On the enumeration of quintic fields with small discriminant

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Dedicated to Professor Helmut Hasse on the occasion of his 75th birthday

Introduction

In his paper "A numerical study of quintics of small discriminant" [2], H. Cohn gives a table of irreducible polynomials of degree 5 with integral coefficients, arranged in increasing order of their discriminant. For a given discriminant, there are up to 6 different polynomials and H. Cohn asks whether these polynomials define or not the same number field.

In the course of another investigation [1] we had to determine explicitly the decomposition laws of the prime numbers in some totally real number fields. When we came across the fields of degree 5 and used Cohn's table, we found that the fields generated by the different equations of a given discriminant exhibited identical decomposition laws (for the primes $p \leq 500$).

This gave us a strong suspicion that the corresponding fields could be identical That is indeed true[1]).

The question was to find a convenient algorithm to decide whether two irreducible equations of a given degree generate the same field. Fortunately such a method has been already given by Zassenhaus [3]. We give here in tabular form the transformations between Cohn's equations of the same discriminant, as calculated on the electronic high speed computer IBM 360/44 of the Department of Mathematics at Strasbourg.

In a forthcoming publication, we shall apply our numerical results to the search of fundamental systems of units in the fields under consideration.

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1) According to H. Cohn, this was checked in most cases by Emil Artin as early as 1955, using various arithmetical tricks.

Tables

For a given D, we give a polynomial

$$F(X) = X^5 + \alpha_1 X^4 + \alpha_2 X^3 + \alpha_3 X^2 + \alpha_4 X + \alpha_5$$

of discriminant D and a variable number of polynomials

$$G(X) = X^5 + \beta_1 X^4 + \beta_2 X^3 + \beta_3 X^2 + \beta_4 X + \beta_5$$

of discriminant D. The coefficients $\alpha_1, \ldots, \alpha_5$ and $\beta_1, \ldots, \beta_5$ listed below are taken from Cohn's tables [2]. The last column lists the coefficients a_0, a_1, a_2, a_3, a_4 of polynomials

$$T(X) = a_0 X^4 + a_1 X^3 + a_2 X^2 + a_3 X + a_4$$

with the following property: if $x_1, \ldots, x_5$ are the roots of $F(X)$, then $T(x_1), \ldots, T(x_5)$ are the roots of the polynomial $G(X)$ tabulated in the same line as $T(X)$. We assume that r_1 of the roots of $F(X)$ are real, and the other roots comprise together r_2 pairs of complex conjugate numbers $y_1, \bar{y}_1, \ldots, y_{r_2}, \bar{y}_{r_2}$.

Table 1

$r_1 = 1,\ r_2 = 2$

D	$F(X)$					$G(X)$					$T(X)$				
	α_1	α_2	α_3	α_4	α_5	β_1	β_2	β_3	β_4	β_5	a_0	a_1	a_2	a_3	a_4
1609	0	−1	1	1	−1	2	1	−2	−2	−1	1	1	−1	0	1
						0	−3	0	2	−1	1	1	0	0	1
						−1	3	−3	2	−1	1	0	−1	0	1
						−2	3	−3	3	−1	0	0	1	0	0
						−3	1	−2	3	−1	1	1	−1	1	2
1649	0	−3	1	3	−1	1	−1	0	1	−1	1	0	−2	0	1
						1	1	0	−1	−1	0	0	−1	0	1
1777	0	−2	1	2	−1	1	−2	−1	1	−1	1	1	−2	−1	2
						−1	1	−2	1	−1	1	0	−1	1	1
						−2	2	−3	2	−1	1	0	−2	0	2
						2	3	−1	−3	−1	1	1	−1	0	1
2209	0	−1	−2	−2	−1	1	1	−1	−2	−1	1	−1	0	−1	−1
						−1	−3	2	3	−1	1	0	−2	−1	−1
						−1	2	−3	3	−1	0	−1	1	1	1
2297	0	1	−1	1	−1	−1	−3	1	2	−1	1	0	1	0	1
						−1	−2	−1	3	−1	0	1	1	1	0
						1	−1	−3	−2	−1	1	1	1	0	0
2617	0	−2	−3	−2	−1	0	1	−2	0	−1	0	1	−1	−1	−1
						−1	−2	1	1	−1	−1	1	2	1	0
						1	−1	2	3	−1	1	−1	−2	0	0
2665	0	1	0	−2	−1	1	−1	−2	3	−1	−1	1	−2	2	1
						1	0	−3	−3	−1	2	−1	3	−1	−3
						−1	1	−1	2	−1	1	−1	2	−1	−1
						1	3	1	2	−1	2	−1	3	−2	−3
2869	0	−2	0	1	−1	0	0	0	−1	−1	0	1	0	−1	0

