A COURSE ON DETERMINANTS

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INTRODUCTION

Determinants enter the field of quantum physics via *Pauli's exclusion principle*. The mathematical expression of this principle is as follows: denoting by $\vec{r}_j$ the coordinates of the j-th electron (three space coordinates taking arbitrary real values, plus a spin coordinate taking the values $+\frac{1}{2}$ and $-\frac{1}{2}$), the wave function $\psi(\vec{r}_1,\ldots,\vec{r}_p)$ for a system of p electrons is *antisymmetric in its arguments* $\vec{r}_1,\ldots,\vec{r}_p$. If, for instance, $\psi_1(\vec{r}),\ldots,\psi_p(\vec{r})$ are normalized one-electron wave-functions, *mutually orthogonal*, corresponding to energy levels $E_1,\ldots,E_p$, then the normalized p-electron wave function $\psi(\vec{r}_1,\ldots,\vec{r}_p) = (p!)^{-1/2} \det \psi_i(\vec{r}_j)$ satisfies the exclusion principle and corresponds to the total energy $E_1+\ldots+E_p$. Similar constructions occur in Fock space or statistical mechanics. It is not the place to review the manifold physical implications of the exclusion principle; it suffices to say that the stability of matter as we know it depends strongly on this principle. At the mathematical level, the basic estimate was provided by Hadamard: if $A = (a_{ij})$ is a p x p matrix with complex elements, and if C is the maximum of the numbers $|a_{ij}|$, then the determinant D of A satisfies $|D| \leq p^{p/2}C^p$. This result shows a remarkable compensation occuring among the p! products of size C^p (approximately) which compose D, since $p^{p/2}$ is roughly of the order of $(p!)^{1/2}$ for large p. Nothing similar could occur in the case of *Bose-Einstein statistics*, where $\psi(\vec{r}_1,\ldots,\vec{r}_p)$ is symmetrical in its arguments $\vec{r}_1,\ldots,\vec{r}_p$ and the determinant should be replaced by a *permanent*

$$\sum_{\sigma\in S_p} \psi_1(\vec{r}_{\sigma(1)})\ldots\psi_p(\vec{r}_{\sigma(p)})$$

(symmetrization of the product $\psi_1(\vec{r}_1)\ \ldots\ \psi_p(\vec{r}_p)$).

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A glance of the table of contents will reveal the organization of this paper. It is essentially a leisurely exposition of the basic properties of determinants, with special emphasis on the infinite-dimensional case. In a venerable subject like this, it is hard to innovate. In *part one*, we mostly review the properties of finite determinants in a form most suitable for generalizations. One of the novel features is our use of *volume forms* in subsections 1.4 and 1.5; we aim at giving characterizations not only of ordinary determinants, but also of powers of (absolute value of) determinants, and we provide a link with a non-commutative determinant introduced by Dieudonné around 1940. The connection between determinants and *antisymmetric tensors* is well-known. We made some efforts to present this (classical) theory in the spirit of *supersymmetry*. The analogy between the symmetric (Bose-Einstein statistics) and antisymmetric (Fermi-Dirac statistics) cases is especially transparent in the so-called *Mac Mahon's master theorem*, connecting various generating series of interest in statistical mechanics. We end part one by reviewing various formulas about Gaussian integrals; they are all classical and provide useful integral representations for various determinants. These formulas should be compared with the ones derived in part three using Berezin integration of functions of Grassmann variables.

Part two is devoted to the infinite-dimensional determinants which occur as variants of the *Fredholm determinants* for integral operators. We begin an exposition of the classical results of Fredholm. We endeavoured at motivating, as far as possible, the definitions by analogy with the finite-dimensional case. With the notable exception of *Fredholm's alternative*, we favoured the constructive proofs over the purely existential ones. *The basic formula*, which is used to define the determinant in various contexts is the following

$$\det(1 + A) = \sum_{n\geq 0} \mathrm{Tr}(\Lambda^n A) \quad ,$$

where $\Lambda^n A$ is the operator acting on the antisymmetric tensor space $\Lambda^n V$ by mapping $x_1 \wedge \ldots \wedge x_n$ into $Ax_1 \wedge \ldots \wedge Ax_n$. In physical slang, $\Lambda^n V$ is the n-particle Fock space for fermions, if V is the one-particle state space. The theory is especially smooth in the case of operators in Hilbert spaces - incidentally, this is the case of greatest relevance in quantum physics. But the Hilbert space theory does not contain the original case of integral operators with continuous kernels. The main difficulty to be overcome is that the series $\sum_n \lambda_n$ of the eigenvalues of an integral operator does not always converge, but instead $\sum_n |\lambda_n|^2$ is finite. Various authors (Grothendieck, Ruston) made efferts to extend the definition of Fredholm determinants to suitable classes of operators in Banach spaces. We present here a novel version, which depends strongly on the properties of Hilbert-Schmidt operators. This can be considered as the beginning of a theory of *renormalized determinants*.

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PART ONE:

DETERMINANTS IN THE FINITE DIMENSIONAL CASE

1. A Review of the Elementary Theory

1.1. Let A be a *square matrix* of size n, with complex entries a_{ij} (for $1 \leq i \leq n$, $1 \leq j \leq n$). We denote by $a_1,\dots,a_n$ its column vectors, so

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \quad \dots, \quad \vec{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Therefore $\vec{a}_1,\dots,\vec{a}_n$ are elements of the complex n-space $\mathbb{C}^n$. We also denote by I (or I_n) the *unit matrix*, by δ_{ij} its elements (Kronecker symbol) and $\vec{e}_1,\dots,\vec{e}_n$ the column vectors of I_n. Therefore e_i is the vector whose only nonzero component is the i-th one, which is equal to 1. Any vector $\vec{x}$ in $\mathbb{C}^n$ with components $x_1,\dots,x_n$ is therefore written as the linear combination $x_1\vec{e}_1 + \dots + x_n\vec{e}_n$, and $\vec{e}_1,\dots,\vec{e}_n$ is the so-called *canonical basis* of $\mathbb{C}_n$.

1.2. The *determinant* of A is a complex number, denoted usually by det A (or sometimes $|A|$). Viewed as a function of the columns $\vec{a}_1,\dots,\vec{a}_n$ of A, we denote it as $\Delta(\vec{a}_1,\dots,\vec{a}_n)$. We can express as follows the basic properties of the determinant:

a) $\Delta(\vec{e}_1,\dots,\vec{e}_n) = 1$ *normalisation*

b) for $\vec{a}_1,\dots,\vec{a}_{i-1}, \vec{a}_{i+1},\dots\vec{a}_n$ fixed, $\Delta(\vec{a}_1,\dots,\vec{a}_i,\dots,\vec{a}_n)$ is a linear function of the vector $\vec{a}_i$, hence a linear combination *n-linearity* of $a_{1i},\dots,a_{ni}$ with coefficients depending only on the other columns.

c) if we exchange $\vec{a}_i$ and $\vec{a}_{i+1}$ and keep the other vectors fixed, $\Delta(\vec{a}_1,\dots,\vec{a}_n)$ gets multiplied by -1. *antisymmetry.*

These properties completely characterize the determinant.

