# A general theory of fibre spaces with structure sheaf

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Introduction. When one tries to state in a general algebraic formalism the various notions of fibre space : general fibre space (without structure group, and maybe not even locally trivial) ; or fibre bundle with topological structure group $G$ as expounded in the book of Steenrod (The Topology of Fibre Bundles, Princeton University Press) ; or the "differentiable" and "analytic" (real or complex) variants of these notions ; or the notions of algebraic fibre spaces (over an abstract field $k$), one is led in a natural way to the notion of fibre space with a structure sheaf $G$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space $X$, with abelian structure group $G$, as the elements of the first cohomology group of $X$ with coefficients in the sheaf $G$ of germs of continuous maps of $X$ into $G$ ; the word "continuous" being replaced by "analytic" respectively "regular" if $G$ is supposed an analytic respectively an algebraic group (the space $X$ being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when $G$ is not abelian, to denote by $H^1(X, G)$ the set of classes of fibre spaces on $X$ with structure sheaf $G$, $G$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of $X$ into $G$. Here we develop systematically the notion of fibre space with structure sheaf $G$, where $G$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $H^1(X, G)$ of $X$ with coefficients in $G$.

The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including the sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the last chapter, we define the cohomology set $H^1(X, G)$ of $X$ with coefficients in the sheaf of groups $G$, so that the expected classification theorem for fibre spaces with structure sheaf $G$ is valid. We then proceed to a careful study of the exact cohomology sequence associated with an exact sequence of sheaves $e \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow e$. This is the main part, and in fact the origin, of this paper. Here $G$ is any sheaf of groups, $F$ a subsheaf of groups, $H = G/F$, and according to various supplementary hypotheses on $F$ (such as $F$ normal, or $F$ normal abelian, or

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$F$ in the center) we get an exact cohomology sequence going from $H^0(X, F)$ (the group of sections of $F$) to $H^1(X, G)$ respectively $H^1(X, H)$ respectively $H^2(X, H)$, with more or less additional algebraic structures involved.

The formalism thus developed is quite suggestive, and as it seems useful, in particular in dealing with the problem of classification of fibre bundles with a structure group $G$ in which we consider a sub-group $F$, or the problem of comparing say the topological and analytic classification for a given analytic structure group $G$. However, in order to keep this exposition in reasonable bounds, no examples have been given. Some complementary facts, examples, and applications for the notions developed will be given in the future. This report has been written mainly in order to serve the author for future reference; it is hoped that it may serve the same purpose, or as an introduction to the subject, to somebody else.

Of course, as this report consists in a fortunately straightforward adaptation of quite well known notions, no real difficulties had to be overcome and there is no claim for originality whatsoever. Besides, at the moment to give this report for mimeography, I hear that results analogous to those of chapter 5 were known for some years to Mr. Frenkel, who did not publish them till now. The author only hopes that this report is more pleasant to read than it was to write, and is convinced that anyhow an exposition of this sort had to be written.

Remark (added for the second edition). It has appeared that the formalism developed in this report, and specifically the results of Chapter V, are valid (and useful) also in other situations than just for sheaves on a given space $X$. A generalization for instance is obtained by supposing that a fixed group $\Pi$ is given acting on $X$ as a group of homeomorphisms, and that we restrict our attention to the category of fibre spaces over $X$ (and especially sheaves) on which $\Pi$ operates in a manner compatible with its operations on the base $X$ (See for instance A. Grothendieck, Sur le mémoire de Weil; Généralisation des fonctions abéliennes, Séminaire Bourbaki Décembre 1956). When $X$ is reduced to a point, one gets (instead of sheaves) sets, groups, homogeneous spaces etc, admitting a fixed group $\Pi$ of operators, which leads to the (commutative and non-commutative) cohomology theory of the group $\Pi$. One can also replace $\Pi$ by a fixed Lie group (operating on differentiable varieties, on Lie groups, and homogeneous Lie spaces). Or $X, \Pi$ are replaced by a fixed ground field $k$, and one considers algebraic spaces, algebraic groups, homogeneous spaces defined over $k$, which leads to a kind of cohomology theory of $k$. All this suggests that there should exist a comprehensive theory of non-commutative cohomology in suitable categories, an exposition of which is still lacking. (For the "commutative" theory of cohomology, see A. Grothendieck, sur quelques points d'Algèbre Homologique, Tohoku Math. Journal, 1958).

# 1 General fibre spaces

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

## 1.1 Notion of fibre space.

Definition 1.1.1. A fibre space over a space $X$ is a triple $(X, E, p)$ of the space $X$, a space $E$ and a continuous map $p$ of $E$ into $X$.

We do not require $p$ to be onto, still less to be open, and if $p$ is onto, we do not require the topology of $X$ to be the quotient topology of $E$ by the map $p$. For abbreviation, the fibre space $(X, E, p)$ will often be denoted by $E$ only, it being understood that $E$ is provided with the supplementary structure consisting of a continuous map $p$ of $E$ into the space $X$. $X$ is called the base space of the fibre space, $p$ the projection, and for any $x \in X$, the subspace $p^{-1}(x)$ of $E$ (which is closed if $\{x\}$ is closed) is the fibre of $x$ (in $E$).

Given two fibre spaces $(X, E, p)$ and $(X', E', p')$, a homomorphism of the first into the second is a pair of continuous maps $f : X \longrightarrow X'$ and $g : E \longrightarrow E'$, such that $p'g = fp$, i.e, commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

Then $g$ maps fibres into fibres (but not necessarily onto ! ) ; furthermore, if $p$ is surjective, then $f$ is uniquely determined by $g$. The continuous map $f$ of $X$ into $X'$ being given, $g$ will be called also a $f$-homomorphism of $E$ into $E'$. If, moreover, $E''$ is a fibre space over $X''$, $f'$ a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a $f'$-homomorphism, then $g'g$ is a $f'f$-homomorphism. If $f$ is the identity map of $X$ onto $X$, we say also $X$-homomorphism instead of $f$-homomorphism. If we speak of homomorphisms of fibre spaces over $X$, without further comment, we will always mean $X$-homomorphisms.

The notion of isomorphism of a fibre space $(X, E, p)$ onto a fibre space $(X', E', p')$ is clear : it is a homomorphism $(f, g)$ of the first into the second, such that $f$ and $g$ are onto-homeomorphisms.

## 1.2 Inverse image of a fibre space, inverse homomorphisms.

Let $(X, E, p)$ be a fibre space over the space $X$, and let $f$ be a continuous map of a space $X'$ into $X$. Then the inverse image of the fibre space $E$ by $f$ is a fibre space $E'$ over $X'$. $E'$ is defined as the subspace of $X' \times E$ of points $(x', y)$ such that $fx' = py$, the projection $p'$ of $E'$ into the base $X'$ being given by $p'(x', y) = x'$. The map $g(x', y) = y$ of $E'$ into $E$ is then an $f$-homomorphism, inducing for each $x' \in X'$ a homeomorphism of the fibre of $E'$ over $x'$ onto the fibre of $E$ over $f'x'$.

Suppose now, moreover, given a continuous map $f' : X'' \longrightarrow X'$ of a space $X''$ into $X'$. Then there is a canonical isomorphism of the fibre space $E''$ over $X''$, inverse image of the fibre space $E$ by $ff'$, and the inverse image of the fibre space $E'$ (considered above) by $f'$ (transitivity of inverse images). If $(x'', y) \in E''(x'' \in X'', y \in E, ff'x'' = py)$, it is mapped by this isomorphism into $(x'', (f'x'', y))$.

