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MODULAR FORMS OF WEIGHT 1/2

by

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INTRODUCTION

In his Annals paper on modular forms of half integral weight [8], Shimura mentions several open questions. One of them is the following : is every form of weight 1/2 a linear combination of theta series in one variable ?

We show that the answer is positive. The precise statements are given in §2, Theorems A and B; they give an explicit basis of modular forms (and cusp forms) of weight 1/2 and given level. The proof uses the fact that, for weight 1/2, the formula defining the Hecke operator $T(p^2)$ introduces unbounded powers of p in the denominators of the coefficients – unless some remarkable cancellations take place (§5). But it is a familiar fact that coefficients of modular forms (on congruence subgroups) have bounded denominators. Hence the above cancellations do hold, and they give us the information we need, when combined with basic properties of "newforms" à la Atkin-Lehner-Li (§§ 3,4). The details are carried out in §§ 6,7. As an Appendix, we have included a letter from Deligne sketching an alternative method, using the "group-representation" point of view.

In the above proofs, arithmetic arguments play an essential role. It would be interesting to have a more analytic proof; a natural line of attack would be to adapt Shimura's Main Theorem ([8], §3) to weight 1/2, but we have not investigated this.

We mention a possible application of Theorems A and B : since the weights 1/2 and 3/2 occur together in dimension formulae and trace formulae ([9], §5), the explicit knowledge of forms of weight 1/2 gives a way of computing these dimensions and traces for weight 3/2.

§1. SOME NOTATION

1.1. Upper half-plane and modular groups.

We use standard notations, cf. [3], [7]. The letter H denotes the upper half-plane $\{z \mid \mathrm{Im}(z) > 0\}$. If $z \in H$, we put $q = e^{2\pi iz}$. Let $\mathbf{GL}_2(\mathbf{R})^+$ be the subgroup of $\mathbf{GL}_2(\mathbf{R})$ consisting of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) > 0$; we make $\mathbf{GL}_2(\mathbf{R})^+$ act on H by

$$z \mapsto Az = (az+b)/(cz+d).$$

Let N be a positive integer divisible by 4. We denote by $\Gamma_0(N)$ and $\Gamma_1(N)$ the subgroups of $\mathbf{SL}_2(\mathbf{Z})$ defined by :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \iff c \equiv 0 \pmod N$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \iff a \equiv d \equiv 1 \pmod N \text{ and } c \equiv 0 \pmod N.$$

The group $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$, and the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$ induces an isomorphism of $\Gamma_0(N)/\Gamma_1(N)$ onto $(\mathbf{Z}/N\mathbf{Z})^*$.

1.2. Characters.

If $t \in \mathbf{Z}$, we denote by χ_t the primitive character of order $\leqslant 2$ corresponding to the field extension $\mathbf{Q}(t^{1/2})/\mathbf{Q}$. If t is a square, we have $\chi_t = 1$. It t is not a square, and the discriminant of $\mathbf{Q}(t^{1/2})/\mathbf{Q}$ is D, then χ_t is a quadratic character of conductor $|D|$, and we have

$$\chi_t(m) = \left(\frac{D}{m}\right) \qquad \text{(Kronecker symbol).}$$

In particular, $\chi_t(m) = 0$ if and only if $(m,D) \neq 1$. (Recall that, if $t = u^2 d$, with $u \in \mathbf{Z}$, and d is square-free, we have $D = d$ if $d \equiv 1 \pmod 4$, and $D = 4d$ otherwise.)

1.3. Theta multiplier.

Let $\theta(z) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2 = \sum_{-\infty}^{+\infty} q^{n^2} = 1 + 2q + 2q^4 + \dots$ be the standard theta function. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma_0(4)$, we have

$$\theta(Az) = j(A,z)\theta(z),$$

where $j(A,z)$ is the "θ-multiplier" of A. Recall (cf. for instance [8]) that, if $c \neq 0$, we have

$$j(A,z) = \varepsilon_d^{-1} \chi_c(d)(cz+d)^{1/2},$$

where
$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4), \end{cases}$$

and $(cz+d)^{1/2}$ is the "principal" determination of the square root of $cz + d$, i.e. the one whose real part is > 0 (more generally, all fractional powers in this paper have to be understood as principal values). If $c = 0$, we have $A = \pm 1$, and $j(A,z)$ is obviously equal to 1.

1.4. Modular forms of half integral weight.

Let $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$ be a character (mod N), and let κ be a positive odd integer. A function f on H is called a modular form of type $(\kappa/2,\chi)$ on $\Gamma_0(N)$ if :

a) $f(Az) = \chi(d)\, j(A,z)^{\kappa} f(z)$ for every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(N)$; this makes sense since $4|N$;

b) f is holomorphic, both on H and at the cusps (see [8]).

One then calls $\kappa/2$ the weight of f, and χ its character. The space of such functions will be denoted by $M_0(N,\kappa/2,\chi)$; it is clear that $M_0(N,\kappa/2,\chi)$ consists only of 0 unless χ is even, i.e. $\chi(-1) = 1$. We put

$$M_1(N,\kappa/2) = \bigoplus_{\chi} M_0(N,\kappa/2,\chi),$$

where the sum is taken over all (even) characters of $(\mathbf{Z}/N\mathbf{Z})^*$; this space is the space of modular forms of weight $\kappa/2$ on $\Gamma_1(N)$.

A modular form which vanishes at all cusps is called a cusp form. The subspace of $M_0(N,\kappa/2,\chi)$ (resp. $M_1(N,\kappa/2)$) made up by cusp forms will be denoted by $S_0(N,\kappa/2,\chi)$ (resp. $S_1(N,\kappa/2)$).

EXAMPLE : theta series with characters.

Let ψ be an even primitive character of conductor $r = r(\psi)$. We put

$$\theta_\psi(z) = \sum_{-\infty}^{\infty} \psi(n)q^{n^2}.$$

When $\psi = 1$, θ_ψ is equal to θ. When $\psi \neq 1$, θ_ψ is equal to :

$$2\sum_{\substack{n\geqslant 1\\ (n,r)=1}} \psi(n)q^{n^2} = 2(q + \psi(2)q^4 + \ldots) .$$

We have $\theta_\psi \in M_0(4r^2,1/2,\psi)$, cf. [8], p.457. This implies that, if t is an integer $\geqslant 1$, the series $\theta_{\psi,t}$ defined by

$$\theta_{\psi,t}(z) = \theta_\psi(tz) = \sum_{-\infty}^{\infty} \psi(n)q^{tn^2}$$

belongs to $M_0(4r^2t,1/2,\chi_t\psi)$, see for instance Lemma 2 below.

Warning. One should not confuse θ_ψ with the series $\sum \psi(n)^2q^{n^2}$ obtained by twisting θ with the character ψ, cf. §7.

1.5. Petersson scalar product.

If $z \in H$, we put $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. The measure $dxdy/y^2$ is invariant by $\mathbf{GL}_2(\mathbf{R})^+$. If f,g belong to $M_1(N,\kappa/2)$, the function

$$F_{f,g}(z) = f(z)\overline{g(z)}y^{\kappa/2}$$

is invariant by $\Gamma_1(N)$. Hence $F_{f,g}(z)y^{-2}dxdy$ is invariant by $\Gamma_1(N)$ and defines a measure $\mu_{f,g}$ on $H/\Gamma_1(N)$. One checks immediately that $\mu_{f,g}$ is a <u>bounded measure</u> in each of the following two cases :

i) one of the forms f,g is a <u>cusp form</u>;

ii) $\kappa = 1$ (this was first noticed by Deligne).

In each case, the <u>Petersson scalar product</u> $\langle f,g \rangle$ of f and g is defined as the (absolutely convergent) integral :

$$\langle f,g \rangle = \frac{1}{c(N)} \int \mu_{f,g} = \frac{1}{c(N)} \int_{H/\Gamma_1(N)} f(z)\overline{g(z)}\, y^{\kappa/2-2}dxdy,$$

where c(N) is the index of $\Gamma_1(N)$ in $\mathbf{SL}_2(\mathbf{Z})$.

This is a hermitian scalar product. One has $\langle f,f \rangle > 0$ if $\langle f,f \rangle$ is defined and $f \neq 0$.

§2. STATEMENT OF RESULTS

2.1. Basis of modular forms of weight 1/2.

Our main result (Theorem A below) states that every modular form of weight 1/2 is a linear combination of theta series with characters. More precisely, let χ be an even character (mod N); let $\Omega(N,\chi)$ be the set of pairs (ψ,t), where t is an integer $\geqslant 1$, and ψ is an even primitive character with conductor $r(\psi)$, such that :

(i) $4r(\psi)^2t$ divides N,

(ii) $\chi(n) = \psi(n)\chi_t(n)$ for all n prime to N.

Condition (ii) is equivalent to saying that ψ is the primitive character associated with $\chi\chi_t$; hence ψ is determined by t and χ. Conversely, t and ψ determine χ.

THEOREM A. *The theta series* $\theta_{\psi,t} = \sum_{-\infty}^{\infty} \psi(n)q^{tn^2}$, *with* $(\psi,t) \in \Omega(N,\chi)$, *make up a basis of* $M_0(N,1/2,\chi)$.

This will be proved in §6.

Call $\Omega(N)$ the set of pairs (ψ,t) satisfying condition (i) above; this set is the union of the $\Omega(N,\chi)$, for all even characters χ (mod N); hence Theorem A implies :

COROLLARY 1. *The series* $\theta_{\psi,t}$, *with* $(\psi,t) \in \Omega(N)$, *make up a basis of the space* $M_1(N,1/2)$ *of modular forms of weight* 1/2 *on* $\Gamma_1(N)$.

In particular :

COROLLARY 2. *If* $f = \sum_{n=0}^{\infty} a(n)q^n$ *is a modular form of weight* 1/2 *on* $\Gamma_1(N)$, *then* $a(n) = 0$ *if* n *is not of the form* tm^2, *where* t *is a divisor of* N/4, *and* $m \in \mathbf{Z}$.

COROLLARY 3. *Let* $f = \sum_{n=0}^{\infty} a(n)q^n$ *be a formal power series with complex coefficients. The following properties are equivalent* :

1) f *is a modular form of weight* 1/2 *on some* $\Gamma_1(N)$.

