HARMONIC ANALYSIS ON TREES

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1. Introduction. Following the method of Bruhat and Tits in [**1**] one associates to a reductive p-adic group G of rank one a simplicial complex of dimension 1 which appears to be a tree. The set of vertices of this tree is the discrete space $G_K/G_{\mathfrak{O}}$ (K a local field, $\mathfrak{O}$ its ring of integers). Serre used this tree in [**10**] and [**11**] to study the arithmetical properties of the subgroup of integral points $G_{\mathfrak{O}}$, namely congruence subgroups and amalgamation. We sketch here a method to deal with the representation theory of G_K using the combinatorics of the tree $G_K/G_{\mathfrak{O}}$. A detailed exposition will be submitted to the *Journal of Combinatorial Theory*.

2. Geometry of a tree ([**3**], [**4**], [**10**]). We consider a tree X as defined by its *vertex* set S and a set A of two-element subsets of S, the *edges*. By definition of a tree, for given vertices s and s' there exists a unique chain joining s to s', namely a sequence $[s_0, s_1, \ldots, s_m]$ of distinct vertices such that $s_0 = s$, $s_m = s'$ and $\{s_{j-1}, s_j\} \in A$ for $1 \leqq j \leqq m$. The integer m is the distance from s to s', written as $d(s, s')$. We assume moreover that X is *homogeneous of degree* $q+1$; that is, each vertex is adjacent to exactly $q+1$ edges ($q \geqq 2$).

By adjunction of a set S_∞ of *boundary points* to S, one gets a compact space $\hat{S}$ containing S as a discrete open dense subspace. A (one-sided) *geodesic* issued from the vertex s is an infinite sequence $[s_0, s_1, \ldots]$ of distinct vertices such that $s_0 = s$ and $\{s_{j-1}, s_j\} \in A$ for any $j \geqq 1$. Given s there is a bijective correspondence between the geodesics $[s_0, s_1, \ldots]$ issued from s and the boundary points b, expressed by the relation $b = \lim_{m \to \infty} s_m$ in $\hat{S}$.

Let b be a boundary point. Let s and s' be two vertices, $[s_0, s_1, \ldots]$ the geodesic from s to b, and $[s'_0, s'_1, \ldots]$ the geodesic from s' to b. There exist integers $r \geqq 0$

AMS (MOS) subject classifications (1970). Primary 22E35; Secondary 05C05, 60C05.

and $r' \geq 0$ such that $[s_0, s_1, \ldots, s_r = s'_{r'}, s'_{r'-1}, \ldots, s'_1, s'_0]$ is the chain from s to s' and $s_{r+n} = s'_{r'+n}$ for any $n \geq 0$. One has, therefore, $d(s, s') = r + r'$ and one puts $\delta_b(s, s') = r - r'$. From the identity $\delta_b(s, s'') = \delta_b(s, s') + \delta_b(s', s'')$ one deduces the existence of a partition $(H_n)_{n \in \boldsymbol{Z}}$ of S such that $\delta_b(s, s') = m - m'$ for s in H_m and s' in $H_{m'}$. This partition is unique up to a shift in the index. The sets H_n are the *horocycles* of center b. The shift V in the set of horocycles is defined by $V(H_n) = H_{n+1}$.

The space of all horocycles provided with a natural topology appears as a principal bundle P of basis S_∞ and structure group $\boldsymbol{Z}$ (acting via the powers of V). For each vertex s the set of horocycles going through s is a continuous cross section of P over S_∞.