Let $Y$ be a subspace of the base $X$ of a fibre space $E$ ; consider the injection $f$ of $Y$ into $X$ ; the inverse image $E'$ of $E$ by $f$ is called fibre-space induced by $E$ on $Y$, or the restriction of $E$ to $Y$, and is denoted by $E|Y$. This is canonically homeomorphic to a subspace of $E$, namely the set of elements mapped by $p$ into $Y$ ; the projection of $E|Y$ into $Y$ is induced by $p$. By what has been said above, if $Z$ is a subspace of $Y$, the restriction of $E|Y$ to $Z$ is the restriction $E|Z$ of $E$ to $Z$.
Again let $(X, E, p)$ and $(X', E', p')$ be two fibre spaces, $f$ a continuous map $X \longrightarrow X'$. An inverse homomorphism associated with $f$ is an $X$-homomorphism $g$ of the fibre space $E_0$ into $E$, where $E_0$ denotes the inverse image of the fibre space $E'$ by $f$. That means that $g$ is a continuous map, of the subspace $E_0$ of $X \times E'$ of pairs $(x, y')$ such that $fx = p'y'$, into $E$, mapping for any $x \in X$ the fibre of $x$ into $E_0$ (homeomorphic to the fibre of $fx$ in $E'$!) into the fibre $p^{-1}(x)$ of $x$ in $E$. For instance, if $E$ is itself the inverse image of $E'$ by $f$, then there is a canonical inverse homomorphism of $E'$ into $E$ associated with $f$ : the identity! (Though somewhat trivial, this is the most important case of inverse homomorphisms.)

## 1.3 Subspace, quotient, product.

Let $(X, E, p)$ be a fibre space, $E'$ any subspace of $E$, then the restriction $p'$ of $p$ to $E'$, defines $E'$ as a fibre space with the same basis $X$, called a sub-fibre-space of $E$. So the sub-fibre-spaces of $E$ are in one to one correspondence with the subsets of $E$ ; in particular, for them the notions of union, intersection etc. are defined. (Of course, in most cases we are only interested in fibre spaces the projection of which is onto ; this imposes then a condition on the subspaces of $E$ considered, which may be fulfilled for two subspaces and not for the intersection.)

Let now $R$ be an equivalence relation in $E$ compatible with the map $p$, i.e. such that two elements of $E$ congruent mod $R$ have the same image under $p$. Then $p$ defines a continuous map $p'$ of the quotient space $E' = E/R$ into $X$, which turns $E'$ into a fibre space with base $X$, called a quotient fibre space of $E$. So the latter are in one-to-one correspondence with the equivalence relations in $E$ compatible with $p$. A quotient fibre space of a quotient fibre space of $E$ is a quotient fibre space.

Let $(X, E, p)$ and $(X', E', p')$ be two fibre spaces, then $(p, p')$ defines a continuous map of $E \times E'$ into $X \times X'$, so that $E \times E'$ appears as a fibre space over $X \times X'$, called the

product of the fibre spaces $E, E'$.
The fibre of $(x, x')$ in $E \times E'$ is the product of the fibres of $x$ in $E$, respectively $x'$ in $E'$. Suppose now $X = X'$, and consider the inverse image of $E \times E'$ under the diagonal map $X \longrightarrow X \times X$, we get a fibre space over $X$, called the fibre product of the fibre spaces $E, E'$ over $X$, denoted by $E \underset{(X)}{\times} E'$. The fibre of $x$ in this fibre-product is the product of the fibres of $x$ in $E$ respectively $E'$. Of course, product of an arbitrary family of fibre spaces can be considered, and the usual formal properties hold.

## 1.4 Trivial and locally trivial fibre spaces.

Let $X$ and $F$ be two spaces, $E$ the product space, the projection of the product on $X$ defines $E$ as a fibre space over $X$, called the trivial fibre space over $X$ with fibre $F$.

All fibres are canonically homeomorphic with $F$. Let us determine the homomorphisms of a trivial fibre space $E = X \times F$ into another $E' = X \times F'$. More generally, we will only assume that the projection of $X \times F$ onto $X$ is the natural one and continuous for the given topology of $X \times F$, which induces on the fibres the given topology (but the topology of $X \times F$ may not be the product topology, for instance : $X$ and $F$ are algebraic varieties with the Zariski topology) ; same hypothesis on $X \times F'$. Then a homomorphism $u$ of $E$ into $E'$, inducing for each $x \in X$ a continuous map of the fibre of $E$ over $x$ into the fibre of $E'$ over $x$, defines a function $x \longrightarrow f(x)$ of $X$ into the set of all continuous maps of $F$ into $F'$, and of course the homomorphism is well determined by this map by the formula

$$u(x, y) = (x, f(x).y) \qquad (x \in X, y \in F). \tag{1.4.1}$$

So the homomorphisms of $E$ into $E'$ can be identified with those maps $f$ of $X$ into the set of continuous maps of $F$ into $F'$ such that the map (1.4.1) is continuous. If the topologies of $E$ and $E'$ are the product topologies, this means that $(x, y) \longrightarrow f(x).y$ is continuous ; as is well known, if moreover $F$ is locally compact or metrizable, this means also that $f$ is continuous when we take on the set of all continuous maps from $F$ into $F'$ the topology of compact convergence. If we consider a homomorphism $v$ from $E'$ into $E'' = X \times F''$ given by a map $g$ of $X$ into the set of all continuous maps of $F'$ into $F''$ the homomorphism $vu$ is given by the map $x \longrightarrow g(x)f(x)$. In order that the map (1.4.1) be injective (respectively surjective, bijective) it is necessary and sufficient that for each $x \in X$, $f(x)$ has the same property. In the bijective case, the inverse map is then defined by the function $x \longrightarrow f(x)^{-1}$. It follows that $u$ is an isomorphism onto if and only if for each $x \in X$, $f$ is a homeomorphism of $F$ onto $F'$, and the map $(x, y') \longrightarrow (x, f(x)^{-1}.y')$ continuous. So we get in particular (coming back to the case of trivial fibre spaces) :

Proposition 1.4.1. Let $E = X \times F$ and $E' = X \times F'$ be two trivial fibre spaces over $X$, then the isomorphisms of $E$ onto $E'$ can be identified with the maps $f$ of $X$ into the set of homeomorphisms of $F$ onto $F'$ such that $f(x).y$ and $f(x)^{-1}.y'$ be continuous functions from $X \times F$ into $F'$ respectively $X \times F'$ into $F$. If $E = E'$, this identification is compatible with the group structures on the set of automorphisms of $E$ respectively the set of maps of $X$ into the group of automorphisms of $F$.

Two fibre spaces $E, E'$ over $X$ are said to be locally isomorphic if each point $x$ of $X$ has a neighborhood $U$ (which can be assumed open) such that the restrictions of $E$ and $E'$ to $U$ are isomorphic. This is clearly an equivalence relation. A fibre space $E$ over $X$ is said locally trivial with fibre $F$ ($F$ being a given space) if it is locally isomorphic to the trivial space $X \times F$.