It is easy to give an inductive construction of the determinant. Denote by $A^{(i)}$ (for $1 \le i \le n$) the square matrix of size n-1 obtained by erasing from A its first column and its i-th row. Then we have

$$\det A = \sum_{i=1}^{n} (-1)^{i-1} a_{i1} \cdot \det A^{(i)} \quad . \qquad (1.1)$$

1.3. A slightly more invariant presentation is as follows. We consider a vector space V of finite dimension n over the field $\mathbb{C}$ of complex numbers. A *volume form* on V is a function $\omega(\vec{x}_1,\dots,\vec{x}_n)$ depending on n vectors $\vec{x}_1,\dots,\vec{x}_n$ in V, with complex values, which is multilinear and antisymmetric, namely

$$\omega(\dots,\vec{x}_i,\dots) = \lambda'\omega(\dots,\vec{x}_i',\dots) + \lambda''\omega(\dots,\vec{x}_i'',\dots) \qquad (1.2)$$

if $\vec{x}_i = \lambda'\vec{x}_i' + \lambda''\vec{x}_i''$,

$$\omega(\dots,\vec{x}_i,\vec{x}_{i+1},\dots) = -\,\omega(\dots,\vec{x}_{i+1},\vec{x}_i,\dots) \quad . \qquad (1.3)$$

Given any basis $\vec{e}_1,\dots,\vec{e}_n$ of V, there exists a unique volume form ω_0 normalized by $\omega_0(\vec{e}_1,\dots,\vec{e}_n) = 1$. Then, for any volume form, we get

$$\omega(\vec{x}_1,\dots,\vec{x}_n) = t\omega_0(\vec{x}_1,\dots,\vec{x}_n) \qquad (1.4)$$

for $\vec{x}_1,\dots,\vec{x}_n$ arbitrary in V, with a constant $t = \omega(e_1,\dots,e_n)$. Hence, up to a scaling factor, volume forms are unique.

If there is a non-trivial linear relation $\lambda_1\vec{x}_1 + \dots + \lambda_n\vec{x}_n = 0$ among the vectors $\vec{x}_1,\dots,\vec{x}_n$, it follows immediately from (1.2) and (1.3) that $\omega(\vec{x}_1,\dots,\vec{x}_n) = 0$ for any volume form ω on V. On the other hand, if a volume from ω assumes a nonzero value on some basis $\vec{e}_1,\ldots,\vec{e}_n$, then it is nonzero on any basis $\vec{x}_1,\ldots,\vec{x}_n$ whatsoever.

1.4. Let us denote by B(V) the set of all (ordered) basis $\vec{x}_1,\ldots,\vec{x}_n$ of V, by $\Omega(V)$ the set of volume forms, and $\Omega^*(V)$ the set of nonzero volume forms. Any element ω of $\Omega^*(V)$ can be viewed as a function from B(V) to the set $\mathbb{C}^\times$ of nonzero complex numbers. As such, it is characterized by the following variants of properties (1.2) and (1.3)

$$\omega(\lambda_1\vec{x}_1,\ldots,\lambda_n\vec{x}_n) = \lambda_1 \ldots \lambda_n\, \omega(\vec{x}_1,\ldots,\vec{x}_n) \tag{1.5}$$

if $\lambda_1,\ldots,\lambda_n$ are in $\mathbb{C}^\times$, and

$$\omega(\vec{x}_1,\ldots,\vec{x}_i + \vec{x}_k, \ldots, \vec{x}_n) = \omega(\vec{x}_1,\ldots,\vec{x}_i,\ldots,\vec{x}_n) \tag{1.6}$$

if k is different from i.

Let us check for instance the antisymmetry; it suffices to write the proof in the case n = 2. Hence

$$\begin{aligned}\omega(\vec{x}_2,\vec{x}_1) &= \omega(\vec{x}_2+\vec{x}_1,\vec{x}_1) = \omega((\vec{x}_2+\vec{x}_1), -\vec{x}_2 + (\vec{x}_2+\vec{x}_1))\\ &= \omega((\vec{x}_2+\vec{x}_1),-\vec{x}_2) = \omega(\vec{x}_2+\vec{x}_1 +(-\vec{x}_2), -\vec{x}_2)\\ &= \omega(\vec{x}_1, -\vec{x}_2)\end{aligned}$$

by repeated application of (1.6) and $\omega(\vec{x}_1,-\vec{x}_2) = -\omega(\vec{x}_1,\vec{x}_2)$ by (1.5).

To construct a volume form ω in $\Omega^*(V)$, we can proceed inductively as follows. Decompose V as the direct sum of a line D (of dimension 1) and a hyperplane H (of dimension n-1). Choose a vector $\vec{e} \neq 0$ in D and a volume form φ in $\Omega^*(H)$. Then there exists a unique ω in $\Omega^*(V)$, such that

$$\omega(\vec{e}, \vec{y}_1,\ldots,\vec{y}_{n-1}) = \varphi(\vec{y}_1,\ldots,\vec{y}_{n-1}) \tag{1.7}$$

for any basis $\vec{y}_1,\ldots,\vec{y}_{n-1}$ of H. Let $\vec{x}_1,\ldots,\vec{x}_n$ be a basis of V, and express $\vec{e}$ as the linear combination $\lambda_1\vec{x}_1,+\ldots+\lambda_n\vec{x}_n$.

From (1.5) and (1.6) it follows that one doesn't change the value $\omega(\vec{x}_1,\ldots,\vec{x}_n)$ if one adds to any $\vec{x}_i$ a linear combination of the other vectors. From (1.7) one derives

$$\omega(\vec{x}_1,\ldots,\vec{x}_n) = (-1)^{i-1}\lambda_i^{-1}\ \varphi(p(\vec{x}_1),\ldots,p(\vec{x}_{i-1}),p(\vec{x}_{i+1}),\ldots,p(\vec{x}_n)) \tag{1.8}$$

for any index i such that $1 \leq i \leq n$ and $\lambda_i \neq 0$; for every vector $\vec{x}$ in V, $p(\vec{x})$ is the unique vector in H such that $\vec{x}-p(\vec{x})$ lies in D (that is, is proportional to $\vec{e}$). It is easy to check that *the right-hand side of* (1.8) *is independent of the index* i *as long as* $\lambda_i \neq 0$, hence we have a construction of ω; checking properties (1.5) and (1.6) for ω is easy if a little tedious.

If $\vec{e}_1,\ldots,\vec{e}_n$ is a fixed basis of V, let $\vec{e} = \vec{e}_1$ and H be the subspace of V with basis $\vec{e}_2,\ldots,\vec{e}_n$. If we accept that there exists a unique φ_0 in $\Omega^*(H)$ normalized by $\varphi_0(\vec{e}_2,\ldots,\vec{e}_n) = 1$, it follows from (1.7) that there exists a unique ω_0 in $\Omega^*(V)$ normalized by $\omega_0(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n) = 1$ and that $\omega_0(\vec{e}_1,\vec{y}_2,\ldots,\vec{y}_n) = \varphi_0(\vec{y}_2,\ldots,\vec{y}_n)$ for every basis $\vec{y}_2,\ldots,\vec{y}_n$ of H.