3. The tree associated to $PGL_2(K)$ **[10].** Let K be a field complete under a discrete valuation, $\mathfrak{O}$ its ring of integers and $\mathfrak{p}$ the nonzero prime ideal in $\mathfrak{O}$. One assumes that the residue field $k = \mathfrak{O}/\mathfrak{p}$ is finite with q elements. One lets $\mathfrak{O}^\times = \mathfrak{O} - \mathfrak{p}$. The image of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $PGL_2(K)$ shall be denoted by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

Let $\boldsymbol{V}$ be the vector space of column vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ with coordinates taken from K. By a lattice in $\boldsymbol{V}$ we mean any $\mathfrak{O}$-submodule of $\boldsymbol{V}$ generated by two linearly independent vectors. Two lattices Λ and Λ' are called equivalent if there exists a nonzero element x in K such that $\Lambda' = x \cdot \Lambda$. The set S consists of the equivalence classes of lattices. An edge is a pair $\{s, s'\}$ of vertices with representative lattices Λ and Λ' such that $\Lambda \supset \Lambda'$ and $(\Lambda : \Lambda') = q$. One gets a tree X, homogeneous of degree $q+1$, whose boundary S_∞ can be canonically identified with the projective line $\boldsymbol{P}^1(K)$.

We can now interpret some of the standard homogeneous spaces of $G_K = PGL_2(K)$. As usual the *Borel subgroup* B_K has elements $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$, the *split torus* H_K has elements $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$, the subgroup $G_\mathfrak{O}$ consists of the integral elements $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with a, b, c, d in $\mathfrak{O}$ and $ad - bc$ in $\mathfrak{O}^\times$, while the *Hecke subgroup* $\Gamma_0(\mathfrak{p}^m)$ is the subgroup of $G_\mathfrak{O}$ defined by the congruence $c \equiv 0 \bmod \mathfrak{p}^m$. Then S is isomorphic to $G_K/G_\mathfrak{O}$ and S_∞ to G_K/B_K while $G_K/\Gamma_0(\mathfrak{p}^m)$ is the space of pairs of vertices at the distance m and G_K/H_K is the set of pairs of distinct boundary points, in bijective correspondence with the two-sided geodesics in the tree X. Similarly, the space P of horocycles is isomorphic to G_K/M where M consists of the elements $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$ with a in $\mathfrak{O}^\times$ and b in K.

4. Spherical functions [7]. We denote by $X = (S, A)$ any homogeneous tree of degree $q+1$ and Γ its full automorphism group. The vector space $C_c^\infty(S)$ consists of the complex-valued functions on S with finite support. The *Hecke algebra* $\mathscr{H}$ is the commuting algebra of the natural representation of Γ in $C_c^\infty(S)$. One describes as follows a basis $(\Theta_0, \Theta_1, \ldots)$ of the complex vector space $\mathscr{H}$: The operator Θ_m transforms a function f into the function whose value at s is the sum of the values of f at the vertices at distance m from s. The coefficient of Θ_p in the product $\Theta_m \cdot \Theta_n$ is the number of "triangles" stu in S with given basis su of length p and two other sides of lengths m and n. In particular, one gets the relations

$$\Theta_1^2 = \Theta_2 + (q+1)\,\Theta_0, \tag{1}$$

$$\Theta_1 \cdot \Theta_m = \Theta_{m+1} + q\Theta_{m-1} \qquad (\text{for } m \geqq 2), \tag{2}$$

conveniently collected into the generating series

$$\sum_{m=0}^{\infty} \Theta_m u^m = \frac{1-u^2}{1-u\Theta_1 + qu^2}. \tag{3}$$

In particular, $\mathscr{H}$ is the polynomial algebra $\boldsymbol{C}[T]$ with $T = \Theta_1$ the Hecke operator.

Fix a vertex s and let Γ_s be the stabilizer of s in Γ. We denote by $\mathscr{H}$ the Hilbert space consisting of the functions on S invariant under Γ_s, with the norm $[\sum_{t\in S} |f(t)|^2]^{1/2}$. One describes as follows an orthonormal basis $e_0, e_1, \dots$ for $\mathscr{H}$: The function e_0 is equal to 1 at s and vanishes elsewhere, and

$$e_m = [q^{m-1}(q+1)]^{-1/2}\,\Theta_m e_0 \quad \text{for } m \geqq 1.$$

The Hecke operator T acts on $\mathscr{H}$ with a matrix given by

$$\begin{pmatrix} 0 & (q+1)^{1/2} & 0 & 0 & 0 & \dots \\ (q+1)^{1/2} & 0 & q^{1/2} & 0 & 0 & \dots \\ 0 & q^{1/2} & 0 & q^{1/2} & 0 & \dots \\ 0 & 0 & q^{1/2} & 0 & q^{1/2} & \dots \\ 0 & 0 & 0 & q^{1/2} & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix}.$$

Our next concern will be the spectral decomposition of T.