## 1.5 Definition of fibre spaces by coordinate transformations.

Let $X$ be a space, $(U_i)$ a covering of $X$, for each index $i$, let $E_i$ be a fibre space over $U_i$, and for any couple of indices $i, j$ such that $U_{ij} = U_i \cap U_j \neq \varnothing$, let $f_{ij}$ be a $U_{ij}$-isomorphism of $E_j|U_{ij}$ onto $E_i|U_{ij}$. On the topological sum $\mathscr{E}$ of the spaces $E_i$, let us consider the relation

(1.5.1.) $\quad y_i \in E_i|U_{ij}$ and $y_j \in E_j|U_{ij}$ are equivalent means $y_i = f_{ij} y_j$.

This is an equivalence relation, as easily checked, if and only if we have, for each triple $(i, j, k)$ of indices such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \varnothing$, the relation

(1.5.2.) $\quad f_{ik} = f_{ij} f_{jk}$

(where, in order to abbreviate notations, we wrote simply $f_{ik}$ instead of : the isomorphism of $E_k|U_{ijk}$ onto $E_i|U_{ijk}$ induced by $f_{ik}$ and likewise for $f_{ij}$ and $f_{jk}$). Supposing this condition satisfied, let $E$ be the quotient space of $\mathscr{E}$ by the preceding equivalence relation. The projections $p_i$ of $E_i$ into $U_i$ define a continuous map of the topological sum $\mathscr{E}$ into $X$, and this map is compatible with the equivalence relation in $\mathscr{E}$, so that there is a continuous map $p$ of $E$ into $X$ (which is onto if the $p_i$'s are all onto).

Definition 1.5.1. The fibre space over $X$ just constructed is called the fibre space defined by the "coordinate transformations" $(f_{ij})$ between the fibre spaces $E_i$.

The identity map of $E_i$ into $\mathscr{E}$ defines a map $\mathscr{S}_i$, of $E_i$ into $E$, which by virtue of (1.5.1.) is a one to one $U_i$-homomorphism of $E_i$ onto $E|U_i$. The topology of $E$ (by a well known transitivity property for topologies defined as the finest which ...) is the finest topology on $E$ for which the maps $\mathscr{S}_i$ are continuous. Moreover, it is easy to show that in case the interiors of the $U_i$'s already cover $X$, the maps $\mathscr{S}_i$ are homeomorphisms into. Henceforth, for simplicity we will only work with open coverings of $X$, so that the preceding properties are automatically satisfied. Then $\mathscr{S}_i$ can be considered as a $U_i$-isomorphism of $E_i$ onto $E|U_i$. Clearly

(1.5.3.) $\quad f_{ij} = \mathscr{S}_i^{-1} \mathscr{S}_j$

(where again, in order to abbreviate, we wrote $\mathscr{S}_i$ instead of the restriction of $\mathscr{S}_i$ to $E_i|U_{ij}$, $\mathscr{S}_j$ instead of the restriction of $\mathscr{S}_j$ to $E_j|U_{ij}$). Conversely, let $E$ be a fibre space over $X$, and suppose that for each $i$, there exists a $U_i$-isomorphism $\mathscr{S}_i$ of $E_i$ onto $E|U_i$, then (1.5.3.) defines, for each pair $(i,j)$ such that $U_i \cap U_j = U_{ij} \neq \varnothing$, a $U_{ij}$-isomorphism of $E_j|U_{ij}$ onto $E_i|U_{ij}$, and the system $(f_{ij})$ satisfies obviously (1.5.2.). Therefore we can consider the fibre space $E'$ defined by the coordinate transformations $f_{ij}$. Then it is obvious that the map of $\mathscr{E}$ into $E$ defined by the maps $\mathscr{S}_i$ is compatible with the equivalence relation in $\mathscr{E}$, therefore defines a continuous map $f$ of $E'$ into $E$ which is of course an $X$-homomorphism. Let $\mathscr{S}'_i$ be the natural isomorphism of $E_i$ onto $E'|U_i$ defined above ; it is checked at once that the map of $E'|U_i$ into $E|U_i$ induced by $f$ is $\mathscr{S}_i\mathscr{S}'^{-1}_i$, hence an isomorphism onto. It follows that $f$ itself is an isomorphism of $E'$ onto $E$, by virtue of the following easy lemma (proof left to the reader) :

<u>Lemma 1.</u> Let $E, E'$ be two fibre spaces over $X$, and $f$ an $X$-homomorphism of $E$ into $E'$, such that for any $x \in X$, exists a neighborhood $U$ of $x$ such that $f$ induces an isomorphism of $E|U$ onto (respectively, into) $E'|U$. Then $f$ is an $X$-isomorphism of $E$ onto (respectively, into) $E'$.

What precedes shows the truth of :

<u>Proposition 1.5.1.</u> The open covering $(U_i)$ and the fibre spaces $E_i$ over $U_i$ being given, the fibre spaces over $X$ which can be obtained by means of suitable coordinate transformations $(f_{ij})$ are exactly those, up to isomorphism, for which $E|U_i$ is isomorphic to $E_i$ for any $i$.

Consider now two systems of coordinate transformations $(f_{ij}), (f'_{ij})$ corresponding to the same covering $(U_i)$, and to two systems $(E_i)$, $(E'_i)$ of fibre spaces over the $U_i$'s. Let $E$ be the fibre space defined by $(f_{ij})$ and $E'$ the fibre space defined by $(f'_{ij})$ ; we will determine all homomorphisms of $E$ into $E'$. If $f$ is such a homomorphism, then for each $i$, $f_i = \mathscr{S}'^{-1}_i f \mathscr{S}_i$ (where $f$ stands for the restriction of $f$ to $E|U_i$) is a homomorphism of $E_i$ into $E'_i$, and the system $(f_i)$ satisfies clearly, for each pair $(i,j)$ such that $U_{ij} \neq \varnothing$ :

(1.5.4) $\qquad f_i f_{ij} = f'_{ij} f_j$

(where we write simply $f_i$ instead of the restriction of $f_i$ to $E_i|U_{ij}$, and likewise for $f_j$). The homomorphism $f$ is moreover fully determined by the system $(f_i)$ since $f_i$ determines the restriction of $f$ to $E|U_i$ ; and moreover the system $(f_i)$ subject to (1.5.4) can be chosen otherwise arbitrarily, for this relation expresses exactly that the map of the topological sum $\mathscr{E}$ of the $E_i$'s into the topological sum $\mathscr{E}'$ of the $E'_i$'s transforms equivalent points into equivalent points, and therefore defines an $X$-homomorphism $f$ of $E$ into $E'$ ; and it is clear that the system $(f_i)$ is nothing else but the one which is defined as above in terms of the homomorphism $f$. Of course, in view of lemma 1, in order that $f$ be an isomorphism onto, (respectively, into) it is necessary and sufficient that each $f_i$ be an isomorphism of $E_i$ onto (respectively, into) $E'_i$. Thus we get :

Proposition 1.5.2. Given two fibre spaces over $X$, $E$ and $E'$, defined by coordinate transformations $(f_{ij})$ respectively $(f'_{ij})$ relative to the same open covering $(U_i)$, the $X$-homomorphisms $f$ of $E$ into $E'$ are in one to one correspondence with systems $(f_i)$ of $U_i$-homomorphisms $E_i \longrightarrow E'_i$ satisfying (1.5.4.). $f$ is an onto-isomorphism if and only if the $f'_i$'s are, i.e. $E'$ is isomorphic to $E$ if and only if we can find onto-isomorphisms $f_i : E_i \longrightarrow E'_i$ such that, for any pair $(i, j)$ of indices satisfying $U_{ij} \neq \varnothing$, we have

(1.5.5.) $$f'_{ij} = f_i f_{ij} f_j^{-1}$$

(where as usual $f_i$ and $f_j$ stand for restricted maps).