1.5. The previous construction of volume forms has some advantages on the more orthodox ones. For instance, we can modify the homogeneity property (1.5) into

$$\omega(\lambda_1\vec{x}_1,\ldots,\lambda_n\vec{x}_n) = |\lambda_1 \ldots\lambda_n|^s\ \omega(\vec{x}_1,\ldots,\vec{x}_n) \tag{1.9}$$

where s is a complex number. If s is an integer, we can also consider $(\lambda_1 \ldots \lambda_n)^s$ instead of $|\lambda_1 \ldots \lambda_n|^s$. Then it follows that any function $\omega\colon B(V) \to \mathbb{C}^\times$ satisfying (1.9) and (1.6) is of the form

$$\omega(\vec{x}_1,\ldots,\vec{x}_n) = t|\omega_0(\vec{x}_1,\ldots,\vec{x}_n)|^s \tag{1.10}$$

where ω_0 is as before and $t = \omega(\vec{e}_1,\ldots,\vec{e}_n)$ is a constant.

The previous considerations are valid *verbatim* in the real case. If we denote by $\Delta(\vec{x}_1,\vec{x}_2)$ the area of the parallelogram built in the plane $\mathbb{R}^2$ on the vectors $\vec{x}_1$, $\vec{x}_2$, it follows from geometric reasons that Δ satisfies the following rules (fig. 1):

$$\Delta(\lambda_1\vec{x}_1,\lambda_2\vec{x}_2) = |\lambda_1||\lambda_2|\ \Delta(\vec{x}_1,\vec{x}_2) \quad , \tag{1.11}$$

$$\Delta(\vec{x}_1,\vec{x}_2) = \Delta(\vec{x}_1 + \vec{x}_2,\ \vec{x}_2) = \Delta(\vec{x}_1,\ \vec{x}_2 + \vec{x}_1) \quad . \tag{1.12}$$

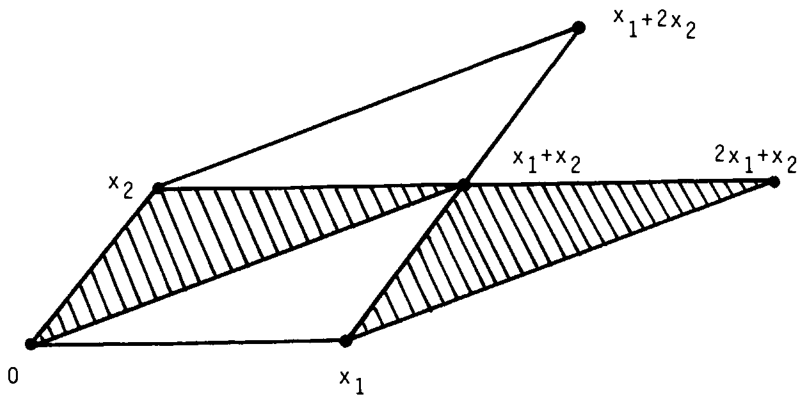

Fig. 1

Hence $\Delta(\vec{x}_1,\vec{x}_2) = |\det X|$ where X is the 2 x 2 matrix with columns $\vec{x}_1$ and $\vec{x}_2$ (a well-known result !). This extends immediately to the volume of the parallelotop in the space $\mathbb{R}^n$ spanned by n vectors $\vec{x}_1,\ldots,\vec{x}_n$; again this is equal to $|\det X|$ if X is the n x n matrix with columns $\vec{x}_1,\ldots,\vec{x}_n$.

We can also consider vectors in the *quaternionic* space $\mathbb{H}^n$, whose n components are quaternions. If $\vec{e}_1,\ldots,\vec{e}_n$ are the columns of the unit matrix I_n(as in section 1.1), we can associate to any invertible n x n matrix $A = (a_{ij})$ with quaternionic entries a determinant $\Delta(A)$ characterized by the following properties

$$\Delta(\vec{e}_1,\ldots,\vec{e}_n) = 1 \quad , \tag{1.13}$$

$$\Delta(\vec{a}_1\lambda_1,\ldots,\vec{a}_n\lambda_n) = |\lambda_1\ \ldots\ \lambda_n|\Delta(\vec{a}_1,\ldots,\vec{a}_n) \tag{1.14}$$

$$\Delta(\vec{a}_1,\ldots,\vec{a}_i + \vec{a}_k,\ldots,\vec{a}_n) = \Delta(\vec{a}_1,\ldots,\vec{a}_i,\ldots,\vec{a}_n) \quad (i \neq k). \tag{1.15}$$

Here the determinant $\Delta(A)$ is expressed as a function of the columns $\vec{a}_1,\ldots,\vec{a}_n$ of A, a quaternionic vector $\vec{a} = {}^t(a_1,\ldots,a_n)$ is multiplied on the right by a quaternion λ, hence

$$\vec{a}\lambda = \begin{pmatrix} a_1\lambda \\ \vdots \\ a_n\lambda \end{pmatrix} \tag{1.16}$$

and the modulus of a quaternion $q = a + bi + cj + dk$ is as usual $|q| = (a^2 + b^2 + c^2 + d^2)^{1/2}$. This quaternionic determinant is a particular case of a non-commutative determinant defined by Dieudonné [5, tome II, p.67].

1.6. The basic property of determinants is *multiplicativity*. Let again V be a vector space of finite dimension n. The volume forms on V form a one-dimensional vector space $\Omega(V)$. If A is any linear operator on V, it acts on $\Omega(V)$ by

$$(\omega\cdot A)(\vec{x}_1,\ldots,\vec{x}_n) = \omega(A\vec{x}_1,\ldots,A\vec{x}_n) \tag{1.17}$$

for $\vec{x}_1,\ldots,\vec{x}_n$ in V; to verify this, it suffices to check that if ω satisfies the conditions (1.2) and (1.3), so does $\omega\cdot A$, and this is obvious. Since $\Omega(V)$ is one-dimensional, A acts on $\Omega(V)$ via multiplication by a scalar det A, hence

$$\omega(A\vec{x}_1,\ldots,A\vec{x}_n) = (\det A)\ \omega(\vec{x}_1,\ldots,\vec{x}_n) \quad . \tag{1.18}$$

The multiplicative property

$$\det(AB) = \det A \cdot \det B \tag{1.19}$$

follows immediately.

