is <u>smooth</u> on S, because under these conditions a formal variant of Poincaré's lemma shows that

$$H^q(Q^{\circ}(\underline{\Omega}^{\bullet}_{X/S} \otimes F)_{strat}) = \begin{cases} 0 \text{ if } q > 0 \\ F_{strat} \text{ if } q = 0 \end{cases} .$$

The spectral sequence $(*)$ then degenerates, showing that $(**)$ is an isomorphism, as asserted. Taking $F = \underline{O}_X$, we have therefore proven Theorem 4.1.

The following example shows that $(**)$ is not an isomorphism, in general. Let X/S be an abelian scheme, where S is the spectrum of a local artinian ring with residue field of characteristic $p > 0$, and G the formal group associated with X/S. Then the spectral sequence of the end of (5.1) yields :

$$E_2^{pq} = H^p(G, H^q(X_{zar}, \underline{O}_X)) \Longrightarrow \underline{H}^*(X_{strat}, \underline{O}_{X_{strat}}) \quad ,$$

and if G is a formal torus, $E_2^{pq} = 0$ if $p > 0$, so that the spectral sequence degenerates, yielding an isomorphism

$$H^i(X_{strat}, \underline{O}_{X_{strat}}) \xrightarrow{\sim} H^i(X_{zar}, \underline{O}_X)$$

and hence

$$H^i(X_{strat}, \underline{O}_{X_{strat}}) = 0 \quad \text{if } i > \dim X$$

But the De Rham cohomology of X/S is certainly not zero in dimension 2 dim X

7. The crystalline topos and connecting topos.

7.1. **Inadequacy of infinitesimal topos.** Let X_0 be a scheme above a perfect field k of characteristic $p > 0$. Then, regarding X_0 as being above $S = \text{Spec}W(k)$ instead of k, the infinitesimal cohomology

$$H^*((X_0/S)_{inf}, \underline{O}_{X_0})$$

is a graded module on $W(k)$. One might hope that it would be a good p-adic cohomology, at least for X_0 proper over k. However, this is not so. For example, if X_0/k is an abelian scheme, which from the formal point of view is a torus, then one finds, using the example given in the last section, that

$$H^i((X_0/S)_{inf}, \underline{O}_{X_0}) = 0 \quad \text{if } i > \dim X_0 \quad (\text{instead of } i > 2\dim X_0),$$

contrarily to what we expect from p-adic cohomology.

7.2. **The crystalline topos and connecting topos.** Let X be a scheme on any base S. We modify the definition of the infinitesimal site Inf(X/S) to obtain the crystalline site Cris(X/S) as follows. The underlying category of Cris(X/S) has as objects all nilpotent S-immersions $U \hookrightarrow T$, U being an open set of X, and the ideal on T

defining this immersion being endowed with a nilpotent divided power structure [9] (*). We again take the topology on Cris(X/S) to be that of Zariski. If S is of characteristic 0, Cris(X/S) and Inf(X/S) are the same, since every ideal admits a unique divided power structure.

In complete analogy with the relation between the infinitesimal site and the stratifying site, we define the connecting site Conn(X/S) of X on S as follows. The underlying category of Conn(X/S) is the full subcategory of Cris(X/S) consisting of those objects $U \longrightarrow T$ "for which there exists" an extension of $U \hookrightarrow X$ to an S-morphism $T \longrightarrow X$. Thus, if X is smooth on S, Cris(X/S) and Conn(X/S) are equal.

It seems likely that the theory given in these notes for Inf(X/S) and Strat(X/S) will hold for Cris(X/S) and Conn(X/S), without the hypothesis that S be of characteristic 0. The principal points to be proven are the following.

a) If we are given X/S and X_0/S, and a nilpotent immersion $X_0 \hookrightarrow X$, there is a canonical isomorphism

$$H^*(X_{cris}, \underline{0}_{X_{cris}}) \xrightarrow{\sim} H^*(X_{0cris}, \underline{0}_{X_{0cris}}) \ .$$

b) If X is smooth on S, there is a canonical isomorphism

$$H^*(X_{cris}, \underline{0}_{X_{cris}}) \xrightarrow{\sim} H^*_{DR}(X/S) \ .$$

Once these two results have been established, we would know that we have a good definition of p-adic cohomology, at least for

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(*) A divided power structure $(\gamma_n)_{n \geq 1}$ is called nilpotent, if $\gamma_{n_1}(\lambda_1) \cdots \gamma_{n_r}(\lambda_r) = 0$ for $\lambda_1, \ldots \lambda_r \in \mathbf{J}$ and $\Sigma \, n_i = n$ large.

(**) This statement is not correct as given, a slightly more sophisticated form turns out to be correct. Since these notes were written, P. Berthelot has proved b) and the correct version of a).

algebraic schemes in characteristic $p > 0$, which are proper.