We proceed to the comparison of fibre spaces $E, E'$ defined by coordinate transformations corresponding to different coverings, $(U_i)$ and $(U'_i)$, in particular to the determination of the homomorphisms of $E$ into $E'$ and hence of the $X$-isomorphisms of $E$ and $E'$, and therefore to the determination of whether $E$ and $E'$ are isomorphic. Let $(V_j)$ be an open covering of $X$ which is a refinement of both preceding coverings ; we will show that $E$ and $E'$ are isomorphic to fibre spaces defined by coordinate transformations relative to this same covering $(V_j)$, so that the problem is reduced to one already dealt with.

So let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be two open coverings of $X$, the second finer than the first, that is any $V_j$ is contained in some $U_i$, i.e. there exists at least one map $\tau : J \longrightarrow I$ such that $V_j \in U_{\tau(j)}$ for any $j \in J$. For each $i \in I$, let $E_i$ be a fibre space over $U_i$, and let $(f_{ii'})$ be a system of coordinate transforms relative to the system $(E_i)$. For each $j \in J$, let $F_j = E_{\tau(j)}|V_j$, and let $g_{jj'}$ be the restriction of $f_{\tau(j),\tau(j')}$ to $F_j|V_{jj'}$ ; so $g_{jj'}$ is an isomorphism of $F_j|V_{jj'}$ onto $F_j|V_{jj'}$, and the system $(g_{jj'})$ is a system of coordinate transformations, as follows at once from the definition and (1.5.2.) applied to the system $(f_{ii'})$. Let $F$ be the fibre space defined by the system of coordinate transformations $(g_{jj'})$ ; we shall define a canonical $X$-isomorphism of $F$ onto $E$. For $j \in J$, let $g_j$ be the injection map of $F_j$ into $E_{\tau(j)}$ ; it is hence a map of $F_j$ into the topological sum $\mathscr{E}$ of the $E_i$'s ; the system $(g_j)$ defines a continuous map $g'$ of the topological sum $\mathcal{F}$ of the $f_j$'s into $\mathscr{E}$, and as easily seen $g'$ maps equivalent points into equivalent points. Hence $g'$ induces a continuous map $g$ of $F$ into $E$, which clearly is an $X$-homomorphism. Moreover, for any $j$, $g$ induces an isomorphism of $F|V_j$ onto $E|V_j$, for if we compose it with the natural isomorphism of $E|U_i$ onto $E_i$, we get the injection map of $E|V_j$ into $E_i$ (we put $i = \tau(j)$). Now applying lemma 1, we see that $g$ is an isomorphism of $F$ onto $E$.

## 1.6 The case of locally trivial fibre spaces.

The method of the preceding section for constructing fibre spaces over $X$ will be used mainly in the case where we are given a fibre space $T$ over $X$, and where, given an open covering $(U_i)$ of $X$, we consider the fibre spaces $E_i = T|U_i$ over $U_i$ and coordinate transformations $(f_{ij})$ with respect to these. Then $f_{ij}$ is an $U_{ij}$-automorphism of $T|U_{ij}$. The fibre space defined by the system $(f_{ij})$ of coordinate transformations will be locally isomorphic (cf. 1.4.) to $T$, and in virtue of proposition 1.5.1., we obtain in this way exactly (up to isomorphism) all fibre spaces over $X$ which are locally isomorphic to $T$ (by taking the open sets $U_i$ small enough, and then a suitable system $(f_{ij})$.

In case $T$ is a trivial fibre space, $T = X \times F$, we have $E_i = U_i \times F$, and $E_i|U_{ij} = U_{ij} \times F$. Thus $f_{ij}$ is an automorphism of the trivial fibre space $U_{ij} \times F$, and therefore, in view of proposition 1.4.1. given by a map $x \longrightarrow f_{ij}(x)$ of $U_{ij}$ into the group of homeomorphisms of $F$ onto itself. The equations (1.5.2.) expressing that $(f_{ij})$ is a system of coordinate transformations then translate into

(1.6.1.) $$f_{ik}(x) = f_{ij}(x) f_{jk}(x) \qquad \text{for } x \in U_{ijk}$$

Moreover, it must not be forgotten that $x \longrightarrow f_{ij}(x)$ is submitted to the continuity condition of proposition 1.4.1. Such a system then defines in a natural way a fibre space $E$ over $X$, and by what has been said it follows that this fibre bundle is locally isomorphic to $X \times F$, i.e. locally trivial with fibre $F$, and that (for suitable choice of the covering and the coordinate transformations), we get thus, up to isomorphism, all locally trivial fibre spaces over $X$ with fibre $F$.

Let in the same way $T' = X \times F'$, and consider for the same covering $(U_i)$ a system $(f_{ij})$ and a system $(f'_{ij})$ of coordinate transformations, the first relative to the fibre $F$ and the second to the fibre $F'$. Let $E$ and $E'$ be the corresponding fibre spaces over $X$. The homomorphisms of $E$ into $E'$, by proposition 1.5.2., correspond to homomorphisms $f_i$ of $E_i = U_i \times F$ into $E'_i = U_i \times F'$, satisfying conditions (1.5.4). Now, (proposition 1.4.1.) such a homomorphism $f_i$ is determined by a map $x \longrightarrow f_i(x)$ of $U_i$ into the set of continuous maps of $F$ into $F'$ by $f_i(x, y) = (x, f_i(x).y)$, subject to the only requirement that $f_i(x).y$ is continuous with respect to the pair $(x, y) \in U_i \times F$. Then the equation (1.5.4.) translates into

(1.6.2.) $$f_i(x) f_{ij}(x) = f'_{ij}(x) f_j(x) \qquad (x \in U_{ij})$$

Thus are determined the homomorphisms of $E$ into $E'$. In particular, the isomorphisms of $E$ onto $E'$ are obtained by systems $(f_i)$ such that $f_i(x)$ be a homeomorphism of $F$ onto $F'$ for any $x \in U_i$, and that $x \longrightarrow f_i^{-1}(x)$ satisfies the same continuity requirement as $x \longrightarrow f_i(x)$. The compatibility condition (1.6.2.) can then be written

(1.6.3.) $$f'_{ij}(x) = f_i(x) f_{ij}(x) f_j(x)^{-1} \qquad (x \in U_{ij})$$

## 1.7 Sections of fibre spaces.

Definition 1.7.1. Let $(X, E, p)$ be a fibre space ; a section of this fibre space (or, by pleonasm, a section of $E$ over $X$) is a map $x$ of $X$ into $E$ such that $ps$ is the identity map of $X$. The set of continuous sections of $E$ is noted $H^0(X, E)$.

It amounts to the same to say that $s$ is a function the value of which at each $x \in X$ is in the fibre of $x$ in $E$ (which depends on $x$!). The existence of a section implies of course that $p$ is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A section of $E$ over a subset $Y$ of $X$ is by definition a section of $E|Y$. If $Y$ is open, we write $H^0(Y, E)$ for the set $H^0(Y, E|Y)$ of all continuous sections of $E$ over $Y$.