A matrix $A = (a_{ij})$ of size n can be viewed as an operator acting on $\mathbb{C}^n$, transforming the vector $\vec{x}$ with components $x_1,\ldots,x_n$ into the vector $\vec{y} = A\vec{x}$ with components $y_1,\ldots,y_n$ given by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \quad . \qquad (1.20)$$

In particular, the columns of A are $\vec{a}_1 = A\vec{e}_1, \ldots, \vec{a}_n = A\vec{e}_n$, and the determinant of A as an operator in $\mathbb{C}^n$ is the one defined in section 1.2. The multiplicative property (1.19) therefore applies to matrices.

1.7. It is not the place to review the *numerical methods* used to evaluate determinants. Needless to say, the inductive definition afforded by (1.1) is not practical unless n is very small, since it requires n! operations.

The multiplicative property can be used to give various characterizations of determinants. By the elementary matrix $M_{ij}(\lambda)$ we mean the matrix differing from the unit matrix I_n by the entry in row i and column j being put equal to λ (here $i \neq j$). If we denote the diagonal matrix with diagonal entries $c_1, \ldots, c_n$ as $\mathrm{diag}(c_1, \ldots, c_n)$, we have

$$\det M_{ij}(\lambda) = 1 \quad , \qquad (1.21)$$

$$\det \mathrm{diag}(c_1, \ldots, c_n) = c_1 \ldots c_n \quad . \qquad (1.22)$$

If a square matrix A has columns $\vec{a}_1, \ldots, \vec{a}_n$, the columns of $AM_{ij}(\lambda)$ are $\vec{a}_1, \ldots, \vec{a}_n$ except that $\vec{a}_j$ is replaced by $\vec{a}_j + \vec{a}_i\lambda$. Similarly, if $\vec{r}_1, \ldots, \vec{r}_n$ are the rows of A, the matrix $M_{ij}(\lambda)A$ differs from A by replacing the row $\vec{r}_i$ by $\vec{r}_i + \lambda\vec{r}_j$. By pre- or postmultiplying the matrix A by elementary matrices, we can therefore achieve to transform A into a diagonal matrix. Noticing that the matrices $AM_{ij}(\lambda)$, $M_{ij}(\lambda)A$ and A have all the same determinant, we conclude that the multiplicativity property together with formulas (1.21) and (1.22) characterizes the determinant.

1.8. A systematic procedure of this kind is known as *Gauss' pivoting method*. Put generally, assume that A is given in block form

$$A = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

where the sizes of the blocks are as follows

$$M \text{ is } p \times p, \quad N \text{ is } p \times q, \qquad (p+q=n)$$
$$P \text{ is } q \times p, \quad Q \text{ is } q \times q \quad .$$

If X if any q x p matrix and Y any p x q matrix, one gets

$$\begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix} = \begin{pmatrix} M & MY+N \\ XM+P & XMY+XN+PY+Q \end{pmatrix} .$$

Moreover, the matrices $\begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix}$ and $\begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}$ are products of elementary matrices, hence of determinant 1. If M is invertible, choose X and Y such that $XM+P = 0$ and $MY+N = 0$. Hence

$$\begin{pmatrix} I_p & 0 \\ -PM^{-1} & I_q \end{pmatrix} \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \begin{pmatrix} I_p & -M^{-1}N \\ 0 & I_q \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & Q - PM^{-1}N \end{pmatrix}$$

and the final result reads as follows

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det M \cdot \det(Q - PM^{-1}N) \quad . \tag{1.23}$$

Hence an n-th order determinant is expressed as product of determinants of order p and q respectively.

The case $p = 1$ is worth recording. After permuting if necessary some rows and columns, one obtains the rule:

- choose a nonzero entry a_{ij} in A, known as *pivot* ;
- erase from A the row and column containing the pivot;
- replace any remaining element a_{kl} by

  $b_{kl} = a_{kl} - a_{kj}\, a_{ij}^{-1}\, a_{il}$;
- compute the determinant b of the $(n-1)(n-1)$ matrix with entries b_{kl};
- multiply b by $(-1)^{i+j}\, a_{ij}$ to get the determinant of A.

1.9. Two final remarks are in order. The previous methods can be used to compute the *quaternionic determinants* with two small changes. Namely change $c_1 \ldots c_n$ into $|c_1 \ldots c_n|$ into formula (1.22) and use $\det A = b \cdot |a_{ij}|$ in the last step of the pivoting method.

Moreover, in formula (1.23) assume that $p = q$ and that the square matrices M, N, P, Q of size p commute pairwise. Then one gets

$$\det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det (MQ - NP) \quad . \tag{1.24}$$

Using a limiting argument, one gets rid of the restriction $\det M \neq 0$. Now the determinant of a 2×2 matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ is $mq - np$.

Therefore the rule: use the formula to compute a 2×2 determinant replacing the scalar entries by the commuting $p \times p$ matrices, then take the determinant of the resulting $p \times p$ matrix. This rule can be generalized when the matrix A is in block form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix}$$

whenever the $p \times p$ matrices A_{ij} commute pairwise (and $pr = n$). This is known as *Williamson's theorem.*

2. Symmetry Properties of Tensors

2.1. Let us recall the notion of *tensor product* of two vector spaces V and W. This is a new vector space denoted by $V \otimes W$; moreover, to any pair of vectors (x,y) in $V \times W$ is associated an element $x \otimes y$ of $V \otimes W$ and the following properties hold:

a) *for any* y *in* W, *the map* $x \mapsto x \otimes y$ *from* V *into* $V \otimes W$ *is linear;*

b) *as* a) *with* V *and* W *interchanged;*

c) *if* (e_α) *is a basis of* V *and* (f_β) *a basis of* W, *then the vectors* $e_\alpha \otimes f_\beta$ *form a basis of* $V \otimes W$.

With the previous notations, any element x of V has components x^α such that $x = \sum_\alpha x^\alpha e_\alpha$; similarly for any element $y = \sum_\beta y^\beta f_\beta$ of W and any element $t = \sum_{\alpha,\beta} t^{\alpha\beta} e_\alpha \otimes f_\beta$ of $V \otimes W$. Moreover if $t = x \otimes y$, one gets $t^{\alpha\beta} = x^\alpha y^\beta$.

Notice that we do not assume the dimensions of V and W to be finite. In a similar way, one defines the triple tensor product $U \otimes V \otimes W$ as $(U \otimes V) \otimes W$, etc.

Tensor products of spaces help transform multilinear functions into linear functions. For instance, if $\varphi(x,y)$ is an element of a space T depending linearly on x in V for y fixed in W, and symmetrically depending linearly on y in W for x fixed in V, there exists a unique linear map Φ from $V \otimes W$ into T such that $\varphi(x,y) = \Phi(x \otimes y)$. This follows from c) above, namely

$$\Phi(t) = \sum_{\alpha,\beta} t^{\alpha\beta}\varphi(e_\alpha, f_\beta) \tag{2.1}$$

for $t = \sum_{\alpha,\beta} t^{\alpha\beta} e_\alpha \otimes f_\beta$.