Let t be any complex number, which we write in the form $t = q^{1/2}(\lambda + \lambda^{-1})$. There exists on S a unique eigenfunction of T with the eigenvalue t, invariant under the group Γ_s and normalized by taking the value 1 at s. For s' at the distance $m \geqq 1$ from s, its value $F_t(s, s')$ at s' is equal to $\chi_t(\Theta_m)/q^{m-1}(q+1)$ where χ_t is the algebra homomorphism from $\mathscr{H}$ into $\boldsymbol{C}$ taking T to t. From (3) one deduces the explicit formula

$$F_t(s, s') = \frac{q(\lambda^{m+1} - \lambda^{-m-1}) - (\lambda^{m-1} - \lambda^{-m+1})}{q^{m/2}(q+1)\,(\lambda - \lambda^{-1})} \tag{4}$$

where $m = d(s, s')$.

The *Plancherel measure* is the unique probability measure μ on the real line $\boldsymbol{R}$ such that $\int \chi_t(\Theta_m)\,d\mu(t) = 0$ for any $m \geqq 1$. This moment problem is solved by the classical method; namely μ is the weak limit for $\varepsilon \to 0$ of

$$\pi^{-1} \operatorname{Im} \langle e_0 | (T-t-i\varepsilon)^{-1} | e_0 \rangle.$$

The resolvent $(T-z)^{-1}$ is easily computed from (3) and one gets

$$d\mu(t) = \frac{((4q-t^2)_+)^{1/2}}{(q+1)^2-t^2}\, dt. \tag{5}$$

The *Plancherel formula for spherical functions* reads as follows:

$$\int_{-2q^{1/2}}^{2q^{1/2}} \frac{(4q-t^2)^{1/2}}{(q+1)^2-t^2} F_t(s, s')\, dt = 1 \quad \text{for} \quad s = s', \tag{6}$$

$$= 0 \quad \text{otherwise.}$$

The following table collects information about the spherical functions.

Spectrum of T	$\lvert\lambda\rvert = 1$	$-2q^{1/2} \leqq t \leqq 2q^{1/2}$
F_t bounded	$q^{-1/2} \leqq \lvert\lambda\rvert \leqq q^{1/2}$	$\left(\frac{\operatorname{Re} t}{q+1}\right)^2 + \left(\frac{\operatorname{Im} t}{q-1}\right)^2 \leqq 1$
F_t positive-definite	$\lvert\lambda\rvert = 1$ *or* λ real and $q^{-1/2} \leqq \lvert\lambda\rvert \leqq q^{1/2}$	$-q-1 \leqq t \leqq q+1$

Whenever X is the tree associated to $PGL_2(K)$, the previous results agree with the known results due mainly to Mautner [7].

5. Principal and supplementary spherical series. Let t be a real number in the interval $[-2q^{1/2}, 2q^{1/2}]$. We let $\mathscr{H}_t$ be the Hilbert space of functions on S with reproducing kernel $F_t(s, s')$. Any function f in $\mathscr{H}_t$ satisfies $Tf = tf$. The automorphism group Γ acts on $\mathscr{H}_t$ and one gets in this way an irreducible unitary continuous[1] representation of Γ. This provides the construction of the *principal spherical series* of representations of Γ. The *supplementary spherical series* obtains for t real in either of the intervals $]2q^{1/2}, q+1[$ and $]-q-1, -2q^{1/2}[$. Finally, for $t = q+1$ or $t = -q-1$, one gets one-dimensional representations of Γ. Whenever the tree X is associated to $PGL_2(K)$ one obtains by restriction from Γ to $PGL_2(K)$ the familiar series of representations. Notice that our proof of the fact that F_t is positive-definite for t real in $[-q-1, q+1]$ is the same in all cases and provides therefore a unified construction of the principal and supplementary series parametrized by characters of $K^\times$ trivial on the unit group $\mathfrak{O}^\times$.