$H^0(X, E)$ as a functor. Let $E, E'$ be two fibre spaces over $X$, $f$ an $X$-homomorphism of $E$ into $E'$. For any section $s$ of $E$, the composed map $fs$ is a section of $E'$, continuous if $s$ is continuous. We get thus a map, noted $f$, of $H^0(X, E)$ into $H^0(X, E')$. The usual functor properties are satisfied :

a. If the two fibre spaces are identical and $f$ is the identity, then so is $f$

b. if $f$ is an $X$-homomorphism of $E$ into $E'$ and $f'$ an $X$-homomorphism of $E'$ into $E''$ ($E, E', E''$ fibre spaces over $X$) then $(f'f) = f' f$.

Let $(X, E, p)$ be a fibre space, $f$ a continuous map of a space $X'$ into $X$, and $E'$ the inverse image of $E$ under $f$. Let $s$ be a section of $E'$ consider the map $s'$ of $X'$ into $E'$ given by $s'x' = (x', sfx')$ (the second member belongs to $E'$, since $fx' = psfx'$ because $px$ = identity), this is a section of $E'$, continuous if $s$ is continuous. Thus we get a canonical map of $H^0(X, E)$ into $H^0(X', E')$ ($E'$ being the inverse image of $E$ by $f$). In case $X' \subset X$ and $f$ is the inclusion map, therefore $E' = E|X'$, then the preceding map is nothing but the restriction map (of $H^0(X, E)$ into $H^0(X', E)$ if $X'$ open). We leave to the reader statement and proof of an evident property of transitivity for the canonical maps just considered.

The two sorts of homomorphisms for sets of continuous sections are compatible in the following sense. Let $\mathscr{S}$ be a fixed continuous map of a space $X'$ into $X$, then to any fibre space $E$ over $X$ corresponds its inverse image $E'$ under $\mathscr{S}$, which is a fibre space over $X'$; moreover, given an $X$-homomorphism $f : E \longrightarrow F$, it defines in a natural way an $X'$-homomorphism $f'$ of $E'$ into $F'$. (We could go further and state that, for fixed $\mathscr{S}$, $E'$ is a "functor" of $E$ by means of the preceding definitions.)

Then the following diagram

$$\begin{array}{ccc} H^0(X, E) & \xrightarrow{f_*} & H^0(X, F) \\ \downarrow & & \downarrow \\ H^0(X', E') & \xrightarrow{f'_*} & H^0(X', F') \end{array}$$

is commutative, where the vertical arrows stand for the canonical homomorphisms defined above. The checking of course is trivial. Particular case : replacing $X$ by an open subset $U$ of $X$, and taking for $X'$ an open subset $V$ of $U$ and $\mathscr{S}$ the inclusion map

$V \longrightarrow U$, we get that for any two fibre spaces $E, F$ over $X$ and $X$-homomorphism $f : E \longrightarrow F$, the following diagram is commutative :

$$\begin{array}{ccc} H^0(U,E) & \longrightarrow & H^0(U,F) \\ \downarrow & & \downarrow \\ H^0(V,E) & \longrightarrow & H^0(V,F) \end{array}$$

where the vertical arrows are the restriction maps, and the horizontal arrows are the maps defined by $f$ (or, strictly speaking, by the restrictions of $f$ to $E|U$ respectively $E|V$). In words : the "homomorphisms" between spaces of sections over open sets defined by $X$-homomorphisms of fibre spaces commute with the restriction operators.

Determination of sections. Let us come back to the conditions of the definition 1.5.1. ; we keep the notations of that section. Let $s$ be a section of the fibre space $E$, and for any $i$, let $s_i = \mathscr{S}_i^{-1} s$ ; then $s_i$ is a section of $E_i$ over $U_i$, and from $s = \mathscr{S}_i s_i = \mathscr{S}_j s_j$ over $U_{ij}$ we get $s_i = \mathscr{S}_i^{-1} \mathscr{S}_j s_j = f_{ij} s_j$ :

(1.7.3.) $\qquad s_i = f_{ij} s_j$

where again we write $s_i, s_j$ instead of : restriction of $s_i, s_j$ to $U_{ij}$.

Of course, $s$ is entirely determined by the system $(s_i)$, for $s$ is given over $U_i$ by $s = \mathscr{S}_i s_i$. On the other hand, the system $(s_i)$ subject to (1.7.3.) can be otherwise arbitrary, for these conditions express precisely that for $x \in X$, the element $\mathscr{S}_i s_i(x)$ of $E$ obtained by taking a $U_i$ containing $x$ does not depend on $i$, and may therefore be denoted by $s(x)$ : Then the $\mathscr{S}_i^{-1} s$ determined by the above definition are of course nothing else than the $s_i$'s we started with. Let us note also that in order that the section $s$ be continuous, it is necessary and sufficient that each $s_i$ be continuous. We thus obtain :

Proposition 1.7.1. Let $E$ be the fibre space defined by coordinate transformations $(f_{ij})$ relative to an open covering $(U_i)$ of $X$ and fibre spaces $E_i$ over $U_i$. Then there is a canonical one to one correspondence between sections of $E$ and systems $(s_i)$ of sections of $E_i$ over $U_i$, $i$ satisfying conditions (1.7.3.). Continuous sections correspond to systems of continuous sections.

Let again, as in section 1.5, be given two systems $(E_i)$ and $(E'_i)$ of fibre spaces over the $U_i$'s and two corresponding systems of coordinate transformations $(f_{ij})$ and $(f'_{ij})$ let $E$ and $E'$ be the corresponding fibre spaces, and $f$ an $X$-homomorphism of $E$ into $E'$, defined by virtue of proposition 1.5.2., by a system $(f_i)$ of $U_i$-homomorphisms of $E$, into $E_i$ satisfying (1.5.4.). Let $s$ be a section of $E$, given by a system $(s_i)$ of sections of $E_i$ over $U_i$. Then the system $(f_i s_i)$ of sections of $E'_i$ over $U_i$ defines the section $fs$ (trivial).

The reader may check, as an exercise, how the canonical maps of spaces of sections considered above in this section, can be made explicit for fibre spaces given by means of coordinate transformations.

# 2 Sheaves of sets

Throughout this exposition, we will now use the word "section" for "continuous section".

## 2.1 Sheaves of sets.

Definition 2.1.1. Let $X$ be a space. A sheaf of sets on $X$ (or simply a sheaf) is a fibre space $(E, X, p)$ with base $X$, satisfying the condition : each point $a$ of $E$ has an open neighborhood $U$ such that $p$ induces a homeomorphism of $U$ onto an open subset $p(U)$ of $X$.

This can be expressed by saying that $p$ is an interior map and a local homeomorphism. It should be kept in mind that, even if $X$ is separated, $E$ is not supposed separated (and will in most important instances not be separated).

With the notations of definition 2.1.1, let $x = p(a)$. If $f$ is a section of $E$ such that $fx = a$, then $V = f^{-1}(U) \cap p(U)$ is an open set containing $x$, and on this neighborhood $V$ of $x$, $f$ must coincide with the inverse of the homeomorphism $p|U$ of $U$ onto $p(U)$. In particular

Proposition 2.1.1. Two sections of a sheaf $E$ defined in a neighborhood of $x$ and taking the same value at $x$ coincide in some neighborhood of $X$.