2) f *is a linear combination of theta series*

$$\theta_{n_0,r,t} = \sum_{\substack{n \equiv n_0 \pmod r \\ n \in \mathbf{Z}}} q^{tn^2}$$

3) *For each square-free integer* $t \geq 1$, *there is a periodic function* ε_t *on* $\mathbf{Z}$ *such that* :

3.1) $a(tn^2) = \varepsilon_t(n)$ *for every* $n \geq 1$;

3.2) *each* ε_t *is even* (*i.e.* $\varepsilon_t(n) = \varepsilon_t(-n)$ *for all* $n \in \mathbf{Z}$);

3.3) ε_t *is* 0 *for all but finitely many* t;

3.4) $a(0) = \frac{1}{2} \sum_t \varepsilon_t(0)$.

PROOF. The equivalence of 2) and 3) is elementary. The fact that a theta series is a modular form is well known (cf. for instance [8], §2); hence 2) implies 1). Corollary 2 above shows that 1) implies 3).

COROLLARY 4. Let $f = \sum_{n=0}^{\infty} a(n)q^n$ be a non-zero modular form of weight 1/2 on some $\Gamma_1(N)$. Then :

a) $|a(n)| = O(1)$;

b) for every $\rho \geqslant 0$, there is a constant $c_\rho > 0$ such that

$$\sum_{n \leqslant x} |a(n)|^\rho = c_\rho x^{1/2} + O(1) \text{ for } x \to \infty.$$

(If $\rho = 0$ and $a(n) = 0$, we put $|a(n)|^\rho = 0$.)

PROOF. This follows from Corollary 3.

REMARK. If f and g are modular forms of weight 1/2 on $\Gamma_1(N)$, their product $F = f.g$ is a modular form of weight 1. By Tneorem A, F is a linear combination of series

$$\sum_{n,m} \alpha(n)\beta(m)\, q^{an^2+bm^2} ,$$

where α and β are characters. This shows that F is a linear combination of Eisenstein series and cusp forms of dihedral type associated with imaginary quadratic fields (cf. [3], §4). Hence, one cannot use products of forms of weight 1/2 to construct "exotic" modular forms of weight 1.

2.2. Cusp forms of weight 1/2.

If ψ is a character with conductor r, one may write ψ in a unique way as $\psi = \prod_{p|r} \psi_p$, where the conductor of ψ_p is the highest power of p dividing r; we call ψ_p the p^{th}-component of ψ (in the Galois interpretation of characters, ψ_p is just the restriction of ψ to the inertia group at p). We say that ψ is <u>totally even</u> if all the ψ_p's are even, i.e. if $\psi_p(-1) = 1$ for all $p|r$; this is equivalent to saying that ψ is the <u>square</u> of a character (which can be chosen of conductor r, if r is odd, and of conductor 2r, if r is even).

Denote by $\Omega_e(N,\chi)$ the subset of $\Omega(N,\chi)$ (see above) made up of the (ψ,t) such that ψ is totally even, and put

$$\Omega_c(N,\chi) = \Omega(N,\chi) - \Omega_e(N,\chi).$$

Define similarly

$$\Omega_e(N) = \bigcup_{\chi} \Omega_e(N,\chi) \ , \ \Omega_c(N) = \bigcup_{\chi} \Omega_c(N,\chi).$$

<u>THEOREM</u> B. <u>The series</u> $\theta_{\psi,t}$, <u>with</u> $(\psi,t) \in \Omega_c(N,\chi)$, <u>make up a basis of the space</u> $S_0(N,1/2,\chi)$ <u>of cusp forms of</u> $M_0(N,1/2,\chi)$. <u>The series</u> $\theta_{\psi,t}$, <u>with</u> $(\psi,t) \in \Omega_e(N,\chi)$, <u>make up a basis of the orthogonal complement of</u> $S_0(N,1/2,\chi)$ <u>in</u> $M_0(N,1/2,\chi)$ <u>for the Petersson scalar product</u>.

This theorem will be proved in §7. It implies :

<u>COROLLARY</u> 1. <u>The series</u> $\theta_{\psi,t}$, <u>with</u> $(\psi,t) \in \Omega_c(N)$, <u>make up a basis of the space</u> $S_1(N,1/2)$ <u>of cusp forms of weight</u> 1/2 <u>on</u> $\Gamma_1(N)$.

<u>COROLLARY</u> 2. <u>We have</u> $S_1(N,1/2) \neq 0$ <u>if and only if</u> N <u>is divisible by either</u> $64p^2$ <u>where</u> p <u>is an odd prime</u>, <u>or</u> $4p^2p'^2$, <u>where</u> p <u>and</u> p' <u>are distinct odd primes</u>.

Indeed, Cor. 1 shows that $S_1(N,1/2)$ is non-zero if and only if there exists an even character ψ with conductor $r(\psi)$, which is not totally even, and which is such that $r(\psi)^2$ divides N/4. Since ψ is even, at least two p^{th}-components of ψ are odd; this shows that $r(\psi)$ is divisible by either 4p, where p is an odd prime, or by pp', where p and p' are distinct odd primes; hence N is divisible by either $4.(4p)^2 = 64p^2$ or $4(pp')^2 = 4p^2p'^2$. Conversely, if N is divisible by $64p^2$ (resp. by $4p^2p'^2$), one takes for ψ the product of an odd character of conductor p by an odd character of conductor 4 (resp. p'); it is clear that ψ has the required properties.

<u>EXAMPLES</u>. The above results allow an easy determination of the spaces of modular form of weight 1/2 on $\Gamma_0(N)$ and $\Gamma_1(N)$: all one has to do is to make a list of the divisors t of $N/4$, and, for each such t, determine the even characters ψ with conductor $r(\psi)$ such that $r(\psi)^2$ divides $N/4t$. The pairs (ψ,t) thus obtained make up the set $\Omega(N)$. We give two examples :

i) $N = 4p_1\dots p_h$, where the p_i's are distinct primes. In this case t is a product of some of the p_i's, and $r(\psi)$ must be equal to 1, hence $\psi = 1$. Applying Cor. 1 to Th. A, we see that the series

$$\theta(tz) = \sum_{-\infty}^{\infty} q^{tn^2} \qquad \text{(where } t \text{ divides } p_1\dots p_h)$$

make up a basis of $M_1(N,1/2)$. Moreover, we have $\theta(tz) \in M_0(N,1/2,\chi_t)$; since the χ_t's are pairwise distinct, each $M_0(N,1/2,\chi_t)$ is one-dimensional, and we have $M_0(N,1/2,\chi) = 0$ if χ is not equal to one of the χ_t's (in particular if χ is not real).

ii) Let us determine $S_1(N,1/2)$ <u>for</u> $N < 900$. If this space is $\neq 0$, Cor. 2 to Th. B shows that N is divisible by either $64p^2$ or $4p^2p'^2$ where p,p' are distinct odd primes; the first case is possible only if $N = 576 = 64.3^2$; the second one is impossible (since it implies $N \geqslant 4.3^2 5^2 = 900$, which contradicts the assumption made on N). Hence we have N = 576, and it is easy to see that the only element of $\Omega_c(N)$ is the pair (ψ,t) with $t = 1$ and $\psi = \chi_3$ (which has conductor 12). The corresponding theta series is

$$\theta_{\chi_3} = \sum_{n\equiv\pm 1 \pmod{12}} q^{n^2} - \sum_{n\equiv\pm 5 \pmod{12}} q^{n^2}$$
$$= 2(q - q^{25} - q^{49} + q^{121} + q^{169} + \dots).$$

It follows from a classical result of Euler (cf. for instance [4], p. 931 or [8], p. 457) that $\frac{1}{2}\,\theta_{\chi_3}$ is equal to

$$\eta(24z) = q \prod_{n=1}^{\infty} (1-q^{24n}).$$

Up to a scalar factor, this series is thus the only cusp form of weight 1/2 and level $N < 900$.

§3. OPERATORS

3.1. Conventions on characters.

From now on, all characters are assumed to be primitive; this is necessary when dealing with different levels. We say that such a character χ is definable (mod m) when its conductor $r(\chi)$ divides m. The product $\chi\chi'$ of two characters χ and χ' is the primitive character associated with $n \mapsto \chi(n)\chi'(n)$; hence, we have

$$(\chi\chi')(n) = \chi(n)\chi'(n)$$

if n is prime to $r(\chi)r(\chi')$, but maybe not otherwise.

3.2. The group $\underline{G}$.

Following Shimura [8], we introduce the group extension $\underline{G}$ of $\mathbf{GL}_2(\mathbf{R})^+$ whose elements consist of pairs $\{M,\phi(z)\}$, where $M = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ belongs to $\mathbf{GL}_2(\mathbf{R})^+$ and $\phi(z)^2 = \alpha \det(M)^{-1/2}(tz+u)$, with $|\alpha| = 1$. The multiplication law in $\underline{G}$ is given by

$$\{M,\phi(z)\}\{N,\psi(z)\} = \{MN,\phi(Nz)\psi(z)\}.$$

When dealing with forms of weight $\kappa/2$ it is convenient to define the "slash operator" $f|_\kappa\xi = f|\xi$ by :

$$(f|\xi)(z) = \phi(z)^{-\kappa} f(Mz) \qquad \text{where } \xi = \{M,\phi\} \in \underline{G},$$

and, for $\xi_i \in \underline{G}$ and $c_i \in \mathbf{C}$:

$$f|(\sum c_i \xi_i) = \sum c_i (f|\xi_i).$$

If $A \in \Gamma_0(4)$, we define $A^* \in \underline{G}$ by $A^* = \{A, j(A,z)\}$, where $j(A,z)$ is the θ-multiplier of A, cf. §1. Thus, if $f \in M_0(N,\kappa/2,\chi)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have $f|A^* = \chi(d)f$.

It follows from the definition of j that

(1) $$A^* B^* = (AB)^* \qquad \text{if } A,B \in \Gamma_0(4).$$

Computations in $\underline{G}$ are greatly aided by making use of (1) whenever possible.