Let us explain briefly how the *method of horocycles* works in our case (see [6] for an exposition of the classical case). On the space P of horocycles there

[1] The group Γ is locally compact and separable in the natural topology of pointwise convergence.

exists a unique measure ν giving the mass 1 to the set of horocycles going through a fixed vertex. This measure is invariant under the automorphism group Γ and the shift V multiplies ν by q^{-1}. One gets, therefore, a continuous representation of Γ in the Hilbert space $L^2(P, \nu)$ and one defines by $\Lambda f(\gamma) = q^{1/2} f(V\gamma)$ a unitary operator Λ in $L^2(P, \nu)$ commuting with the operators from Γ.

On the other hand, one has the natural representation of Γ in the Hilbert space $L^2(S)$ (each point of S being given the mass 1) and the Hecke operator T in $L^2(S)$. The *Radon transform* Jf of a function f in $C_c^\infty(S)$ is defined by $Jf(\gamma) = \sum_{s \in \gamma} f(s)$ for γ in P. One establishes the following facts:

(a) *The previous map J extends to an isometry J from $L^2(S)$ into $L^2(P, \nu)$ commuting with the action of Γ.*

(b) *One has $JT = q^{1/2}(\Lambda + \Lambda^{-1})J$.*

(c) *There exists a unitary operator W in $L^2(P, \nu)$ such that $W^2 = 1$, $W\Lambda = \Lambda^{-1}W$ and the image of J consists of the functions f with $Wf = f$.*

It is clear how to make explicit the spectral decomposition of Λ in $L^2(P, \nu)$: For any complex number λ of modulus 1, the suitably normalized functions f on P such that $q^{1/2} f(V\gamma) = f(\gamma)$ make a Hilbert space $\mathscr{H}'_\lambda$ which is the carrier of an irreducible unitary continuous representation of Γ. For λ and λ^{-1} one gets equivalent representations. Using J backwards, one recovers the previous results about the spectral decomposition of T.

As one expects, the supplementary series of representations can be realized in spaces of functions on P, but the description of the invariant scalar product is a little more delicate. It is also easy to extend to our case the results of Mautner [**8**] about the integral representation for the eigenfunctions of the Hecke operator.

6. Other series of representations. So far we have been able to extend the construction of the principal and supplementary series of representations to the automorphism group of a tree with an arbitrary degree $q+1 \geqq 3$. We shall be very brief about the construction of the other series.

The solutions of the equation $Tf = (q+1)f$ are the harmonic functions on S, namely the functions whose value at any vertex s is the arithmetic mean of the values at the immediate neighbours of s. One can specialize to a homogeneous tree our results in [**3**]. The harmonic functions are the carrier of the *special representation*, in accordance with the general results of Borel and Serre in [**2**]. The equation $Tf = -(q+1)f$ is treated similarly.

According to a result of Silberger [**9**] made more precise by Casselman [**5**], there exists in any irreducible (admissible) representation of $PGL_2(K)$ a vector invariant under $\Gamma_0(\mathfrak{p}^m)$ for m suitably large. This suggests studying the natural representation of Γ in the space of functions $f(s, s')$ where the vertices s and s' are at a fixed distance m. One can make the decomposition of this representation into irreducible components by combinatorial methods. In this way, one can construct, for instance, irreducible representations for Γ which upon restriction to $PGL_2(K)$ break into finitely many irreducible representations of the discrete series. This provides a *combinatorial construction of the supercuspidal representations of* $PGL_2(K)$.

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