Corollary : Given two sections of $E$ in an open set $V$, the set of points where they are equal is open. (But in general not closed, as would be the case if $E$ were separated!).

## 2.2 $H^0(A, E)$ for arbitrary $A \subset X$.

First let $E$ be an arbitrary fibre space over $X$. Let $A$ be an arbitrary subset of $X$; the open neighborhoods of $A$, ordered by $\supset$, form an ordered filtering set. To each element $U$ of this set is associated a set $H^0(U, E)$ : the set of sections of $E$ over $U$, and if $U \supset V$ ($U$ and $V$ open neighborhoods of $A$), we have a natural map $\mathscr{S}_{VU} : H^0(U, E) \longrightarrow H^0(V, E)$ (restriction map), with the evident transitivity property $\mathscr{S}_{WV}\mathscr{S}_{VU} = \mathscr{S}_{WU}$ when $U \supset V \supset W$. Therefore we can consider the direct limit of the family of sets $H^0(U, E)$ for the maps $\mathscr{S}_{VU}$.

Definition 2.2.1. We put $H^0(A, E) = \varinjlim H^0(U, E)$, ($U$ ranging over the open neighborhoods as explained above). If $A = \{x\}$ ($x \in X$), we simply write $H^0(x, E)$. The elements of $H^0(A, E)$ are called germs of sections of $E$ in the neighborhood of $A$.

If $A$ is open, we find of course nothing else but the set of continuous sections of $E$ over $A$, already denoted by $H^0(A, E)$. If $A \supset B$, there is a natural map, again noted $\mathscr{S}_{BA}$ of $H^0(A, E)$ into $H^0(B, E)$, (definition left to the reader). When $A$ and $B$ are both open, this is the usual restriction map (therefore it will in general still be called restriction map); when $A$ is open, then this is the natural homomorphism of $H^0(A, E)$ into the direct limit of all $H^0(A', E)$ corresponding to open neighborhoods $A'$ of $B$. Of course $A \supset B \supset C$ implies $\mathscr{S}_{CB}\mathscr{S}_{BA} = \mathscr{S}_{CA}$.

Let $\Gamma(A, E)$ be the set of continuous sections of $E$ over the arbitrary set $A \supset X$, then the restriction maps $H^0(U, E) = \Gamma(U, E) \longrightarrow \Gamma(A, E)$ ($U$, open neighborhood of $A$) define a natural map of $\varinjlim H^0(U, E) = H^0(A, E)$ into $\Gamma(A, E)$. In particular, there is a natural map $H^0(x, E) \longrightarrow E_x$, where $E_x$ is the fibre of $x$ in $E$ (value at $x$ of a germ of section in a neighborhood of $x$). This of course, though frequently an onto-map, will seldom be one-to-one. However :

Proposition 2.2.1. If $E$ is a sheaf on $X$, then for $x \in X$, the canonical map $H^0(x, E) \longrightarrow E_x$, is bijective (i.e, one-to-one and onto). If $A$ is any subset of $X$, then the canonical map $H^0(A, E) \longrightarrow \Gamma(A, E)$ is one-to-one; it is moreover onto if $A$ admits a fundamental system of paracompact neighborhoods.

The one-to-one parts are contained in Proposition 2.1.1 and its corollary. The first onto-assertion results at once from definition 2.1.1. Now let $f$ be a continuous section of $E$ over $A$; for any $x \in A$, let $g_x$ be a continuous section of $E$ on an open neighborhood $V_x$ of $x$ in $X$, such that $g_x(x) = f(x)$ (these exist by first part of proposition 2.2.1.). Moreover, by the first part of proposition 2.1.1. applied to $E|A$, we can suppose $V_x$ small enough so that on $V_x \cap A$, $g_x$ and $f$ coincide. We can suppose that $U = \cup V_x$ is a paracompact neighborhood of $A$. Let $(V_i)_{i \in I}$ be an open locally finite covering of $U$ finer than $(V_x)$, that is each $V_i$ is contained in some $V_x$. Then for each $V_i$ exists $g_i \in H^0(V_i, E)$ such that $g_i$ and $f$ coincide on $V_i \cap A$. $U$ being paracompact, we can find an open covering $(V'_i)$ of $U$ such that the relative closure of $V'_i$ in $U$ be contained in $V_i$. For each $x \in A$, there exists an open neighborhood $W_x$ of $x$ in $U$ meeting only a finite number among the $V_i$'s; taking $W_x$ small enough, we can assume that $x \notin \overline{V'_i}$ implies $V'_i \cap W_x = \varnothing$, and $x \in \overline{V'_i}$ implies $W_x \subset V_i$. Moreover, by virtue of proposition 2.1.1., we can suppose that the corresponding $g_i$'s are identical on $W_x$ since they take the same value $f(x)$ at $x$. Therefore whenever a $V'_i$ encounters $W_x$, then $g_i$ is defined on $W_x$ and does not depend on the choice of $i$, so that we can denote it by $h_x$. It follows that in $W_x \cap W_y$, $h_x$ and $h_y$ are the same, therefore, the $h_x$ are the restrictions of a unique section $h$ of $E$ over $W = \cup W_x$. This is a continuous section of $E$ on an open neighborhood of $A$, and we see at once that its restriction to $A$ is $f$. This ends the proof.

Remark. The last part of proposition 2.2.1. becomes false if we drop the paracompactness restriction. Let for instance $X$ be an infinite set, with the topology in which the open sets are the complements of all finite sets (such spaces are significant in algebraic topology, for instance : irreducible algebraic curve with the Zariski topology). Let $F$ be a discrete space ; consider the trivial fibre space $X \times F$. This is a sheaf ; its sections on a set $A$ are the locally constant maps of $A$ into $F$ (cf. section 2. 6. below, example a). Let $A$ be a finite subset of $X$ ; it is seen at once that any open neighborhood of $A$ is homeomorphic to $X$ and hence connected ; therefore a section of $E$ on such a neighborhood is constant ; but sections on $A$ can have arbitrary distinct values at the points of $A$ and therefore will not in general be restrictions of sections defined in a neighborhood of $A$.

## 2.3 Definition of a sheaf by systems of sets.

As we noticed in the preceding section, any fibre space $E$ (and in particular any sheaf) determines sets $H^0(U, E)$ (for instance for any open $U \subset X$) and maps $H^0(U, E) \longrightarrow H^0(V, E)$ for $U \supset V$, satisfying an evident transitivity property. Proposition 2.2.1. suggests that conversely such a system should define a sheaf. Indeed, let $\mathcal{V}$ be an open covering of $X$, and suppose defined a function $U \longrightarrow E_U$ on the set of open sets which are small of order $\mathcal{V}$ (i. e. contained in some set element of $\mathcal{V}$), each $E_U$ being a set. Suppose given moreover, if $U$ and $V$ are $\mathcal{V}$-small and $U \supset V$, a map $\mathscr{S}_{VU} : E_U \longrightarrow E_V$, these maps satisfying the transitivity condition

(2.3.1.) $$\mathscr{S}_{WV}\mathscr{S}_{VU} = \mathscr{S}_{WU} \qquad \text{(if } U \supset V \supset W\text{)},$$

For any $x \in X$, let $E_x = \varinjlim E_U$, $U$ ranging over the ordered filtering set of open neighborhoods of $x$ (ordered by $\supset$). Let $E$ be the union of the $E_x$'s, and $p$ the map of $E$ into $X$ mapping $E_x$ in $x$. Define in $E$ a topology as follows : for any $f \in E_U$ and $x \in U$, we consider the canonical image $f_x \in E$ of $f$ in the direct limit $E_x$ of the sets $E'_U$ corresponding to all open neighborhoods $U'$ of $x$. Let $O(f)$ be the set of all elements $f_x \in E$ when $x$ ranges over $U$. When $U$ and $f \in E_U$ vary, we get a family of subsets $O(f)$ of $E$, which generate a topology on $E$. It is easily checked that $(E, X, p)$ form a sheaf, that is that $p$ is continuous, interior and a local homeomorphism.