3.3. Hecke operators.

For a prime p, with $p \nmid N$, we define $T(p^2)$ on $M_0(N,\kappa/2,\chi)$ as in Shimura [8] by :

$$T(p^2) = p^{\kappa/2-2}\Big[\sum_{j=0}^{p^2-1} \{\begin{pmatrix} 1 & j \\ 0 & p^2 \end{pmatrix}, p^{1/2}\} + \chi(p) \sum_{j=1}^{p-1} \{\begin{pmatrix} p & j \\ 0 & p \end{pmatrix}, \varepsilon_p^{-1} \chi_{-j}(p)\}$$

$$+ \chi(p^2)\{\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/2}\}\Big]$$

where $\varepsilon_p = 1$ or i according as $p \equiv 1$ or $3 \pmod 4$, cf. §1. For a prime p with $p \mid N$ (for instance $p = 2$), we define $T(p^2)$ by

$$T(p^2) = p^{\kappa/2-2} \sum_{j=0}^{p^2-1} \{\begin{pmatrix} 1 & j \\ 0 & p^2 \end{pmatrix}, p^{1/2}\},$$

and, if $4p \mid N$, we define $T(p)$ by

$$T(p) = p^{\kappa/4-1} \sum_{j=0}^{p-1} \{\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}, p^{1/4}\}.$$

LEMMA 1. *Let* $f = \sum_{n=0}^{\infty} a(n)q^n$ *be an element of* $M_0(N,\kappa/2,\chi)$, *and let* $f|T(p^2) = \sum_{n=0}^{\infty} b(n)q^n$. *Then* $f|T(p^2)$ *belongs to* $M_0(N,\kappa/2,\chi)$ *also, and* *we have*

$$b(n) = \begin{cases} a(np^2) & \text{if } p \mid N, \\ a(np^2) + p^{(\kappa-3)/2}\chi(p)\chi_{-4}(p)^{(\kappa-1)/2}(\frac{n}{p})a(n) + \\ \quad + p^{\kappa-2}\chi(p^2)a(n/p^2) & \text{if } p \nmid N, \end{cases}$$

where $(\frac{n}{p})$ *is the Legendre symbol. If* $4p \mid N$, *then* $f|T(p)$ *belongs to* $M_0(N,\kappa/2,\chi\chi_p)$ *and is equal to* $\sum_{n=0}^{\infty} a(np)q^n$. *Any two such operators commute.*

PROOF. The statements about $T(p^2)$ are proved in Shimura, *loc. cit.* Those about $T(p)$, when $4p \mid N$, are proved by a simple computation.

3.4. *Other operators.*

We need the *shift* $V(m) = m^{-\kappa/4}\ \{(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}), m^{-1/4}\}$ which acts by

$$[f|V(m)](z) = f(mz).$$

We need also the *symmetry* $W(N) = \{(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}), N^{1/4}(-iz)^{1/2}\}$, which acts by

$$[f|W(N)](z) = N^{-\kappa/4}(-iz)^{-\kappa/2}f(-1/Nz),$$

so that $[f|W(N)]|W(N) = f$ for all f.

The *conjugation* operator H is defined by :

$$(f|H)(z) = \overline{f(-\bar{z})} = \sum_{n=0}^{\infty} \overline{a(n)}q^n \quad \text{if} \quad f = \sum_{n=0}^{\infty} a(n)q^n.$$

LEMMA 2. *The operators* $V(m)$, $W(N)$ *and* H *take* $M_0(N,\kappa/2,\chi)$ *to* $M_0(Nm,\kappa/2,\chi\chi_m)$, $M_0(N,\kappa/2,\bar{\chi}\chi_N)$ *and* $M_0(N,\kappa/2,\bar{\chi})$ *respectively. Further, if* f *belongs to* $M_0(N,\kappa/2,\chi)$, *we have* :

$$[f|V(m)]|T(p^2) = [f|T(p^2)]|V(m) \quad \text{when } p \nmid m,$$

$$[f|H]|T(p^2) = [f|T(p^2)]|H,$$

$$[f|W(N)]|T(p^2) = \bar{\chi}(p^2)[f|T(p^2)]|W(N) \quad \text{when } p \nmid N.$$

PROOF. Again, the proof involves simple computations in $\underline{G}$ and is left to the reader. Care should be exercised in the commutativity results since the definition of $T(p^2)$ depends on the character appearing in the space containing the function to which $T(p^2)$ is applied.

The following operators will be used in §4 only. To define the first one, suppose the prime p_0 divides $N/4$, and write $\Gamma_0(N/p_0)$ as a disjoint union of cosets modulo $\Gamma_0(N)$:

$$\Gamma_0(N/p_0) = \coprod_{j=1}^{\mu} \Gamma_0(N)A_j, \text{ with } A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \text{ and } \mu = (\Gamma_0(N/p_0) : \Gamma_0(N)).$$

We define the trace operator $S'(\chi) = S'(\chi,N,p_0)$ on $M_0(N,\kappa/2,\chi)$ by

$$S'(\chi) = \sum_{j=1}^{\mu} \chi(a_j)A_j^* = \sum_{j=1}^{\mu} \overline{\chi}(d_j)A_j^*.$$

It is easily seen that this operator does not depend on the choice of the A_j's. Moreover, if χ is definable (mod N/p_0), $S'(\chi)$ takes $M_0(N,\kappa/2,\chi)$ to $M_0(N/p_0,\kappa/2,\chi)$ and commutes with $T(p^2)$ for $p \nmid N$; if f belongs to $M_0(N/p_0,\kappa/2,\chi)$, we have

$$f|S'(\chi) = \mu f.$$

For our purposes, it is more important to find an operator which goes from level N to level N/p_0 and which undoes the action of the shift operator $V(p_0)$. To do this, we define $S(\chi) = S(\chi,N,p_0)$ on $M_0(N,\kappa/2,\chi)$ by :

$$S(\chi) = \frac{1}{\mu}\, p_0^{\kappa/4}\; W(N)\; S'(\overline{\chi}\chi_N)W(N/p_0).$$

LEMMA 3. *Let p_0 be a prime such that $4p_0|N$, and $\chi\chi_{p_0}$ is definable (mod N/p_0). Then :*

a) *The operator $S(\chi,N,p_0)$ maps $M_0(N,\kappa/2,\chi)$ into $M_0(N/p_0,\kappa/2,\chi\chi_{p_0})$.*

b) <u>If</u> m <u>is prime to</u> p_0, <u>and</u> f <u>belongs to</u> $M_0(N,\kappa/2,\chi)$, <u>then</u>

$$f|S(\chi,N,p_0) = f|S(\chi,Nm,p_0).$$

c) $S(\chi)$ <u>commutes with all</u> $T(p^2)$, <u>for</u> $p \nmid N$.

d) <u>If</u> $g \in M_0(N/p_0,\kappa/2,\chi\chi_{p_0})$, <u>then</u> $g|V(p_0) \in M_0(N,\kappa/2,\chi)$ <u>and</u>

$$[g|V(p_0)]|S(\chi,N,p_0) = g.$$

e) <u>Let</u> p <u>be a prime such that</u> $4p|N$, $p \neq p_0$, <u>and</u> $\chi\chi_p$ <u>is definable</u> (mod N/p). <u>If</u> $g \in M_0(N/p,\kappa/2,\chi\chi_p)$, <u>we have</u>

$$[g|V(p)]|S(\chi,N,p_0) = [g|S(\chi\chi_p,N/p,p_0)]|V(p).$$

<u>PROOF</u>. Assertion a) follows from Lemma 2 and from the fact that

$$\overline{\chi}\chi_N = \overline{\chi\chi_{p_0}}\chi_{N/p_0}$$

is definable (mod N/p_0).

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma_0(Nm/p_0)$, with $(m,p_0) = 1$, then

$$W(Nm)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* W(Nm/p_0) = \{m,1\}\ W(N)\begin{pmatrix} a & bm \\ c/m & d \end{pmatrix}^* W(N/p_0),$$

and b) follows, since $f|\{m,1\} = f$.

Assertion c) follows from the commutativity of the $T(p^2)$, $p \nmid N$, with $W(N), S'(\overline{\chi}\chi_N)$ and $W(N/p_0)$.

As for d), we have

$$\{\begin{pmatrix} p_0 & 0 \\ 0 & 1 \end{pmatrix}, p_0^{-1/4}\}W(N) = \{p_0,1\}W(N/p_0),$$

hence

$$[g|V(p_0)]|W(N) = p_0^{-\kappa/4}\ g|W(N/p_0).$$

This is invariant by $\frac{1}{\mu}\ S'(\overline{\chi}\chi_N)$, and is sent to $p_0^{-\kappa/4}$ g by $W(N/p_0)$, whic proves d).

As for e), we have $4p_0p|N$, and $\chi\chi_{p_0}\chi_p$ is definable (mod N/pp_0). Further :

$$\{(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}), p^{-1/4}\} W(N) = \{p,1\} W(N/p),$$

$$W(N/p_0) = W(N/pp_0)\{(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}), p^{-1/4}\},$$

and $\overline{\chi}\chi_N = \overline{\chi\chi_p}\chi_{N/p}$. The formula

$$[g|V(p)]|S(\chi,N,p_0) = [g|S(\chi\chi_p,N/p,p_0)]|V(p)$$

follows from this, after a simple computation.

Let p be any prime. We shall need the operator

$$K(p) = 1 - T(p,Np)V(p),$$

where T(p,Np) is the Hecke operator T(p) relative to the level Np (see above).

LEMMA 4. If $f = \sum_{n=0}^{\infty} a(n)q^n$ belongs to $M_0(N,\kappa/2,\chi)$, then $f|K(p)$ belongs to $M_0(Np^2,\kappa/2,\chi)$ and is equal to $\sum_{(n,p)=1} a(n)q^n$. Further, if $p' \nmid Np$, then $T(p'^2)$ and $K(p)$ commute.

PROOF. This is immediate.

REMARK. All the above operators take cusp forms to cusp forms.

§4. NEWFORMS

4.1. Definitions.

Let $f \in M_0(N,\kappa/2,\chi)$ be an eigenform of all but finitely many $T(p^2)$. We say that f is an oldform (compare [1], [5]) if there exists a prime p dividing N/4 such that :

either χ is definable (mod N/p) and f belongs to $M_0(N/p,\kappa/2,\chi)$,

or $\chi\chi_p$ is definable (mod N/p) and $f = g|V(p)$, with $g \in M_0(N/p,\kappa/2,\chi\chi_p)$.