Namely, if X_0 is a scheme on a perfect field k of characteristic
$p > 0$, the p-adic cohomology of X would be

$$H^*((X_0/S)_{cris}, \underline{O}_{X_0}) \quad ,$$

where, as usual, $S = \text{Spec } W(k)$ (*). This p-adic cohomology would then
satisfy the following crucial test. If X_0/k lifts to a proper and
smooth scheme X/S, then there is a canonical isomorphism

$$H^*((X_0/S)_{cris}, \underline{O}_{X_0}) \xrightarrow{\sim} H^*_{DR}(X/S) \quad .$$

This can be deduced from a) and b) as follows. Let $S_n = \text{Spec } W_n(k)$,
and $X_n = X \times_S S_n$. Then there is a canonical isomorphism

$$H^*((X_0/S)_{cris}, \underline{O}_{X_0}) \xrightarrow{\sim} \varprojlim_n H^*((X/S_n)_{cris}, \underline{O}_X) \quad .$$

Applying a), we obtain a canonical isomorphism

$$\varprojlim_n H^*((X/S_n)_{cris}, \underline{O}_X) \xrightarrow{\sim} \varprojlim_n H^*((X_n/S_n)_{cris}, \underline{O}_X) ,$$

and applying b), a canonical isomorphism

$$\varprojlim_n H^*((X_n/S_n)_{cris}, \underline{O}_{X_n}) \xrightarrow{\sim} \varprojlim_n H^*_{DR}(X_n/S_n) \quad ,$$

and since there is a canonical isomorphism (EGA III 4)

$$\varprojlim H^*_{DR}(X_n/S_n) \xrightarrow{\sim} H^*_{DR}(X/S)$$

the assertion follows. Of course, a) would also allow us (in analogy
to 5.3) to give an intrinsic interpretation of the canonical connection
on the De Rham cohomology, etc.

(*) We would get essentially the same crystalline site if we took
$S = \text{Spec } Z$, so that the crystalline cohomology is defined without
any reference to such a thing as the ring of Witt vectors !

7.3. Once a) and b) have been verified, it will easily follow

that all requirements for a reasonable p-adic cohomology theory will be

satisfied, for proper and smooth schemes X_0 over k which "lift" to

W, and the need of systematically developping the formal properties

of the crystalline cohomology of an arbitrary scheme X/S, modelled on

those proven for the ℓ-adic cohomology, will be quite clear. It can

be hoped it will yield good invariants (and correct Betti numbers)

even if X_0 does not lift, or is not smooth.

7.4. **Motivation for definition of crystalline site.** Finally,

we note that introduction of divided powers in the definition of the

crystalline site was practically imposed by the need to define the

first Chern class $c(L) \in H^2(X_{cris}, 0_{X_{cris}})$ of an invertible sheaf L

on X_{zar} in arbitrary characteristic, by analogy with the classical

definition in characteristic 0. There is an obvious "forgetful" functor

$$u^* : \text{Cris}(X/S) \longrightarrow \text{Zar}(X/S)$$

and thus an exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \underline{0}_{X_{cris}} \longrightarrow u_* \underline{0}_{X_{zar}} \longrightarrow 0$$

and similarly an exact sequence of multiplicative groups

$$0 \longrightarrow (1+\mathfrak{I}) \longrightarrow \underline{0}_{X_{cris}}^* \longrightarrow u_* \underline{0}_{X_{zar}}^* \longrightarrow 0 .$$

If $(U \hookrightarrow T, \Theta)$ is any object of $\text{Cris}(X/S)$, where Θ denotes the divided

power structure on the ideal defining the immersion $U \hookrightarrow T$, then

$\mathfrak{I}(U \hookrightarrow T)$ = global sections of the ideal defining the immersion

$U \hookrightarrow T$. Utilizing the divided power structure Θ (which, more explicitely,

consists of the family of mappings $x \longmapsto x^{(n)}$ for $n \geqslant 1$) on this

ideal, we can define the logarithmic and exponential homomorphisms

on $\mathfrak{I}$ resp $1 + \mathfrak{I}$ by

$$\exp(x) = 1 + \sum_{n \geqslant 1} x^{(n)}, \quad \log(1 + x) = \sum_{n \geqslant 1} (n-1)!(-1)^n x^{(n)} ,$$

and these homomorphisms establish isomorphisms

(*) $$(1 + \mathfrak{I}) \xrightarrow{\sim} \mathfrak{I} \quad .$$

To the invertible sheaf L on X_{zar}, there corresponds the canonical

element of $H^1(X_{zar}, \underline{O}_{X_{zar}}^*)$ and so a canonical element of

$H^1(X_{cris}, u_*\underline{O}_{X_{zar}}^*)$. Utilizing the isomorphism (*), and the exact sequences

of cohomology, one immediately obtains a canonical element in

$H^2(X_{cris}, \mathfrak{I})$ and hence an element $c(L) \in H^2(X_{cris}, \underline{O}_{X_{cris}})$, which is

the required Chern class of L. It can be thus viewed as an obstruction to

lifting L to an inversible sheaf on the crystallin site.