Definition 2.3.1. The sheaf $E$ thus defined is called the sheaf defined by the system of sets $E_U$ and maps $\mathscr{S}_{VU}$.

Consider now an open set $U \subset X$, $\mathcal{V}$-small ; for any $f \in E_U$, the map $x \longrightarrow f_x$ is clearly a section of the sheaf $E$, and moreover continuous, which we denote by $\widetilde{f}$. We get thus a natural map $f \longrightarrow \widetilde{f}$ of $E_U$ into $H^0(U, E)$.

Proposition 2.3.1. In order that $f \longrightarrow \widetilde{f}$ be a one-to-one map, it is necessary and sufficient that for any open covering $(U_i)$ of $U$, and two elements $f, g$ of $E_U$, $\mathscr{S}_{U_iU} f = \mathscr{S}_{U_iU} g$ for each $i$ implies $f = g$. In order that $f \longrightarrow \widetilde{f}$ be onto, it is necessary and sufficient that for any open covering $(U_i)$ of $U$, and any system $(f_i) \in \cap E_{U_i}$ satisfying

(2.3.2.) $$\mathscr{S}_{U_i \cap U_j, U_i} f_i = \mathscr{S}_{U_i \cap U_j, U_j} f_j \qquad \text{when } U_i \cap U_j \neq \varnothing$$

there exists $a f \in E_U$ such that $f_i = \mathscr{S}_{U_iU} f$ for each $i$.

Corollary. In order that $f \longrightarrow \widetilde{f}$ be bijective, it is necessary and sufficient that for any open covering $(U_i)$ of $U$, the natural map $E_U \longrightarrow \cap E_{U_i}$ (the components of which are the maps $\mathscr{S}_{U_iU}$) be a one-to-one map of $E_U$ onto the subset of the product of all $(f_i)$ satisfying condition (2.3.2.).

Proof left to the reader, as well as the proof of the following :

Proposition 2.3.2. Let $E$ be a sheaf on $X$, consider the system of sets $H^0(U,E)$ and of restriction maps $\mathscr{S}_{VU} : H^0(U,E) \longrightarrow H^0(V,E)$ for $U \supset V$ $(U,V$ open sets). Then the sheaf $E'$ defined by these data (definition 2.3.1.) is canonically isomorphic to $E$, this isomorphism, transforming for each $x \in X$, $E'_x = \varinjlim H^0(U,E) = H^0(x,E)$ into $E_x$, being the isomorphism considered in proposition 2.2.1.

The two preceding propositions show essential equivalence of the notion of sheaf on the space $X$, and the notion of a system of sets $(E_U)$ $(U$ open $\subset X)$ and of maps $\mathscr{S}_{VU}$ for $U \supset V$, satisfying conditions (2.3.1.) and the condition of corollary of proposition 2.3.1. Both pictures are of importance, the second more intuitive, but the first often technically more simple.

Exercise. Given a system of sets $E_U$ ($U$ open and $\mathcal{V}$-small) and of homomorphisms $\mathscr{S}_{VU}$ $(U \supset V)$ satisfying (2.3.1.), prove that if we restrict to those $U$ which are $\mathcal{V}'$-small (where $\mathcal{V}'$ is an open covering of $X$ finer than $\mathcal{V}$), the sheaf defined by this new system is canonically isomorphic to the sheaf defined by the first.

## 2.4 Permanence properties.

Let $E$ be a sheaf on the space $X$, and let $f$ be a continuous map of a space $X'$ into $X$, then the inverse image of the fibre space $E$ by $f$ (cf 1.2.) is again a sheaf. In particular, if $X' \subset X$, $E$ induces a sheaf on $X'$.

If $E$ is a sheaf on $X$, $F$ a sheaf on $Y$, then $E \times F$ is a sheaf on $X \times Y$ ; therefore, if $E$ and $F$ are two sheaves on $X$, then their fibre-product $E \underset{X}{\times} F$ (cf. 1.3) is again a sheaf; this extends to the product of a finite number of sheaves.

Under the conditions of 1.5. suppose that the fibre spaces $E_i$ on the open sets $U_i$ are sheaves, then the fibre space $E$ obtained by means of coordinate transforms $f_{ij}$ is again a sheaf. This results at once from the more general remark : if $E$ is a fibre space such that each $x \in X$ has a neighborhood $U$ such that $E|U$ be a sheaf, then $E$ is a sheaf (trivial).

### 2.5 Subsheaf, quotient sheaf. Homomorphisms of sheaves.

Proposition 2.5.1. Let $E$ be a sheaf on the space $X$. In order that a subset $F$ of $E$, considered as a fibre space over $X$, be a sheaf, it is necessary and sufficient that it be open. In order that the quotient of $E$ by an equivalence relation $R$ compatible with the fibering, be a sheaf, it is necessary and sufficient that the set of equivalent pairs $(z, z')$ be open in the fibered product $E \underset{X}{\times} E$.

These conditions can be stated also equivalently : if a section $f$ of $E$ in a neighborhood of $x \in X$ is such that $fx \in F$, then $fy \in F$ for $y$ in a neighborhood of $x$ ; if two sections $f, g$ of $E$ in a neighborhood of $x \in X$ are such that $fx$ and $gx$ are equivalent mod $R$, then $fy$ and $gy$ are equivalent mod $R$ for $y$ in a neighborhood of $x$.

Proposition 2.5.2. Let $E$ be a sheaf on $X$, $E'$ a sheaf on $X'$, $f$ a continuous map of $X$ into $X'$ and $g$ a map from $E$ into $E'$ such that $p'g = fp$ ($p, p'$ being the projections of $E, e'$). In order for $g$ to be an $f$-homomorphism (i.e. to be continuous) it is necessary and sufficient that for any section $s$ of $E$ over an open set $U$, $gs$ be a section of $E'$ over $f(U)$.

Corollary 1. Let $f$ be a bijective $X$-homomorphism of a sheaf $E$ in a sheaf $F$, then $f$ is an isomorphism of $E$ onto $F$.

Corollary 2. Let $E, F$ be two sheaves on $X$, $f$ an $X$-homomorphism of $E$ into $F$. Then $f$ is an interior map, and $f(E)$ is a subsheaf of $F$. The quotient of $E$ by the equivalence relation defined by the map $f$ is again a sheaf, and $f$ defines an isomorphism of this quotient onto the sheaf $f(E)$.