We denote by $M_0^{old}(N,\kappa/2,\chi)$ the subspace of $M_0(N,\kappa/2,\chi)$ spanned by old forms. If $f \in M_0(N,\kappa/2,\chi)$ is an eigenform of all but finitely many $T(p^2)$, and f does not belong to $M_0^{old}(N,\kappa/2,\chi)$, we say that f is a newform of level N.

LEMMA 5. The symmetry operator $W(N) : M_0(N,\kappa/2,\chi) \to M_0(N,\kappa/2,\bar{\chi}\chi_N)$ and the conjugation operator $H : M_0(N,\kappa/2,\chi) \to M_0(N,\kappa/2,\bar{\chi})$ take oldforms to oldforms and newforms to newforms.

PROOF. By Lemma 2, W(N) and H take eigenforms to eigenforms. If f is an oldform of the first type above, i.e. $f \in M_0(N/p,\kappa/2,\chi)$, then

$$f|W(N) = p^{\kappa/4}[f|W(N/p)]|V(p)$$

is an oldform of the second type. Conversely, if $f = g|V(p)$ is an oldform of the second type, then $f|W(N) = p^{-\kappa/4}\, g|W(N/p)$ is an oldform of the first type. Hence W(N) takes oldforms to oldforms; the same is obviously true for the conjugation operator H. That W(N) and H take newforms to newforms follows from this, and from the fact that their square is the identity.

LEMMA 6. Let $h \in M_0^{old}(N,\kappa/2,\chi)$ be a non-zero eigenform of all but finitely many $T(p^2)$. Then there is a divisor N_1 of N, with $N_1 < N$, a character ψ definable (mod N_1) and a newform g in $M_0(N_1,\kappa/2,\psi)$ such that h and g have the same eigenvalues for all but finitely many $T(p^2)$.

PROOF. We use induction on N. By construction, $M_0^{old}(N,\kappa/2,\chi)$ has a basis (f_i) consisting of forms of the type g, or $g|V(p)$, where g is an eigenform of all but finitely many $T(p^2)$, and is of lower level. Hence h is a linear combination with non-zero coefficients of some of the f_i's, and each f_i occurring in h has the same eigenvalue for $T(p^2)$ as h does. The Lemma then follows from the induction assumption.

LEMMA 7. *Let* p *be a prime, and let* $f = \sum_{n=0}^{\infty} a(n)q^n$ *be a non-zero element of* $M_0(N,\kappa/2,\chi)$ *such that* $a(n) = 0$ *for all* n *not divisible by* p. *Then* p *divides* $N/4, \chi\chi_p$ *is definable* (mod N/p) *and* $f = g|V(p)$ *with* $g \in M_0(N/p,\kappa/2,\chi\chi_p)$.

PROOF. Put

$$g(z) = f(z/p) = \sum_{n=0}^{\infty} a(pn)q^n = p^{\kappa/4}f|\{(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}), p^{1/4}\}.$$

Let $N' = N/p$ if $4p|N$ and $N' = N$ otherwise. Let $\Gamma_0(N',p)$ be the subgroup of $\Gamma_0(N')$ consisting of matrices $(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix})$ with $b \equiv 0 \pmod p$; if $A = (\begin{smallmatrix}a & b\\ c & d\end{smallmatrix})$ is such a matrix, put $A_1 = (\begin{smallmatrix}a & b/p\\ pc & d\end{smallmatrix})$. We have $A_1 \in \Gamma_0(N)$, and

$$\{(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}), p^{1/4}\} \quad A^* = \{1,\chi_p(d)\} \quad A_1^*\{(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}), p^{1/4}\},$$

hence

$$g|A^* = \chi_p(d)\chi(d)g.$$

Since d is relatively prime to both p and N, this can be rewritten as

$$(*) \qquad g|A^* = (\chi\chi_p)(d)g.$$

By hypothesis, g has a q-expansion in integral powers of q, hence (*) holds for $A = (\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix})$. Since $\Gamma_0(N')$ is generated by $\Gamma_0(N',p)$ and $(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix})$, this shows that (*) holds for any $A \in \Gamma_0(N')$. Since g is non-zero, this implies that $\chi\chi_p$ is definable (mod N'); this is easily seen to be possible only if p divides N/4, in which case $N' = N/p$ and (*) shows that g belongs to $M_0(N/p,\kappa/2,\chi\chi_p)$.

REMARKS. (1) If f is a cusp form, it is clear that g is also a cusp form.

(2) The above Lemma gives a characterization of oldforms of the second type.

THEOREM 1. *Let* m *be an integer* ≥ 1, *and let* $f = \sum_{n=0}^{\infty} a(n)q^n$ *be an element of* $M_0(N,\kappa/2,\chi)$ *such that* $a(n) = 0$ *for all* n *with* $(n,m) = 1$. *Then* f *can be written as*

$$f = \sum_p f_p | V(p), \qquad \text{with } f_p \in M_0(N/p,\kappa/2,\chi\chi_p),$$

where p *runs through the primes such that* $p|m$, $4p|N$, *and* $\chi\chi_p$ *is definable* (mod N/p).
If f *is a cusp form, the* f_p *can be chosen to be cusp forms. If* f *is an eigenform of all but finitely many* $T(p'^2)$, *then the* f_p *may be further chosen so that they, too, are eigenforms of all but finitely many* $T(p'^2)$, *and have the same eigenvalues as* f.
(Compare with the integral weight case, in [1] or [5].)

PROOF. Clearly, we may assume that m is square-free. We proceed by induction on the number r of prime factors of m. If $r = 0$, then $m = 1$ and all $a(n)$ are zero by hypothesis; there is nothing to prove. Now suppose $r \geq 1$ and that Theorem 1 has been proved for all m's which are products of strictly less than r primes (and all levels). Let p_0 be a prime divisor of m. Put $m = p_0 m_0$, and

$$h = \sum_{(n,m_0)=1} a(n)q^n = f | \prod_{p|m_0} K(p), \qquad \text{cf. §3.}$$

If $h = 0$, we may replace m by m_0, and Theorem 1 follows from the induction hypothesis. Hence, we may assume that $h \neq 0$. By Lemma 4, we have $h \in M_0(Nm_0^2,\kappa/2,\chi)$. If $(n,m_0) = 1$ and $a(n) \neq 0$, by hypothesis we have $(n,p_0) \neq 1$ and Lemma 7 shows that $4p_0|Nm_0^2$, $\chi\chi_{p_0}$ is definable (mod Nm_0^2/p_0) and $h = g_{p_0} | V(p_0)$ with $g_{p_0} \in M_0(Nm_0^2/p_0,\kappa/2,\chi\chi_{p_0})$. This implies that $4p_0|N$ and that $\chi\chi_{p_0}$ is definable (mod N/p_0).
Moreover, we have

$$f - h = f - g_{p_0} | V(p_0) = \sum_{n=0}^{\infty} b(n)q^n,$$

with $b(n) = 0$ if $(n,m_0) = 1$. By the induction hypothesis (applied to m_0 and to the level Nm_0^2), this shows that $f - g_{p_0}|V(p_0)$ can be written as

$$f - g_{p_0}|V(p_0) = \sum_p g_p|V(p),$$

where p runs through the primes such that $p|m_0$ and $\chi\chi_p$ is definable (mod Nm_0^2/p), with $g_p \in M_0(Nm_0^2/p,\kappa/2,\chi\chi_p)$. We now apply the operator $S(\chi) = S(\chi,N,p_0)$ of §3 to f. Using Lemma 3, the above formula gives

$$f|S(\chi) - g_{p_0} = \sum_p [g_p|S(\chi\chi_p,Nm_0^2/p,p_0)]|V(p).$$

Let now f_{p_0} be $f|S(\chi)$. We have $f_{p_0} \in M_0(N/p_0,\kappa/2,\chi\chi_{p_0})$. Moreover the above formula shows that the n^{th} coefficient of $f_0 = f - f_{p_0}|V(p_0)$ is 0 if $(n,m_0) = 1$; this allows us to apply the induction hypothesis to f_0 and m_0, and we get the required decomposition of f. As for the other assertions of Theorem 1, they follow from the inductive construction of the f_p's and from Lemma 3.

COROLLARY. *If the form f of Theorem 1 is an eigenform of all but finitely many* $T(p'^2)$, *then* f *belongs to* $M_0^{old}(N,\kappa/2,\chi)$.

§5. THE "BOUNDED DENOMINATORS" ARGUMENT

5.1. Coefficients of modular forms of half integral weight.

LEMMA 8. (a) *There is a basis of* $M_0(N,\kappa/2,\chi)$ *consisting of forms whose coefficients belong to a number field.*

(b) *If* $f = \sum a(n)q^n$ *belongs to* $M_0(N,\kappa/2,\chi)$ *and the* $a(n)$ *are algebraic numbers, then the* $a(n)$ *have bounded denominators* (i.e. there exists a non-zero integer D such that D.a(n) is an algebraic integer for all n).

PROOF. The analogous statement for modular forms of integral weight is well known (cf. for instance [7], Th. 3.5.2 or [3], Prop. 2.7). We shall reduce to that case by the familiar device of multiplying by a fixed form f_0. We choose for f_0 the form

$$\theta^{3\kappa} = (1 + 2q + 2q^4 + \dots)^{3\kappa} = 1 + 6\kappa q + \dots \quad .$$

The map $\Phi : f \mapsto \theta^{3\kappa} f$ sends $M_0(N,\kappa/2,\chi)$ into the space $M_0(N,2\kappa,\chi)$ of modular forms of type $(2\kappa,\chi)$ on $\Gamma_0(N)$. By the results quoted above, it follows that, if the coefficients of f are algebraic, those of $\theta^{3\kappa} f$ have bounded denominators; dividing by $\theta^{3\kappa}$ does not increase denominators, hence b) follows. As for a), one has to check that the image $\mathrm{Im}(\Phi)$ of Φ can be defined by linear equations with algebraic coefficients. This is so because θ does not vanish on the upper half-plane (as its expansion shows), nor at any cusp except those congruent mod $\Gamma_0(4)$ to $1/2$; hence a modular form F in $M_0(N,2\kappa,\chi)$ belongs to $\mathrm{Im}(\Phi)$ if and only if it vanishes (with prescribed multiplicities) at these cusps, i.e. if some of the coefficients of its expansions at these cusps are zero; since it is known that these coefficients are algebraic linear combinations of the coefficients of F at the cusp ∞, the result follows.