Table 2

$r_1 = 3,\ r_2 = 1$

D	$F(X)$					$G(X)$					$T(X)$				
	α_1	α_2	α_3	α_4	α_5	β_1	β_2	β_3	β_4	β_5	a_0	a_1	a_2	a_3	a_4
-4511	0	-2	1	0	-1	0	2	1	-2	-1	1	0	-2	0	0
						-1	-3	2	1	-1	-1	0	1	-1	1
						2	-3	-1	3	-1	1	-1	-2	2	-1
-4903	1	-3	-1	2	-1	1	-2	1	1	-1	0	0	-1	-1	1
						-2	-3	2	2	-1	0	0	1	0	-1
-5519	0	-3	1	1	-1	1	-3	0	3	-1	0	0	-1	0	1
						2	-1	-1	1	-1	0	-1	0	2	-1
-5783	-1	-3	3	2	-1	1	-3	1	2	-1	-1	1	3	-2	-2
						1	2	-1	-3	-1	0	1	0	-2	0
						2	-2	-1	2	-1	0	-1	1	2	-2
						3	3	3	2	-1	0	0	1	0	-2
-7031	0	-1	1	-1	-1	-2	-1	1	3	-1	0	0	1	0	0
-7367	-2	-1	-1	3	-1	3	2	1	1	-1	-2	3	3	4	-4
						2	-3	0	2	-1	-4	6	7	7	-8
-7463	2	-1	-3	-3	-1	2	0	-1	-2	-1	-1	-1	2	1	1
						3	1	-2	-3	-1	0	0	0	-1	-1

Table 3

$r_1 = 5,\ r_2 = 0$

D	$F(X)$					$G(X)$					$T(X)$				
	α_1	α_2	α_3	α_4	α_5	β_1	β_2	β_3	β_4	β_5	a_0	a_1	a_2	a_3	a_4
14641	-1	-4	3	3	-1	2	-5	-2	4	-1	1	0	-4	-1	2
24217	-2	-4	3	2	-1	0	-5	1	5	-1	-1	2	3	-1	0
						0	-5	1	3	-1	1	-1	-6	0	3
						3	-4	-5	5	-1	2	-4	-7	4	1
						0	-6	3	2	-1	1	-2	-4	2	2
						-2	-6	3	6	-1	-2	3	9	0	-3
36497	-1	-5	5	4	-3	-1	-5	3	2	-1	1	0	-4	0	2
						0	-6	-1	4	-1	1	0	-5	0	4
						1	-6	1	3	-1	-1	0	5	-1	4
						-1	-5	3	6	-1	0	0	1	0	-2
38569	0	-5	0	4	-1	-1	-5	2	5	-1	-1	-1	5	4	-3
						1	-5	-1	4	-1	1	0	-4	0	1
						5	5	-5	-6	-1	0	0	0	1	-1
						-5	5	5	-6	-1	0	0	0	1	1
						-1	-6	5	1	-1	-1	0	5	1	-3

References

[1] *P. Cartier* and *Y. Roy*, Certains calculs numériques relatifs à l'interpolation p-adique des séries de Dirichlet, in "Modular functions of one variable. III", Lecture notes in Mathematics 350 (1973), 269—349.

[2] *H. Cohn*, A numerical study of quintics of small discriminant, Comm. Pure Appl. Math. 8 (1955), 377—386.

[3] *H. Zassenhaus* and *J. Liang*, On a problem of Hasse, Math. Comp. **23** (1969), 515—519.

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