Consider now a system $(E_U, \mathscr{S}_{VU})$ as in section 2.3., defining a sheaf $E$. Suppose given for each $U$ a subset $E'_U$ of $E_U$, such that $U \supset V$ implies $\mathscr{S}_{VU}(E'_U) \subset E'_V$. Let $\mathscr{S}'_{VU}$ be the map of $E'_U$ into $E'_V$ defined by $\mathscr{S}_{VU'}$, then the system $(E'_U, \mathscr{S}'_{VU}$ defines a sheaf $E'$. For any $x \in X$, the fibres of $x$ in $E$ respectively $E'$ are given by

$$E_x = \varinjlim E_U \qquad E'_x = \varinjlim E'_U$$

the direct limit being taken in the ordered filtering set of open neighborhoods of $x$. Therefore, we have a natural injection $E'_x \subset E_x$, and hence $E' \subset E$. It is easily checked that the injection of $E'$ into $E$ is a homomorphism, (a particular case of a general characterization of homomorphisms to be given below), so that by corollary 1 above, $E'$ is isomorphic to a subsheaf of $E$. Suppose that the conditions of proposition 2.3.1. corollary, are satisfied, which insure that $E_U = H^0(U, E)$. Then clearly the canonical maps $E'_U \longrightarrow H^0(U, E') \subset H^0(U, E)$ are one-to-one.

Proposition 2.3.1. yields that in order that they be onto, it is necessary and sufficient that any $f \in E_U$ such that each $x \in U$ has an open neighborhood $V$ in $U$ such that $\mathscr{S}_{VU} f \in E'_V$ be contained in $E'_U$ ; or shortly speaking that the property, for an element

$f$ of an $E_U$ to belong to the subset $E'_U$, be a property of local character. If conversely we start with an arbitrary subsheaf $E'$ of $E$, and denote by $E'_U$ the subset $H^0(U, E')$ of $E_U = H^0(U, E)$, then these $E'_U$ clearly satisfy to the conditions $\mathscr{S}_{V'U} E'_U \subset E'_V$, and the subsheaf of $E$ defined by them is nothing else but $E'$.

Now let $E, F$ be two sheaves on $X$ defined by systems $(E_U, \mathscr{S}_{VU})$ and $(F_U, \Psi_{VU})$. Suppose given for any $U$ a map $f_U : E_U \longrightarrow F_U$, such that $U \supset V$ implies $\Psi_{VU} f_U = f_U \mathscr{S}_{VU}$. Then this system of maps defines, for each $x \in X$, a map $f_x$ of $E_x = \varinjlim E_U$ into $F_x = \varinjlim F_U$, hence a map $f$ of $E$ into $F$. It is checked easily (using for instance proposition 2.5.2.) that $f$ is a homomorphism of $E$ into $F$. Moreover, $f(E)$ is nothing else but the subsheaf of $F$ defined by the subsets $f_U(E_U)$ of the $F_U$. For any open $U$, the following diagram is commutative.

$$\begin{array}{ccc} E_U & \xrightarrow{fu} & F_U \\ \downarrow & & \downarrow \\ H^0(U,E) & \xrightarrow{f_*} & H^0(U,F) \end{array}$$

In particular, if the vertical maps are bijective, we see that the maps $f_U$ can be identified with the maps $f_* : H^0(X, E) \longrightarrow H^0(X, F)$ defined by the homomorphism $f$. Conversely, if we start with an arbitrary homomorphism $f$ of $E$ into $F$, then the homomorphism defined by the system of maps $f_U$ of $E_U = H^0(U, E)$ into $F_U = H^0(U, F)$ is precisely $f$.

## 2.6 Some examples.

a. Constant and locally constant sheaves

Let $F$ be a discrete space, then the trivial fibre space $X \times F$ is clearly a sheaf on $X$ ; a sheaf isomorphic to such a sheaf is called constant. The sections of this sheaf on a set $A \subset$X are the continuous maps of $A$ in the discrete set $F$, i.e, the maps of $A$ in $F$ which are locally constant. If for instance $A$ is connected, these reduce to the constant maps of $A$ into $F$. Inverse images and products of simple sheaves are simple.

A sheaf $E$ on $X$ is called locally simple, if each $x \in X$ has a neighborhood $U$ such that $E|U$ be simple. Thus a locally simple sheaf on $X$ is nothing else but a covering space of $X$ in the classical sense (but not restricted of course to be connected). Inverse images and products of locally simple sheaves in finite number are locally simple.

b. Sheaf of germs of maps. Let $X$ be a space, $E$ a set. Consider for any open $U \subset X$ the set $\mathscr{F}(U, E)$ of all maps of $U$ into $E$ ; if $U \supset V$, we have a natural map of $\mathscr{F}(U, E)$ into $\mathscr{F}(V, E)$, the restriction map. The transitivity condition of section 2.3 is clearly satisfied, and also the condition of proposition 2.3.1., corollary. Therefore the sets $\mathscr{F}(U, E)$ can be identified with the sets of sections $H^0(U, F)$ of a well determined sheaf $\mathscr{F}$, the elements of which are called germs of maps of $X$ into $E$.

If $A \subset X$, then the elements of $H^0(A, \mathscr{F})$ are called germs of maps of a neighborhood of $A$ into $E$. If now $E$ is a topological space, we can consider for any $U$ the subset $C(U, E)$ of $\mathscr{F}(U, E)$ of the continuous maps of $U$ into $E$. As continuity is a condition of local character, it follows by section 2.5 that the sets $C(U, E)$ are the sets of sections of a well determined subsheaf of $\mathscr{F}$, which is called the sheaf of germs of continuous maps of $X$ into $E$. (If we take on $E$ the coarsest topology, we find again the first sheaf.)

Suppose now that $E$ is a fibre space over $X$, then consider for any $U$ the subset $H^0(U, E)$ of $C(U, E)$ of continuous sections of $E$. The property of being a section is again of local character, so we see that the sets $H^0(U, E)$ are sets of sections of a well determined subsheaf of the sheaf of germs of continuous maps of $X$ into $E$ : the sheaf of germs of sections of the fibre space $E$. If this sheaf is denoted by $\widetilde{E}$, then $H^0(A, \widetilde{E})$ is nothing else but the set of germs of sections of $E$ in the neighborhood of $A$, as defined in definition 2.2.1.

Of course, specializing the spaces $X$ and $E$, we can define a great number of other subsheaves of the sheaf of germs of maps of $X$ into $E$ (germs of differentiable maps, germs of analytic maps, germs of maps which are $L^P$ etc.).

c. Sheaf of germs of homomorphisms of a fibre space into another.

Let $E$ and $F$ be two fibre spaces over $X$, and for any open $U \subset X$ let $H_U$ be the set of homomorphisms of $E|U$ into $F|U$. If $V$ is an open set contained in $U$, there is an evident natural map of restriction $H_U \longrightarrow H_V$. The condition of transitivity as well as the condition of proposition 2.3.1. corollary, are satisfied, so that the sets $H_U$ appear as the sets $H^0(U, H)$ of sections of a well determined sheaf on $X$, the elements of which are called germs of homomorphisms of $E$ into $F$. A section of this sheaf over $X$ is a homomorphism of $E$ into $F$.

d. Sheaf of germs of subsets.

Let $X$ be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of $U$. If $U \supset V$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $H^0(U, P(X))$ of sections of a well determined sheaf on $X$, the elements of which are called germs of sets in $X$. Any condition of a local character on subsets of $X$ defines a subsheaf of $P(X)$, for instance the sheaf of germs of closed sets (corresponding to the relatively closed sets in $U$), or if $X$ is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.