REMARKS. (1) A similar argument shows that $M_1(N,\kappa/2)$ has a basis made up of forms with coefficients in $\mathbf{Z}$, and that the action of $(\mathbf{Z}/N\mathbf{Z})^*$ is $\mathbf{Z}$-linear with respect to that basis. This implies that, if $f = \sum a(n)q^n$ belongs to $M_0(N,\kappa/2,\chi)$ and σ is any automorphism of $\mathbf{C}$, the series

$$f^\sigma = \sum \sigma(a(n))q^n$$

belongs to $M_0(N,\kappa/2,\chi^\sigma)$, just as in the integral weight case ([3], 2.7.4). We will not need these facts.

(2) On noncongruence subgroups, part (a) of Lemma 8 remains true, but part (b) does not, as was first noticed by Atkin and Swinnerton-Dyer [2]. A simple example is

$$f(z) = \theta(z)^{1/2}\,\theta(3z)^{1/2} = 1 + q - \frac{1}{2}q^2 + \frac{3}{2}q^3 + \frac{11}{8}q^4 - \dots ,$$

which is a modular form of weight 1/2 on a subgroup of index 2 of $\Gamma_1(12)$, and whose coefficients have unbounded powers of 2 in denominator (if n is a power of 2, the 2-adic valuation of the n^{th} coefficient of f is $1-n$). Similar examples exist in higher weights, integral as well as half integral : take for instance

$$f_m(z) = \theta(z)^{1/2}\,\theta(3z)^{m/2}, \qquad \text{with } m \text{ odd} \geqslant 1,$$

which is of weight $(m+1)/4$.

5.2. Eigenvectors of the Hecke operators for weight 1/2.

From now on, we restrict ourselves to weight 1/2, i.e. we take $\kappa = 1$.

LEMMA 9. *Let* $f = \sum_{n=0}^{\infty} a(n)q^n$ *be a non-zero element of* $M_0(N,1/2,\chi)$ *and let* p *be a prime, with* $p \nmid N$. *Assume that* $f|T(p^2) = c_p f$, *with* $c_p \in \mathbf{C}$. *Let* $m \geqslant 1$ *be such that* $p^2 \nmid m$. *Then* :

(a) *we have* $a(mp^{2n}) = a(m)\chi(p)^n(\frac{m}{p})^n$ *for every* $n \geqslant 0$.

(b) *If* $a(m) \neq 0$, *then* $p \nmid m$ *and* $c_p = \chi(p)(\frac{m}{p})(1+p^{-1})$.

PROOF. Since $T(p^2)$ maps forms with algebraic coefficients into themselves (cf. Lemma 1), it follows from Lemma 8 that the eigenvalue c_p is algebraic, and that the corresponding eigenspace is generated by forms with algebraic coefficients. Hence we may assume that the coefficients $a(n)$ of f are algebraic numbers. Consider the power series

$$A(T) = \sum_{n=0}^{\infty} a(mp^{2n})T^n,$$

where T is an indeterminate. By [8], p. 452, we have

$$A(T) = a(m)\ \frac{1 - \alpha T}{(1-\beta T)(1-\gamma T)} ,$$

with $\alpha = \chi(p)p^{-1}(\frac{m}{p})$ and $\beta + \gamma = c_p$, $\beta\gamma = \chi(p^2)p^{-1}$ (note the negative exponent of p, which comes from the fact that $\kappa = 1$). This already shows that $a(m) = 0$ implies $A(T) = 0$, i.e. $a(mp^{2n}) = 0$ for all $n \geqslant 0$. Hence we may assume that $a(m) \neq 0$, in which case A(T) is a non-zero rational function of T. If we view A(T) as a p-adic function of T (over a suitable finite extension of the p-adic field $\mathbf{Q}_p$), Lemma 8 (b) shows that A(T) converges in the p-adic unit disk U defined by $|T|_p < 1$; hence A(T) cannot have a pole in U. However, since $\beta\gamma = \chi(p^2)p^{-1}$, either β^{-1} or γ^{-1} belongs to U; assume it is β^{-1}. In order that A(T) be holomorphic at β^{-1}, it is necessary that the factors $1 - \beta T$ and $1 - \alpha T$ cancel each other. We then have $\alpha = \beta$ and

$$A(T) = a(m)/(1-\gamma T), \quad \text{so that} \quad a(mp^{2n}) = \gamma^n a(m).$$

Since $\beta\gamma \neq 0$ we have $\alpha \neq 0$, hence $p \nmid m$. Moreover,

$$\gamma = \beta\gamma/\alpha = \chi(p^2)p^{-1}/\chi(p)p^{-1}(\tfrac{m}{p}) = \chi(p)(\tfrac{m}{p}).$$

This shows that $a(mp^{2n}) = \gamma^n a(m) = a(m)\chi(p)^n(\frac{m}{p})^n$, which proves (a). As for the last assertion of (b), it follows from $c_p = \beta + \gamma = \alpha + \gamma$.

THEOREM 2. Let $f = \sum_{n=0}^{\infty} a(n)q^n$ be a non-zero element of $M_0(N,1/2,\chi)$ and let N' be a multiple of N. Assume that, for all $p \nmid N'$, we have $f|T(p^2) = c_p f$, with $c_p \in \mathbf{C}$. Then there exists a unique square-free integer $t \geqslant 1$ such that $a(n) = 0$ if n/t is not a square. Moreover :

(i) $t|N'$.

(ii) $c_p = \chi(p)(\frac{t}{p})(1+p^{-1})$ if $p \nmid N'$.

(iii) $a(nu^2) = a(n)\chi(u)(\frac{t}{u})$ if $(u,N') = 1$, $u \geqslant 1$.

PROOF. Let m and m' be two integers $\geqslant 1$ such that $a(m) \neq 0$ and $a(m') \neq 0$. We show first that m'/m is a square. Let P be the set of primes p with $p \nmid N'mm'$. If $p \in P$, Lemma 9 shows that

$$\chi(p)(\frac{m}{p})(1+p^{-1}) = c_p = \chi(p)(\frac{m'}{p})(1+p^{-1}),$$

hence $$(\frac{m}{p}) = (\frac{m'}{p}) \text{ for all } p \in P.$$

It is well known that this implies that m'/m is a square. We may write m and m' as $m = tv^2$, $m' = tv'^2$, with $v,v' \geq 1$ and t square-free ≥ 1. This proves the first part of the Theorem, i.e. the existence of t. Write now v as $p^n u$, with $p \nmid N'$ and $(p,u) = 1$, so that $m = tp^{2n}u^2$. By Lemma 9, applied to tu^2, we have $a(m) = \chi(p)^n(\frac{tu^2}{p})^n a(tu^2)$ hence $a(tu^2) \neq 0$ and Lemma 9 (b) shows that $p \nmid tu^2$, hence $p \nmid t$, and $c_p = \chi(p)(\frac{t}{p})(1+p^{-1})$. Hence every prime factor of t divides N'; since t is square-free, this shows that $t|N'$, and (i) and (ii) are proved. As for (iii), it is enough to check it when $u = p$ with $p \nmid N'$; in that case, one writes n as $m_0 p^{2a}$, with $p^2 \nmid m_0$, and applies Lemma 9 (a).

<u>COROLLARY</u>. <u>If</u> $a(1) \neq 0$, <u>then</u> $t = 1$ <u>and</u> $c_p = \chi(p)(1+p^{-1})$ <u>for</u> $p \nmid N'$. (Note that, in this case, the c_p's determine the character χ.)

Let now $\sum_{n=1}^{\infty} a(n)n^{-s}$ be the <u>Dirichlet series</u> associated with f. Let ψ be the character $\chi\chi_t$, so that $\psi(p) = \chi(p)(\frac{t}{p})$ if $p \nmid N'$. Assertions (i) and (iii) of Theorem 2 can be reformulated as :

<u>THEOREM</u> 2'. <u>Under the assumptions of Theorem</u> 2, <u>we have</u>

$$\sum_{n=1}^{\infty} a(n)n^{-s} = t^{-s}\Big(\sum_{n|N'^{\infty}} a(tn^2)n^{-2s}\Big) \prod_{p\nmid N'} (1 - \psi(p)p^{-2s})^{-1}.$$

(The notation $A|B^{\infty}$ means that A divides some power of B, i.e. that every prime factor of A is a factor of B.)

§6. PROOF OF THEOREM A

6.1. Structure of newforms of weight 1/2.

Let $f = \sum_{n=0}^{\infty} a(n)q^n$ be a newform of level N (cf. §4) belonging to $M_0(N,1/2,\chi)$. By Theorem 2, there is a unique square-free integer $t \geq 1$ such that $a(n) = 0$ if n/t is not a square.

LEMMA 10. *We have* $t = 1$ *and* $a(1) \neq 0$.

PROOF. The product expansion of $\sum_{n=1}^{\infty} a(n)n^{-s}$ given in Theorem 2' shows that, if $a(1) = 0$, we have $a(n) = 0$ for every n such that $(n,N') = 1$; the Corollary to Theorem 1 then shows that f belongs to $M_0^{old}(N,1/2,\chi)$, contrary to the assumption that f is a newform. Hence $a(1) \neq 0$, and this implies $t = 1$, cf. the Corollary to Theorem 2.

This Lemma allows us to divide f by $a(1)$; hence we may assume that f is normalized, i.e. that $a(1) = 1$.

LEMMA 11. *Let* $g \in M_0(N,1/2,\chi)$ *be an eigenform of all but finitely many* $T(p^2)$, *with the same eigenvalues as* f. *Then* g *is a scalar multiple of* f.