2.2. We fix now a vector space V of finite dimension n, and if necessary we use coordinates with respect to a fixed basis $e_1,\ldots,e_n$ of V. Fix an integer $k \geq 2$ and denote by $V^{\otimes k}$ the tensor product $V_1 \otimes \ldots \otimes V_k$ where all spaces $V_1,\ldots,V_k$ are put equal to V. By convention we set $V^{\otimes 0} = \mathbb{C}$ and $V^{\otimes 1} = V$. A general element in $V^{\otimes k}$ is written as a tensor

$$t = \sum_{\alpha_1\ldots\alpha_k} t^{\alpha_1\ldots\alpha_k} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \tag{2.2}$$

where the indices $\alpha_1, \ldots, \alpha_k$ run over 1, ..., n independently.

Let us denote by S_k *the group of permutations* of the integers 1,2,...,k. *It operates on* $V^{\otimes k}$ in such a way that

$$\sigma\cdot(x_1\otimes \ldots\otimes x_k) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(k)} \tag{2.3}$$

for $x_1,\ldots,x_k$ in V. In components:

$$(\sigma\cdot t)^{\alpha_1\ldots\alpha_k} = t^{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(k)}} \quad . \tag{2.4}$$

Moreover, the rule of operation is satisfied

$$(\sigma\cdot\tau)\cdot t = \sigma\cdot(\tau\cdot t) \tag{2.5}$$

for t in $V^{\otimes k}$ and σ, τ in S_k.

2.3. The action of the symmetry group on $V^{\otimes k}$ enables one to define the *symmetric part* S^kV of $V^{\otimes k}$ and the *antisymmetric part* Λ^kV of $V^{\otimes k}$.

By definition, S^kV consists of the tensors t in $V^{\otimes k}$ such that $\sigma \cdot t = t$ for all σ in S_k, or what is the same, the components $t^{\alpha_1 \cdots \alpha_k}$ are unchanged by any permutation of the indices $\alpha_1, \ldots, \alpha_k$. If $x_1, \ldots, x_k$ are vectors in V, we denote simply by $x_1 \ldots x_k$ their *symmetric product* $\sum_{\sigma \in S_k} \sigma \cdot (x_1 \otimes \ldots \otimes x_k)$. Then the vector space S^kV has a basis consisting of the symmetric products $e_{\alpha_1} \ldots e_{\alpha_k}$ for $\alpha_1, \ldots, \alpha_k$ in ascending order $\alpha_1 \leq \ldots \leq \alpha_k$. These products can also be expressed as monomials

$$\underbrace{e_1 \ldots e_1}_{\beta_1} \; \underbrace{e_2 \ldots e_2}_{\beta_2} \cdots\cdots \underbrace{e_n \ldots e_n}_{\beta_n}$$

with integers $\beta_1, \ldots, \beta_n$ such that $\beta_1 \geq 0, \ldots, \beta_n \geq 0$, $\beta_1 + \ldots + \beta_n = k$.

Otherwise stated, S^kV can be interpreted as the set of polynomials in $e_1, \ldots, e_n$, homogeneous of degree k. Its dimension is equal to $\binom{n+k-1}{k}$.

2.4. For a permutation σ, the *number* $I(\sigma)$ *of inversions* is the number of pairs of integers i,j such that $1 \leq i < j \leq k$, $\sigma(i) > \sigma(j)$.

The *signature* $\mathrm{sgn}\sigma$ is defined as $(-1)^{I(\sigma)}$; its main property is expressed by

$$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}\sigma \cdot \mathrm{sgn}\tau \quad . \tag{2.6}$$

Then Λ^kV consists of the tensors t such that $\sigma \cdot t = (\mathrm{sgn}\sigma) \cdot t$ for all σ in S_k; alternatively, the component $t^{\alpha_1 \cdots \alpha_k}$ vanishes when two of the indices $\alpha_1, \ldots, \alpha_k$ are equal and is multiplied by - 1 if one interchanges two indices. Hence, up to a sign, the nonzero components of t are the $t^{\alpha_1 \cdots \alpha_k}$ for $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. Put in another form introduce the *wedge product* $x_1 \wedge \ldots \wedge x_k$ of the vectors $x_1, \ldots, x_k$ by

$$x_1 \wedge \ldots \wedge x_k = \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) \, \sigma \cdot (x_1 \otimes \ldots \otimes x_k) \quad . \tag{2.7}$$

Then the vector space $\Lambda^k V$ is spanned by such products and a basis of $\Lambda^k V$ consists of the products $e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_k}$ for $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. Hence $\Lambda^k V$ is of dimension $\binom{n}{k}$.

2.5. *The case* $k=n$ *is particularly interesting*. The vector space $\Lambda^n V$ is one-dimensional, spanned by the products $x_1 \wedge \ldots \wedge x_n$ for $x_1, \ldots, x_n$ running over V, and a basis is given by $e_1 \wedge \ldots \wedge e_n$. We shall follow Grothendieck's convention and denote $\Lambda^n V$ by det V (recall n is the dimension of V).

The volume forms on V can be interpreted as linear forms on det V. More precisely, for any volume form ω on V there exists a unique linear form $\tilde{\omega}$ on det V such that

$$\omega(x_1, \ldots, x_n) = \tilde{\omega}(x_1 \wedge \ldots \wedge x_n) \quad . \tag{2.8}$$

Conversely, for any linear form φ on det V, the formula

$$\omega(x_1, \ldots, x_n) = \varphi(x_1 \wedge \ldots \wedge x_n) \tag{2.9}$$

defines a volume form ω on V.

2.6. Let W be a subspace of V. Denote by m the dimension of W, and by V/W the factor space (of dimension n-m). Let p be the natural projection of V onto V/W. Then one can identify the spaces det V and det W $\otimes$ det(V/W) in such a way that the wedge product $x_1 \wedge \ldots \wedge x_n$ in det V is identified to $(x_1 \wedge \ldots \wedge x_m) \otimes (p(x_{m+1}) \wedge \ldots \wedge p(x_n))$ in case $x_1, \ldots, x_m$ belong to W.

Dually, if ω_W is a volume form on W and $\omega_{V/W}$ a volume form on V/W, there exists a unique volume form ω_V on V such that

$$\omega_V(x_1, \ldots, x_n) = \omega_W(x_1, \ldots, x_m)\, \omega_{V/W}(p(x_{m+1}), \ldots, p(x_n)) \tag{2.10}$$

for $x_1, \ldots, x_m$ in W and $x_{m+1}, \ldots, x_n$ in V.

A variant is obtained by choosing a subspace H of V such that V be a direct sum of W and H. We can identify det V to det W $\otimes$ det H in such a way that $x_1 \wedge \ldots \wedge x_m \wedge y_1 \wedge \ldots \wedge y_{n-m}$ corresponds to $(x_1 \wedge \ldots \wedge x_m) \otimes (y_1 \wedge \ldots \wedge y_{n-m})$ for $x_1, \ldots, x_m$

in W and $y_1,\ldots,y_{n-m}$ in H. Similarly, a volume form ω_V on V is obtained from a volume form ω_W on W and a volume form ω_H on H. Namely

$$\omega_V(x_1,\ldots,x_m,y_1,\ldots,y_{n-m}) = \omega_W(x_1,\ldots,x_m)\omega_H(y_1,\ldots,y_{n-m}) \tag{2.11}$$

for $x_1,\ldots,x_m$ in W and $y_1,\ldots,y_{n-m}$ in H. The construction given in section 1.4 corresponds to the particular case m = 1.