7.5. <u>Remarks on the non proper case.</u> If X_0 is a non proper

(an affine, say) scheme over a perfect field k of char. $p > 0$, and

$S = Spec(W)$, then the calculation of De Rham cohomology made in 2.1

in the case of the formal affine line shows that the crystalline

cohomology $H^*((X_0/S)_{cris}, \underline{O}_{X_{0cris}})$ is no longer reasonable, as it gives

modules of infinite rank over the ring $W(k) = W$ of Witt vectors. This

seems to give clear evidence that some approach making use of Monsky-

Washnitzer's liftings, together with the underlying idea of the

crystalline topos, has to be worked out, whether we like it or not.

A possible first approach is to attach to X_0 the ringed site whose

objects are triples $(U_0 = Spec(A_0), U = Spec(A), D)$ of an affine

open subset U_0 of X_0, a Monsky-Washnitzer algebra A over W, endowed

with a surjective W-homomorphism $A \longrightarrow A_0$ (the augmentation), this A

(or rather its spectrum in a suitable sense) playing the part of the
infinitesimal thickenings in the definition of the crystalline site,
and of a (topologically nilpotent) divided power structure D on
the augmentation ideal J of A. Morphisms between objects are defined
in an evident way, and as topology one may take tentatively the one
deduced from Zariski topology on X_0, as in the definition of the
infinitesimal site (4.1). The corresponding topos could be called
the Monsky-Washnitzer topos of X_0 over k, say $(X_0/k)_{WM}$, and one may
expect that the cohomology of this topos, with coefficients in its
canonical sheaf of local rings, yields the correct Betti numbers of X_0
(namely the same as the ℓ-adic theories for $\ell \neq p$, when k is
algebraically closed). One regrettable feature of such a theory, in
comparison to the theory of crystalline cohomology, which works also
for unequal characteristic schemes, is that it seems so closely tied
up with the assumption of a perfect groundfield of char. $p > 0$.

Appendix

Let X be a scheme above the base S, and F a Module on X. For
each positive integer n, let $\Delta^1(n)$ be the nth infinitesimal neighbour-
hood of the diagonal of $X \times_S X$, and $\Delta^2(n)$ the nth infinitesimal neigh-
bourhood of the diagonal of $X \times_S X \times_S X$. Then there is the usual diagram
of canonical projections

$$X \xleftarrow[p_2(n)]{p_1(n)} \Delta^1(n) \xleftarrow[\substack{p_{32}(n) \\ p_{21}(n)}]{p_{31}(n)} \Delta^2(n) \quad .$$

An n-connection on F relative to S is an isomorphism

$$\varphi : p_1(n)^*(F) \xrightarrow{\sim} p_2(n)^*(F)$$

satisfying the cocycle condition

$$p_{31}^*(\varphi) = p_{32}^*(\varphi) \, p_{21}^*(\varphi) \quad .$$

A stratification on F relative to S is a system of an n-connection

for each positive integer n, in such a fashion that these various

n-connections are compatible or glue together, i.e. if n' $\leq$ n, the

n'-connection induced by the given n-connection is the given

n'-connection.

If X is smooth on S, a 1-connection on F relative to S is

equivalent to a mapping

$$\rho : \underline{Der}_S(\underline{O}_X) \longrightarrow \underline{End}_S(F)$$

satisfying the following condition

$$\rho(\alpha \partial) = \alpha \rho(\partial)$$

$$\rho(\partial + \partial') = \rho(\partial) + \rho(\partial')$$

$$\rho(\partial)(\alpha f) = \partial(\alpha)f + \alpha \, \rho(\partial)(f)$$

where $\alpha \in \Gamma(U, \underline{O}_X)$, $\partial \in \Gamma(U, \underline{Der}_S(\underline{O}_X))$, $f \in \Gamma(U, F)$.

We say that this 1-connection is integrable if

$$\rho(\partial \partial' - \partial' \partial) = \rho(\partial)\rho(\partial') - \rho(\partial') \, \rho(\partial) \quad .$$

There are various other ways of expressing the fact that a sheaf
F has a connection or stratification relative to S, but we do not give
these. We only mention that the classical definition of a connection
on a vector bundle is a particular case of the above, and that, if
X is of characteristic O and smooth over S, a stratification is
equivalent to an integrable 1-connection. Needless to say, this last
assertion is false in non-zero characteristic.

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