PROOF. Let c be the coefficient of q in the q-expansion of g, and set

$$h = g - cf,$$

so that the coefficient of q in the q-expansion of h is 0. Suppose $h \neq 0$. By Lemma 10, h is not a newform; since it is an eigenform of all but finitely many $T(p^2)$, it belongs to $M_0^{old}(N,1/2,\chi)$. Hence, by Lemma 6, there are $N_1 | N$, with $N_1 < N$, a character ψ definable (mod N_1) and a normalized newform g_1 in $M_0(N_1,1/2,\psi)$ with the same eigenvalues c_p as f and h, for all but finitely many $T(p^2)$. Since the c_p's determine the character (cf. the Corollary to Theorem 2) we have $\chi = \psi$ and so g_1 belongs to $M_0^{old}(N,1/2,\chi)$. On the other hand, the coefficient of q in the q-expansion of $f - g_1$ is 0; the same argument as above then shows that $f - g_1$ belongs to $M_0^{old}(N,1/2,\chi)$. Hence $f = g_1 + (f-g_1)$ belongs to $M_0^{old}(N,1/2,\chi)$. This contradicts the assumption that f is a newform. Hence $h = 0$, i.e. $g = cf$.

LEMMA 12. *The form* f *is an eigenform of every* $T(p^2)$. *If we put* $f|T(p^2) = c_p f$, *we have*

$$(*) \qquad \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p|N} (1 - c_p p^{-2s})^{-1} \prod_{p \nmid N} (1 - \chi(p)p^{-2s})^{-1}.$$

Further, if $4p|N$, *then* $c_p = 0$.

PROOF. If we apply Lemma 11 to $g = f|T(p^2)$, we see that g is a multiple of f. Hence f is an eigenform of every $T(p^2)$, and the Euler product $(*)$ follows from this and Theorem 2' (applied with $N' = N$, $t = 1$, $\psi = \chi$).

If $4p|N$, then Lemma 1 shows that

$$f|T(p) = \sum_{n=0}^{\infty} a(np)q^n = \sum_{m=0}^{\infty} a(m^2p^2)q^{pm^2} = c_p f|V(p)$$

belongs to $M_0(N,1/2,\chi\chi_p)$. If $c_p \neq 0$, Lemma 7 applied to $f|T(p)$ and to the character $\chi\chi_p$ shows that χ is definable (mod N/p) and that $f|T(p) = g|V(p)$ with $g \in M_0(N/p,1/2,\chi)$. We have $c_p f|V(p) = g|V(p)$, hence $c_p f = g$; this shows that f belongs to $M_0(N/p,1/2,\chi)$ and contradicts the assumption that f is a newform. Hence $c_p = 0$.

LEMMA 13. *The level* N *of the newform* f *is a square, and* $f|W(N)$ *is a multiple of* $f|H$.

(Recall that $W(N)$ and H are respectively the *symmetry* and *conjugation* operators, cf. §3.)

PROOF. If $p \nmid N$, we have $f|T(p^2) = c_p f$ with $c_p = (1+p^{-1})\chi(p)$, and, by Lemma 2,

$$[f|W(N)]|T(p^2) = \overline{\chi}(p)^2 c_p\, f|W(N) = \overline{c}_p\, f|W(N),$$

$$[f|H]|T(p^2) = (c_p f)|H = \overline{c}_p\, f|H \qquad \text{since } H \text{ is anti-linear.}$$

But $f|W(N)$ and $f|H$ are newforms of level N and characters $\overline{\chi}\chi_N$ and $\overline{\chi}$ respectively, cf. Lemma 5. Since they have the same eigenvalues $\overline{c}_p$ for all $T(p^2)$, $p \nmid N$, and these eigenvalues determine the character (cf. the Corollary to Theorem 2), we have $\overline{\chi}\chi_N = \overline{\chi}$ and N is a square. The fact that $f|W(N)$ and $f|H$ are proportional follows from this and from Lemma 11.

THEOREM 3. If f is a normalized newform in $M_0(N,1/2,\chi)$, and r is the conductor of χ, then $N = 4r^2$ and $f = \frac{1}{2}\theta_\chi$.

PROOF. We write $f = \sum_{n=0}^{\infty} a(n)q^n$ as above, and put

$$F(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p | N} (1 - c_p p^{-2s})^{-1} \prod_{p \nmid N} (1 - \chi(p)p^{-2s})^{-1},$$

$$\overline{F}(s) = \sum_{n=1}^{\infty} \overline{a(n)}n^{-s}.$$

The Dirichlet series F and $\overline{F}$ converge for $\mathrm{Re}(s)$ large enough. Using Mellin transforms, and Lemma 13, we obtain by a standard argument the analytic continuation of F and $\overline{F}$ as entire functions of s (except for a simple pole at $s = 1/2$ if $a(0) \neq 0$), and the functional equation

$$(2\pi)^{-s}\,\Gamma(s)F(s) = C_1\left(\frac{2\pi}{N}\right)^{-(1/2-s)} \Gamma(\tfrac{1}{2}-s)\overline{F}(\tfrac{1}{2}-s),$$

where C_1 (and C_2, C_3, C_4 below) is a non-zero constant.

On the other hand, we know that the functions

$$G(s) = L(2s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-2s} = \prod_{p \nmid r} (1 - \chi(p)p^{-2s})^{-1}$$

$$\overline{G}(s) = L(2s,\overline{\chi})$$

satisfy the functional equation

$$(2\pi)^{-s}\,\Gamma(s)G(s) = C_2\left(\frac{2\pi}{4r^2}\right)^{-(1/2-s)}\,\Gamma(\tfrac{1}{2}-s)\overline{G}(\tfrac{1}{2}-s).$$

Dividing these equations, we find

$$(*)\qquad \prod_{p|m}\left(\frac{1-c_p p^{-2s}}{1-\chi(p)p^{-2s}}\right) = C_3\left(\frac{N}{4r^2}\right)^{-(1/2-s)}\prod_{p|m}\left(\frac{1-\overline{c}_p p^{2s-1}}{1-\overline{\chi}(p)p^{2s-1}}\right),$$

where m is the product of the prime divisors p of N such that $c_p \neq \chi(p)$.

If, for some $p|m$, we have $\chi(p) \neq 0$, then the left side of (*) has an infinity of poles on the line $\mathrm{Re}(s) = 0$, only finitely many of which can appear on the right side. This shows that $p|m$ implies $\chi(p) = 0$, (i.e. $p|r$) and $c_p \neq 0$ since $c_p \neq \chi(p)$. We may now rewrite (*) as :

$$\prod_{p|m}(1-c_p p^{-2s}) = C_4\left(\frac{Nm^2}{4r^2}\right)^{s}\prod_{p|m}(1-c'_p p^{-2s}),$$

where $c'_p = p/\overline{c}_p$. The same argument as above (using zeros instead of poles) shows that, for every $p|m$, we have $c_p = c'_p$, i.e. $|c_p|^2 = p$; the above equation then gives $C_4 = 1$ and $Nm^2 = 4r^2$. But, by Lemma 12, we have $c_p = 0$ when $4p|N$. This shows that $m = 1$ or 2, and that $m = 2$ can occur only when $8 \nmid N$ and $\chi(2) = 0$; in the last case, r is divisible by 4 and the equation $Nm^2 = 4r^2$ shows that N is divisible by 16, which contradicts $8 \nmid N$. Hence only the case $m = 1$ is possible, and we have $N = 4r^2$, $F(s) = G(s)$. This shows that, for every $n \geq 1$, the coefficients of q^n in f and in $\frac{1}{2}\,\theta_\chi$ are the same. Hence $f - \frac{1}{2}\,\theta_\chi$ is a constant, and, since it is a modular form of weight 1/2, it is 0. This concludes the proof.

6.2. Alternative arguments.

(1) To show that the constant term of f and $\frac{1}{2}\,\theta_\chi$ agree, we could have used the well-known fact that they are equal to $-F(0)$ and $-G(0)$

respectively.

(2) Another way to rule out $|c_p|^2 = p$ is to prove *a priori* that $|c_p| \leqslant 1$. This may be done as follows. Choose $D \geqslant 1$ such that p is inert in $\mathbf{Q}(\sqrt{-D})$, and consider the modular form of weight 1 :

$$g(z) = f(z)\theta(Dz) = \left(\sum_{u=0}^{\infty} a(u)q^u\right)\left(\sum_{-\infty}^{\infty} q^{Dv^2}\right) = \sum_{u,v} a(u^2)q^{u^2+Dv^2}.$$

The p^{2n}-th coefficient of g is $a(p^{2n}) = (c_p)^n$. By [3], Cor. 9.2, this coefficient is $O(p^{2n\delta})$ for every $\delta > 0$. This obviously implies $|c_p| \leqslant 1$.

Theorem 3 has a converse :

THEOREM 4. *If χ is an even character of conductor r, then $\frac{1}{2}\,\theta_\chi$ is a normalized newform in $M_0(4r^2,1/2,\chi)$.*

(Recall that all characters are assumed to be *primitive*.)

PROOF. Let $N = 4r^2$. We know that θ_χ belongs to $M_0(N,1/2,\chi)$ and it is easily checked that it is an eigenform of all $T(p^2)$, with eigenvalue

$$c_p = (1+p^{-1})\chi(p) \quad \text{if } p \nmid N \qquad \text{(cf. Lemma 1).}$$

Thus, if θ_χ is not a newform, Lemma 6 shows that there are a divisor N_1 of N, with $N_1 < N$, a character ψ definable (mod N_1) and a newform f in $M_0(N_1,1/2,\psi)$ such that f and θ_χ have the same eigenvalues for all but finitely many $T(p^2)$. We thus have

$$(1+p^{-1})\psi(p) = c_p = (1+p^{-1})\chi(p) \qquad \text{for almost all } p,$$

and this implies $\psi = \chi$, hence $N_1 = 4r^2$ by Theorem 3. This contradicts $N_1 < N$. Hence θ_χ is a newform, and $\frac{1}{2}\,\theta_\chi$ is obviously normalized.

6.3. *Proof of Theorem A.*

Let χ be an even character definable (mod N). With the notations of

§2, we want to prove that the theta series $\theta_{\psi,t} = \theta_\psi|V(t)$, with $(\psi,t) \in \Omega(N,\chi)$, make a basis of $M_0(N,1/2,\chi)$. The proof splits into two parts :

a) Linear independence of the $\theta_{\psi,t}$.