2.7. *Let* A *be a linear operator acting on* V. Then A acts on $V^{\otimes k}$ in such a way that

$$A\cdot(x_1\otimes \ldots \otimes x_k) = Ax_1\otimes \ldots \otimes Ax_k \tag{2.12}$$

for $x_1,\ldots,x_k$ in V. In components, if A is represented by the matrix (a_α^β) in such a way that $Ae_\alpha = \sum_\beta a_\alpha^\beta e_\beta$, then a tensor t with components $t^{\alpha_1\ldots\alpha_k}$ is transformed by A into a tensor At with components

$$(A\cdot t)^{\alpha_1\ldots\alpha_k} = \sum_{\beta_1,\ldots,\beta_k} a_{\beta_1}^{\alpha_1} \ldots a_{\beta_k}^{\alpha_k} t^{\beta_1\ldots\beta_k} . \tag{2.13}$$

This is in accordance with the standard rules of tensor calculus.

It follows from the formulas (2.3) and (2.12) that A acting on $V^{\otimes k}$ commutes with the action of the symmetry operators. Hence it leaves invariant the subspaces S^kV and Λ^kV. As a matter of notation, we denote by $A^{\otimes k}$ the operator in $V^{\otimes k}$ given by the action of A, by S^kA its restriction to S^kV and by Λ^kA its restriction to Λ^kV. For the three kinds of products of vectors, we have therefore the rules

$$A^{\otimes k}(x_1\otimes \ldots \otimes x_k) = Ax_1\otimes \ldots \otimes Ax_k \quad , \tag{2.14}$$

$$S^kA(x_1 \ldots x_k) = Ax_1 \ldots Ax_k \quad , \tag{2.15}$$

$$\Lambda^kA(x_1\wedge \ldots \wedge x_k) = Ax_1\wedge \ldots \wedge Ax_k \quad . \tag{2.16}$$

If we restrict A to the group GL(V) of invertible linear transformations in V, we get thus three linear representations of GL(V) in the spaces $V^{\otimes k}$, S^kV and Λ^kV respectively.

In particular, $\Lambda^n A$ is an operator in the one-dimensional space $\Lambda^n V$ = det V. It follows from the duality of det V and $\Omega(V)$ and from formula (1.20) that $\Lambda^n A$ acts by multiplication by det A on det V. Otherwise stated, one has

$$Ax_1 \wedge \ldots \wedge Ax_n = (\det A)\cdot(x_1 \wedge \ldots \wedge x_n) \tag{2.17}$$

for $x_1,\ldots,x_n$ in V. This is yet another characterization of the determinant.

Assume that W is a subspace of V, stable under A. Choose the basis $e_1,\ldots,e_n$ of V in such a way that $e_1,\ldots,e_m$ constitute a basis of W. The matrix of A in this basis is in block form

$$A = \begin{pmatrix} M & N \\ 0 & P \end{pmatrix} ,$$

where the sizes are as follows: M is m x m, N is m x (n-m), and P is (n-m) x (n-m).

Then the identification of det V and det W $\otimes$ det(V/W) amounts more or less to the classical determinant formula

$$\det A = \det M \cdot \det P \quad . \tag{2.18}$$

2.8. It is the aim of *supersymmetry* to unify bosons and fermions, or in algebraic terms to unify symmetric and antisymmetric tensors. We present here a general method inspired by recent work on *Yang-Baxter equation*.

Fix an integer $k \geq 2$, and define elements $s_1,\ldots,s_{k-1}$ in the symmetric group S_k by

$$s_i(j) = \begin{cases} j & \text{if } j \neq i, i+1 \\ i+1 & \text{if } j = i \\ i & \text{if } j = i+1 \end{cases} \tag{2.19}$$

(s_i is the interchange, or transposition, of i and i+1). It is a classical theorem in group theory (Moore, 1894) that these elements generate the group S_k and that a complete list of relations among these generators is as follows:

$$s_i^2 = 1 \qquad \text{for } i = 1,\dots,k-1 \tag{2.20}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| \geq 2 \tag{2.21}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } i = 1,\dots,k-2 \tag{2.22}$$

In graphical terms, the generators $s_1,\dots,s_{k-1}$ correspond to the nodes of a chain

two generators commute if the corresponding nodes are not directly connected in the chain, and the relation (2.22) holds for pairs of generators corresponding to nodes adjacent to the same edge.

Let now T be any operator acting on $V^{\otimes 2}$. We denote by $T^{i,i+1}$ the operator acting on $V^{\otimes k}$ in such a way that

$$T^{i,i+1}(x_1 \otimes \dots \otimes x_k) = x_1 \otimes \dots \otimes x_{i-1} \otimes T(x_i \otimes x_{i+1}) \otimes \otimes x_{i+2} \otimes \dots \otimes x_k \quad . \tag{2.23}$$

One could in a similar way define more generally operators $T^{i,j}$ acting on the factors of ranks i and j of $V^{\otimes k}$. This construction can be illustrated by a *simple quantum-mechanical model*. We consider a system of k particles labelled 1,2,...,k, subjected to *pair interaction*. If V is the space of one-particle states, $V^{\otimes k}$ describes the states of the system. The elementary interaction law is expressed by a *two-body potential*, an operator T in $V^{\otimes 2}$; then $T^{i,j}$ is the contribution to the potential energy stemming from the pair of particles labelled i and j.

When $|i-j|$ is at least 2, the sets $\{i,i+1\}$ and $j\{j,j+1\}$ are disjoint and the operators $T^{i,i+1}$ and $T^{j,j+1}$ commute. From the equations (2.20) to (2.22), one concludes that there exists a linear representation π_T of the group S_k in the space $V^{\otimes k}$ such that $\pi_T(s_i) = T^{i,i+1}$ (for $1 \leq i \leq k-1$) iff the following conditions are satisfied

We denote by $\Sigma^k V$ the subspace of $V^{\otimes k}$ consisting of the tensors t such that $\sigma \cdot t = t$ for every σ in S_k. We define a product $x_1 \dots x_k$ for vectors by

$$x_1 \dots x_k = \sum_{\sigma \in S_k} \sigma \cdot (x_1 \otimes \dots \otimes x_k) \qquad (2.30)$$

It reduces to the symmetric product of vectors if $x_1, \dots, x_k$ are all even, to the wedge product $x_1 \wedge \dots \wedge x_k$ if $x_1, \dots, x_k$ are all odd. The space $\Sigma^k V$ decomposes as a direct sum of subspaces $\Sigma^{B,F} V$ where $B \geq 0$, $F \geq 0$, $B + F = k$. The number B is the *bosonic number* and F the *fermionic number*, and $\Sigma^{B,F} V$ is generated by the products $(x_1 \dots x_B) \cdot (y_1 \wedge \dots \wedge y_F)$ where $x_1, \dots, x_B$ are even and $y_1, \dots, y_F$ are odd; it can be identified in a natural way with $S^B V^+ \otimes \Lambda^F V^-$. In particular, if $V = V^+$ (that is $\varepsilon = 1$) then $\Sigma^k V$ reduces to $S^k V$, and if $V = V^-$ (that is $\varepsilon = -1$) then $\Sigma^k V$ reduces to $\Lambda^k V$. The elements in $\Sigma^{B,F} V$ are considered as even or odd according as F is even or odd (parity by fermionic number).