Since t and χ determine ψ, every t occurs as the second entry of at most one (ψ,t) in $\Omega(N,\chi)$. Suppose then that we have

$$\lambda_1 \theta_{\psi_1,t_1} + \ldots + \lambda_m \theta_{\psi_m,t_m} = 0,$$

with $t_1 < t_2 < \ldots < t_m$ and $\lambda_i \neq 0$ for all i. The coefficient of q^{t_1} in θ_{ψ_1,t_1} is equal to 2; in θ_{ψ_j,t_j}, $j \geqslant 2$, it is equal to 0. This shows that $2\lambda_1 = 0$, hence $\lambda_1 = 0$. This contradiction proves the linear independence of the $\theta_{\psi,t}$.

b) The $\theta_{\psi,t}$ with $(\psi,t) \in \Omega(N,\chi)$, generate $M_0(N,1/2,\chi)$.

We need :

LEMMA 14. There is a basis of $M_0(N,1/2,\chi)$ consisting of eigenforms for all the $T(p^2)$, $p \nmid N$.

PROOF. Put on $M_0(N,1/2,\chi)$ the Petersson scalar product $<f,g>$, cf. §1. A standard computation shows that, if $p \nmid N$, we have

$$< f|T(p^2),g > = \chi(p^2) < f,g|T(p^2) >,$$

hence $\overline{\chi}(p)T(p^2)$ is hermitian. The Lemma follows from this, and from the fact that the $T(p^2)$ commute.

We can now prove assertion b), using induction on N. By Lemma 14, it is enough to show that any eigenform f of all $T(p^2)$, $p \nmid N$, is a linear combination of the $\theta_{\psi,t}$ with $(\psi,t) \in \Omega(N,\chi)$. If f is a newform, this follows from Theorem 3. If not, we may assume f is an oldform of one of the two types of §4 :

either χ is definable (mod N/p) and f belongs to $M_0(N/p,1/2,\chi)$,

<u>or</u> $\chi\chi_p$ is definable (mod N/p) and $f = g|V(p)$ with $g \in M_0(N,N/p,1/2,\chi\chi_p)$. In the first case, the induction assumption shows that f is a linear combination of the $\theta_{\psi,t}$ with $(\psi,t) \in \Omega(N/p,\chi)$ and <u>a fortiori</u> with $(\psi,t) \in \Omega(N,\chi)$. In the second case, g is a linear combination of the $\theta_{\psi,t}$, with $(\psi,t) \in \Omega(N/p,\chi\chi_p)$, and hence f is a linear combination of the $\theta_{\psi,tp}$, with $(\psi,tp) \in \Omega(N,\chi)$.

<u>REMARK</u>. It is possible to prove Lemma 14 without using Petersson products. Indeed, assume that some $T(p^2)$, $p \nmid N$, is not diagonalizable. Then there exists an eigenvalue c_p of $T(p^2)$ and a non-zero element g of $M_0(N,1/2,\chi)$ such that

$$g|U \neq 0 \quad \text{and} \quad g|U^2 = 0, \quad \text{where } U = T(p^2) - c_p.$$

Using Lemma 8, one may further assume that the coefficients of g are algebraic numbers. A computation similar to that of Lemma 9 then shows that these coefficients have <u>unbounded powers of</u> p in denominators, and this contradicts Lemma 8. Hence, each $T(p^2)$ is diagonalizable. Since these operators commute, Lemma 14 follows.

§7. <u>PROOF OF THEOREM B</u>

7.1. <u>Twists</u>.

Let $f = \sum_{n=0}^{\infty} a(n)q^n$ be a modular form of weight $k = \kappa/2$ on some $\Gamma_1(N)$. Let M be an integer $\geqslant 1$, and ε a function on $\mathbf{Z}$ with period M (i.e. a function on $\mathbf{Z}/M\mathbf{Z}$). We put

$$f * \varepsilon = \sum_{n=0}^{\infty} a(n)\varepsilon(n)q^n.$$

Let $\hat{\varepsilon}$ be the Fourier transform of ε on $\mathbf{Z}/M\mathbf{Z}$, defined by :

$$\hat{\varepsilon}(m) = \frac{1}{M} \sum_{n \in \mathbf{Z}/M\mathbf{Z}} \varepsilon(n) \exp(-2\pi i n m/M).$$

We then have

$$\varepsilon(n) = \sum_{m \in \mathbf{Z}/M\mathbf{Z}} \hat{\varepsilon}(m) \exp(2\pi i n m/M),$$

hence

$$(f * \varepsilon)(z) = \sum_{m \in \mathbf{Z}/M\mathbf{Z}} \hat{\varepsilon}(m) f(z+\tfrac{m}{M}).$$

From this, one deduces easily that $f * \varepsilon$ is a modular form of weight k on $\Gamma_1(NM^2)$.

7.2. Characterization of cusp forms.

We keep the above notation, and we put

$$\phi_f(s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

THEOREM 5. The following properties are equivalent :

i) f vanishes at all cusps m/M, with $m \in \mathbf{Z}$;

ii) for every function ε on $\mathbf{Z}$, with period M, the function $\phi_{f * \varepsilon}(s) = \sum_{n=1}^{\infty} a(n)\varepsilon(n) n^{-s}$ is holomorphic at $s = k$.

(This is also true when k is an integer, instead of a half integer; the proof is the same.)

PROOF. Consider first the case where $M = 1$. Assertion i) then means that f vanishes at the cusp 0, and assertion ii) that $\phi_f(s)$ is holomorphic at $s = k$. If we put

$$g = f|W(N) = \sum_{n=0}^{\infty} b(n) q^n,$$

then i) is equivalent to :

i') g vanishes at the cusp ∞, i.e. $b(0)$ is 0,

while the functional equation relating $\phi_f(s)$ and $\phi_g(k-s)$ shows that ii) is equivalent to :

ii') $(2\pi)^{-s}\Gamma(s)\phi_g(s)$ is holomorphic at $s = 0$, i.e. $\phi_g(0) = 0$.

The equivalence of i') and ii') then follows from the known relation

$$b(0) = -\phi_g(0).$$

Consider now the general case. By applying the above to $f * \varepsilon$ (with N replaced by NM^2), we see that ii) is equivalent to :

iii) for every function ε on $\mathbf{Z}$, with period M, the modular form $f * \varepsilon$ vanishes at the cusp 0.

Using the above formulae, this is in turn equivalent to :

iv) for every $m \in \mathbf{Z}/M\mathbf{Z}$, the modular form $f(z+\frac{m}{M})$ vanishes at the cusp 0,

and it is clear that iv) is equivalent to i).

COROLLARY. The following properties are equivalent :

a) f is a cusp form;

b) for every periodic function ε on $\mathbf{Z}$, the function $\phi_{f * \varepsilon}(s)$ is holomorphic at $s = k$.

Indeed, Theorem 5 shows that b) is equivalent to the fact that f vanishes at all cusps $\neq \infty$; since ∞ is $\Gamma_1(N)$-equivalent to $1/N$, this means that f is a cusp form.

REMARK. When f belongs to some $M_0(N,\kappa/2,\chi)$, it is enough to check property b) for functions ε with period N. Indeed, by Theorem 5, this implies the vanishing of f at all cusps m/N, with $m \in \mathbf{Z}$, and it is known that every cusp is $\Gamma_0(N)$-equivalent to one of these.

We now go back to the case $\kappa = 1$, $k = 1/2$:

LEMMA 15. Let ψ be an even character which is not totally even (cf. §2). Then θ_ψ is a cusp form.

PROOF. Let ε be a periodic function on $\mathbf{Z}$. By the Corollary to Theorem

5, it is enough to prove that the Dirichlet series

$$F_\varepsilon(s) = 2\sum_{n=1}^{\infty} \varepsilon(n^2)\psi(n)n^{-2s}$$

is holomorphic at $s = 1/2$. Let $M \geqslant 1$ be a period of ε, which we may assume to be a multiple of the conductor $r(\psi)$ of ψ. We have

$$F_\varepsilon(s) = 2\sum_{m \in \mathbf{Z}/M\mathbf{Z}} \varepsilon(m^2)\psi(m)F_{m,M}(2s),$$

where

$$F_{m,M}(s) = \sum_{\substack{n \equiv m \pmod M \\ m \geqslant 1}} n^{-s}.$$

It is an elementary fact that $F_{m,M}(s)$ has a simple pole at $s = 1$ with residue $1/M$. Hence $F_\varepsilon(s)$ has at most a simple pole at $s = 1/2$, with residue $R(\varepsilon,\psi)/M$, where

$$R(\varepsilon,\psi) = \sum_{m \in \mathbf{Z}/M\mathbf{Z}} \varepsilon(m^2)\psi(m),$$

and we have to prove that $R(\varepsilon,\psi) = 0$. By assumption, there is a prime ℓ dividing $r(\psi)$ such that the ℓ^{th} component ψ_ℓ of ψ is odd. Let us write M as $\ell^a M'$, with $(\ell,M') = 1$, so that the ring $\mathbf{Z}/M\mathbf{Z}$ splits as $\mathbf{Z}/\ell^a\mathbf{Z} \times \mathbf{Z}/M'\mathbf{Z}$. Let x_ℓ be the element of $\mathbf{Z}/M\mathbf{Z}$ whose first component (in the above decomposition) is -1, and the second component is 1. The fact that ψ_ℓ is odd means that $\psi(x_\ell) = -1$. Since x_ℓ is invertible in $\mathbf{Z}/M\mathbf{Z}$, we have

$$R(\varepsilon,\psi) = \sum_{m \in \mathbf{Z}/M\mathbf{Z}} \varepsilon((x_\ell m)^2)\psi(x_\ell m) = \sum_{m \in \mathbf{Z}/M\mathbf{Z}} \varepsilon(m^2)\psi(x_\ell m)$$

$$= -\sum_{m \in \mathbf{Z}/M\mathbf{Z}} \varepsilon(m^2)\psi(m) = -R(\varepsilon,\psi)$$

which shows that $R(\varepsilon,\psi) = 0$, as wanted.

<u>LEMMA</u> 16. <u>Let</u> ψ <u>be a totally even character</u>, <u>and</u> T <u>a finite set of integers</u> $\geqslant 1$. <u>If the modular form</u> $f = \sum_{t \in T} c_t\theta_{\psi,t}$ $(c_t \in \mathbf{C})$

is a cusp form, then all c_t *are* 0.

PROOF. Assume the c_t are not all 0, and let t_0 be the smallest $t \in T$ such that $c_t \neq 0$. Choose an integer $M \geqslant 1$ which is divisible by $2r(\psi)$ and by all $t \in T$. The first divisibility condition, together with the assumption that ψ is totally even, implies that there is a character α definable (mod M) such that $\alpha^2 = \psi$. Define now a periodic function ε on $\mathbf{Z}$ by

$$\varepsilon(n) = \begin{cases} \bar{\alpha}(n/t_0) & \text{if } t_0 | n \text{ and } n/t_0 \text{ is prime to } M \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\varepsilon(t_0 n^2) = \begin{cases} \bar{\psi}(n) & \text{if } (n,M) = 1 \\ 0 & \text{if } (n,M) \neq 1 \end{cases}$$

and

$$\varepsilon(tn^2) = 0 \text{ if } t \in T,\ t > t_0 \text{ (since } (tn^2, M) \geqslant t > t_0).$$

Using the minimality of t_0, this shows that the Dirichlet series $\Phi_{f * \varepsilon}(s)$ is equal to

$$2c_{t_0} \sum_{\substack{(n,M)=1 \\ n \geqslant 1}} \bar{\psi}(n)\psi(n)(t_0 n^2)^{-s} = 2c_{t_0} t_0^{-s} \sum_{\substack{(n,M)=1 \\ n \geqslant 1}} n^{-2s}.$$

The same argument as in the proof of Lemma 15 shows that the residue of this function at $s = 1/2$ is equal to

$$c_{t_0} t_0^{-1/2}\, \phi(M)/M = c_{t_0} t_0^{-1/2} \prod_{p|M} (1-\tfrac{1}{p}),$$

which is $\neq 0$. By Theorem 5, we thus see that f is not a cusp form.

7.3. Proof of Theorem B.

Let $N, \chi, \Omega_c(N,\chi), \Omega_e(N,\chi)$ be as defined in §2. We have three assertions to prove :

a) The $\theta_{\psi,t}$, with $(\psi,t) \in \Omega_c(N,\chi)$, are cusp forms.
Indeed, Lemma 15 shows that θ_ψ is a cusp form, and this obviously implies the same property for $\theta_{\psi,t}$.

b) No linear combination (except 0) of the $\theta_{\psi,t}$, with $(\psi,t) \in \Omega_e(N,\chi)$, is a cusp form.
Let V be the space of the linear combinations of the $\theta_{\psi,t}$, with $(\psi,t) \in \Omega_e(N,\chi)$, which are cusp forms. It is clear that V is stable under the $T(p^2)$, $p \nmid N$. Hence, if V is non-zero, it contains a common eigenform f of the $T(p^2)$, $p \nmid N$. Since the eigenvalue of $\theta_{\psi,t}$ is $(1+p^{-1})\psi(p)$, the form f has to be a linear combination of the $\theta_{\psi,t}$ for a fixed character ψ, and this contradicts Lemma 16.

c) If $(\psi,t) \in \Omega_c(N,\chi)$ and $(\psi',t') \in \Omega_e(N,\chi)$, then $\theta_{\psi,t}$ and $\theta_{\psi',t'}$ are orthogonal for the Petersson scalar product.
Indeed, since $\psi \neq \psi'$, there is a $p \nmid N$ such that $\psi(p) \neq \psi'(p)$. Hence, $\theta_{\psi,t}$ and $\theta_{\psi',t'}$ are eigenforms of $T(p^2)$ corresponding to different eigenvalues. Since $\overline{\chi}(p)T(p^2)$ is hermitian (cf. the proof of Lemma 14, §6) this implies that these two functions are orthogonal.

7.4. The space $E_1(N,1/2)$.

Let $E_0(N,1/2,\chi)$ be the space of linear combinations of the $\theta_{\psi,t}$ with $(\psi,t) \in \Omega_e(N,\chi)$. By Theorem B, we have the orthogonal decomposition

$$M_0(N,1/2,\chi) = E_0(N,1/2,\chi) \oplus S_0(N,1/2,\chi),$$

where $S_0(N,1/2,\chi)$ is the space of cusp forms. Similarly, if we put

$E_1(N,1/2) = \oplus E_0(N,1/2,\chi)$, we have

$$M_1(N,1/2) = E_1(N,1/2) \oplus S_1(N,1/2).$$

The elements of $E_1(N,1/2)$ can be characterized as follows :

THEOREM 6. Let f be an element of $M_1(N,1/2)$. The following properties are equivalent :

i) f belongs to $E_1(N,1/2)$.

ii) f is a linear combination of $\theta(az+b)$, with $a \in \mathbf{Z}$, $a \geqslant 1$, and $b \in \mathbf{Q}$.

iii) f is orthogonal to all cusp forms of all levels.

PROOF. Clearly ii) implies iii) since θ is in $E_1(M,1/2)$ for every M, and so is orthogonal to all cusp forms; the same is then true of $\theta(az+b)$ for any a and b. We have already shown that iii) implies i). Finally, if ψ is a totally even character, we may write ψ as α^2 where the character α is ramified at the same primes as ψ; we have $\theta_\psi = \theta * \alpha$, hence θ_ψ is a linear combination of the $\theta(z+b)$, with $b \in \mathbf{Q}$; this shows that θ_ψ has property ii), hence that i) implies ii).

REMARK. Maass [6] has shown that $\theta(z)$ can be defined as an "Eisenstein series", by analytic continuation à la Hecke. The same is true for all the $\theta(az+b)$, hence for all the elements of $E_1(N,1/2)$.

APPENDIX

Free translation of a letter from Pierre DELIGNE,

dated March 1, 1976

... Using the same trick as in my Antwerp's paper (vol. II, p.90, proof of 2.5.6), one can deduce directly from your Theorem 2 the structure of the modular forms of weight 1/2 (on congruence subgroups of $\mathbf{SL}_2(\mathbf{Z})$). The final result is :

THEOREM. The q-expansions of the modular forms of weight 1/2 are

$$\sum_t \sum_{u \in \mathbf{Z}} \phi_t(u) q^{tu^2}, \qquad (1)$$

where t runs through a finite subset of $\mathbf{Q}^{*+}$, and, for each t, ϕ_t is a periodic function on $\mathbf{Z}$ (i.e. the restriction of a locally constant function on $\hat{\mathbf{Z}}$).

PROOF. Let H be the space of modular forms of weight 1/2, and Θ the subspace of H consisting of the theta series (1). We put on H the Petersson scalar product (which always converges). The metaplectic 2-covering $\widetilde{\mathbf{SL}}_2(\mathbf{A}_f)$ of $\mathbf{SL}_2(\mathbf{A}_f)$ acts on H, preserves the scalar product, and leaves Θ stable. Under this action, H decomposes into a direct sum of irreducible representations. Let H_i be one of them. We want to prove that H_i is contained in Θ.

One checks immediately that, if N and χ are suitably chosen, H_i has a non-zero intersection with $M_0(N,1/2,\chi)$. The Hecke operators $T(p^2)$ associated with all primes p (including those dividing N) come from the action of (the group ring of) $\widetilde{\mathbf{SL}}_2(\mathbf{A}_f)$, and commute with each other. Hence they have a non-zero common eigenvector f in $H_i \cap M_0(N,1/2,\chi)$. By your Theorem 2, one has

$$f = \sum_{u \in \mathbf{Z}} a(tu^2)q^{tu^2} \qquad (t \text{ square-free, } t|N),$$

and

$$a(mu^2) = a(m)\psi(u) \quad \text{if } (u,N) = 1,\ \psi \text{ being some character (mod 2N)},$$

$$a(mp^2) = \lambda_p a(m) \quad \text{if } p|N \quad \text{(cf. Shimura [8], 1.7).}$$

Consider now

$$g = \sum_{(u,N)=1} a(tu^2)q^{tu^2}.$$

It is clear that g is a non-zero element of Θ. On the other hand, g is (up to a scalar factor) the transform of f by $\prod_{p|N} L_p$, where L_p is the operator which transforms h(z) into $h(z) - \lambda_p h(p^2 z)$. Since L_p can be defined by the element $1 - \lambda_p \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ of the group ring of $\widetilde{\mathbf{SL}}_2(\mathbf{Q}_p)$, this shows that g belongs to H_i, hence $H_i \cap \Theta \neq 0$. Since H_i is irreducible, this implies $H_i \subset \Theta$, q.e.d.

Yours,

P. Deligne

PS. These arguments should extend to any totally real number field.

BIBLIOGRAPHY

[1] A.O.L. ATKIN and J. LEHNER, Hecke operators on $\Gamma_0(m)$, Math. Ann. 185 (1970), p. 134-160.

[2] A.O.L. ATKIN and H.P.F. SWINNERTON-DYER, Modular forms on noncongruence subgroups, Proc. Symp. Pure Math. XIX, p. 1-25, Amer. Math. Soc., 1971

[3] P. DELIGNE and J-P. SERRE, Formes modulaires de poids 1, Ann. Sci. E.N.S. (4) 7 (1974), p. 507-530.

[4] E. HECKE, Mathematische Werke (zw. Aufl.) Vandenhoeck und Ruprecht, Göttingen, 1970.

[5] W. LI, Newforms and Functional Equations, Math. Ann. 212 (1975), p. 285-315.

[6] H. MAASS, Konstruktion ganzer Modulformen halbzahliger Dimension mit θ-Multiplikatoren in einer und zwei Variablen, Abh. Math. Sem. Univ. Hamburg 12 (1937), p. 133-162.

[7] G. SHIMURA, Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, 11, Princeton Univ. Press, 1971.

[8] G. SHIMURA, On modular forms of half integral weight, Ann. of Math. 97 (1973), p. 440-481.

[9] G. SHIMURA, On the trace formula for Hecke operators, Acta Math. 132 (1974), p. 245-281.