3. Mac Mahon's Master Theorem

3.1. We keep the previous notations. For instance, V is a complex vector space of finite dimension n. We defined previously the antisymmetric tensor spaces $\Lambda^0 V = C$, $\Lambda^1 V = V$, $\Lambda^2 V, \dots, \Lambda^n V = \det V$. Moreover, $\Lambda^k V$ is reduced to 0 for $k > n$. Denote by ΛV the direct sum of the space $\Lambda^0 V, \dots, \Lambda^n V$. We define parity on ΛV by the rules

$$\Lambda^+ V = \bigoplus_{k \geq 0} \Lambda^{2k} V \qquad (3.1)$$

$$\Lambda^- V = \bigoplus_{k \geq 0} \Lambda^{2k+1} V \quad . \qquad (3.2)$$

The wedge product of vectors can be extended to a multiplication in ΛV, which is bilinear, associative, complying with the signe rule

$$\xi \wedge \eta = (-1)^{pq} \eta \wedge \xi \qquad (3.3)$$

$$TT = 1 \quad \text{in } V^{\otimes 2} \qquad \textit{"Unitarity"} \tag{2.24}$$

$$T^{12}T^{23}T^{12} = T^{23}T^{12}T^{23} \quad \text{in } V^{\otimes 3} \qquad \textit{"Yang-Baxter equation"} \tag{2.25}$$

2.9. Let us assume that ε is an operator on V such that $\varepsilon^2=1$. Then V is the direct sum of V^+ and V^-, where V^+ corresponds to the eigenvalue +1 of ε and V^- to the eigenvalue -1. A vector in V^+ is called *even*(or bosonic) and a vector in V^- is called *odd*(or fermionic). We consider the operator T in $V^{\otimes 2} \equiv V \otimes V$ characterized by

$$\begin{aligned} T(x_+ \otimes y_+) &= y_+ \otimes x_+ \\ T(x_+ \otimes y_-) &= y_- \otimes x_+ \\ T(x_- \otimes y_+) &= y_+ \otimes x_- \\ T(x_- \otimes y_-) &= -y_- \otimes x_- \end{aligned} \tag{2.26}$$

where x_+ and y_+ are even, and x_-, y_- are odd. This is the well-known *Koszul's sign rule*: "insert a factor -1 each time you permute two odd factors". The properties (2.24) and (2.25) are easily checked, hence we get an action of the symmetric group S_k on $V^{\otimes k}$. Explicitely, one gets

$$\sigma\cdot(x_1\otimes \ldots \otimes x_k) = (-1)^I\, x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(k)} \tag{2.27}$$

where I is the number of pairs (i,j) of integers such that $1 \leq i < j \leq k$, $\sigma(i) > \sigma(j)$ and x_i, x_j are both odd (we assume that $x_1,\ldots,x_k$ have well-defined parities, either even or odd). In particular

$$\sigma\cdot(x_1\otimes \ldots \otimes x_k) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(k)} \quad \text{if } x_1, \ldots, x_k \text{ are all even} \tag{2.28}$$

$$\sigma\cdot(x_1\otimes \ldots \otimes x_k) = \operatorname{sgn}\sigma\cdot(x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(k)}) \quad \text{if } x_1,\ldots,x_k \text{ are all odd.} \tag{2.29}$$

if ξ belongs to $\Lambda^p V$ and η to $\Lambda^q V$. Even elements commute to even or odd elements and $\xi \wedge \eta = -\eta \wedge \xi$ if both ξ and η are odd. If $\xi = x_1 \wedge \ldots \wedge x_p$ and $\eta = y_1 \wedge \ldots \wedge y_q$, then $\xi \wedge \eta$ is by definition the wedge product of vectors $x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_q$.

3.2. Let A be an operator acting linearly on V. We defined the operator $\Lambda^k A$ acting on $\Lambda^k V$. We denote by ΛA the operator on ΛV which coincides with $\Lambda^k A$ on $\Lambda^k V$ for $k = 0,1,\ldots,n$. Then ΛA extends the operator A on $V = \Lambda^1 V$ and respects multiplication

$$\Lambda A(\xi \wedge \eta) = (\Lambda A)\xi \wedge (\Lambda A)\cdot\eta \quad . \tag{3.4}$$

The *first statement* in Mac Mahon's theorem is the following formula

$$\det(1 + tA) = \sum_{k=0}^{n} t^k \mathrm{Tr}(\Lambda^k A) \quad , \tag{3.5}$$

where t is a complex parameter (or a formal variable).

The proof is especially simple if A can be *diagonalized.* Assume that there exists a basis $e_1,\ldots,e_n$ of V consisting of eigenvectors of A, namely

$$Ae_1 = \lambda_1 e_1, \ \ldots, \ Ae_n = \lambda_n e_n \quad . \tag{3.6}$$

Then the tensors $e_{i_1} \wedge \ldots \wedge e_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq n$ form a basis of $\Lambda^k V$ and $\Lambda^k A$ multiplies such a tensor by $\lambda_{i_1} \ldots \lambda_{i_k}$. Hence the trace of $\Lambda^k A$ is the elementary symmetric function $\sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k} = c_k$ of $\lambda_1, \ldots, \lambda_n$. Moreover, the operator $1+tA$ multiplies every basis vector e_i by $1 + t\lambda_i$. Hence formula (3.5) reduces to the classical statement

$$\prod_{i=1}^{n} (1 + t\lambda_i) = \sum_{k=0}^{n} c_k t^k \quad . \tag{3.7}$$

3.3. In the general case, we could argue by continuity since it can be proved that any matrix (or operator) is a limit of matrices conjugate to diagonal ones. Instead, we shall rely on the possibility of finding for the operator A a basis $e_1,\ldots,e_n$

of V such that the matrix (a_i^j) expressing A in this basis be *upper triangular* ($a_i^j = 0$ if $i < j$).

Let us not assume at first that the matrix (a_i^j) be triangular. For every increasing sequence $I = (i_1 < \ldots < i_k)$ of indices between 1 and n, denote by e_I the wedge product $e_{i_1} \wedge \ldots \wedge e_{i_k}$. By definition, we have $Ae_i = \sum_j a_i^j e_j$. It follows

$$(\Lambda^k A)\cdot e_I = \sum_J a_I^J e_J \tag{3.8}$$

where a_I^J is the minor of A corresponding to the set I of columns and the set J of rows. In particular, we get

$$\operatorname{Tr}(\Lambda^k A) = \sum_I a_I^I \quad , \tag{3.9}$$

where the sum is extended over all sequences $I = (i_1 < \ldots < i_k)$ of length k.

Assuming now the matrix (a_i^j) to be triangular with diagonal elements $\lambda_1, \ldots, \lambda_n$, then a_I^I is the determinant of a triangular matrix with diagonal elements $\lambda_{i_1}, \ldots, \lambda_{i_k}$. By a well-known generalization of formula (2.18), we get $a_I^I = \lambda_{i_1} \ldots \lambda_{i_k}$, hence $\sum_I a_I^I = c_k$ and one concludes the proof as before.

We conclude with two remarks:

a) Putting together formulas (3.5) and (3.9) we get

$$\det(1 + tA) = \sum_{k=0}^{n} t^k \sum_{|I|=k} a_I^I \quad , \tag{3.10}$$

a well-known formula for the characteristic determinant. Here $|I|$ is the length of the increasing sequence I.

b) We can put $t = 1$ in formula (3.5). Hence we get

$$\det(1 + A) = \operatorname{Tr}(\Lambda A) = \operatorname{Tr}(\Lambda^+ A) + \operatorname{Tr}(\Lambda^- A) \quad . \tag{3.11}$$

Putting $t = -1$, we get

$$\det(1 - A) = \operatorname{Tr}(\Lambda^+ A) - \operatorname{Tr}(\Lambda^- A) \tag{3.12}$$

The right-hand side is the so-called *supertrace* of A acting on the space ΛV with even part $\Lambda^+ V$ and odd part $\Lambda^- V$.

3.4. We study now the action of A in the *symmetric algebra* $SV = \bigoplus_{k\geq 0} S^kV$. The *symmetric product of vectors* can be extended to a multiplication in SV which is bilinear, associative as well as commutative. Choosing a basis $e_1,\ldots,e_n$ of V enables one to consider the elements of SV as the polynomials with complex coefficients in $e_1,\ldots,e_n$.

The operator A on V defines operators S^kA acting on S^kV. We denote by SA the operator on SV which restricts to S^kA on the subspace S^kV of polynomials homogeneous of degree k. It respects the symmetric product.

With these notations, the *second statement* in Mac Mahon's theorem is the following formula

$$\det(1 - tA)^{-1} = \sum_{k\geq 0} t^k \, \mathrm{Tr}(S^kA) \quad . \tag{3.13}$$

The proof is quite similar to the previous one. Assume first that A can be put in diagonal form as in formula (3.6). Then the monomials $e_1^{\beta_1} \ldots e_n^{\beta_n}$ with $\beta_1 + \ldots + \beta_n = k$ form a basis of S^kV, and A multiplies such a monomial by $\lambda_1^{\beta_1} \ldots \lambda_n^{\beta_n}$. Hence we get for the trace

$$\mathrm{Tr}(S^kA) = \sum_{\beta_1+\ldots+\beta_n=k} \lambda_1^{\beta_1} \ldots \lambda_n^{\beta_n} \quad , \tag{3.14}$$

and therefore

$$\begin{aligned}
\sum_{k\geq 0} t^k \mathrm{Tr}(S^kA) &= \sum_{\beta_1\ldots\beta_n} t^{\beta_1+\ldots+\beta_n} \lambda_1^{\beta_1}\ldots\lambda_n^{\beta_n} \\
&= \sum_{\beta_1\ldots\beta_n} (t\lambda_1)^{\beta_1} \ldots (t\lambda_n)^{\beta_n} \\
&= \sum_{\beta_1=0}^{\infty} (t\lambda_1)^{\beta_1} \ldots \sum_{\beta_n=0}^{\infty} (t\lambda_n)^{\beta_n} \\
&= \frac{1}{1-t\lambda_1} \cdot \ldots \cdot \frac{1}{1-t\lambda_n} \quad .
\end{aligned}$$

But the determinant of 1-tA is the product $(1-t\lambda_1)\ldots(1-t\lambda_n)$, hence formula (3.13).

In the general case, assume that A can be put in triangular form. It means that there exists a basis $e_1,\dots,e_n$ of V and scalars $\lambda_1,\dots,\lambda_n$ such that $Ae_i - \lambda e_i$ be a linear combination of the vectors e_j for $j < i$. The monomial $e_1^{\beta_1} \dots e_n^{\beta_n}$ is transformed by S^kA into $(Ae_1)^{\beta_1} \dots (Ae_n)^{\beta_n}$. Expanding this product, it is easy to see that the coefficient of $e_1^{\beta_1} \dots e_n^{\beta_n}$ into $(Ae_1)^{\beta_1} \dots (Ae_n)^{\beta_n}$ is equal to $\lambda_1^{\beta_1} \dots \lambda_n^{\beta_n}$. Formula (3.14) is still valid, and the determinant of $1-tA$ is still the product $(1-t\lambda_1) \dots (1-t\lambda_n)$. The proof is finished as before.

3.5. It is interesting to put together formulas (3.5) and (3.13). We get

$$1 = \{\sum_{m=0}^{\infty} t^m \mathrm{Tr}(S^mA)\}\cdot\{\sum_{k=0}^{n} (-1)^k t^k \mathrm{Tr}(\Lambda^k A)\} . \tag{3.15}$$

Comparing the coefficients of the powers of t, we get

$$\sum_{B+F=k} (-1)^F \mathrm{Tr}(S^BA)\cdot\mathrm{Tr}(\Lambda^FA) = 0 \quad \text{for } k \geq 1 . \tag{3.16}$$

This formula can be given a *supersymmetric* interpretation.

Denote by W the *double* V x V of V, with a parity operator given by $\varepsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ in matrix form, and a parity changing operator $\pi = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Hence π exchanges W^+ and W^-, and $\pi^2 = 1$. Both W^+ and W^- are copies of V. In physical slang, π associates to every boson state in W^+ its fermionic partner in W^-. From what we saw in section 2.9, we can identify Σ^kW with the direct sum of the spaces $\Sigma^{B,F}W = S^BW^+ \otimes \Lambda^FW^-$ (for $B+F = k$), and in turn to the sum of the spaces $S^BV \otimes \Lambda^FV$.

We extend A to W, in matrix form $\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix}$, that is $A(x,y) = (Ax,Ay)$ for x,y in V. We then extend A to an operator $\tilde{A}$ acting on Σ^kW in such a way that

$$\tilde{A}(x_1 \dots x_k) = Ax_1 \dots Ax_k \tag{3.17}$$

for $x_1,\dots,x_k$ in W. Then the subspaces $\Sigma^{B,F}W$ are stable under

$\tilde{A}$, and if we identify $\Sigma^{B,F}W$ to $S^BV \otimes \Lambda^FV$, the action of $\tilde{A}$ on $\Sigma^{B,F}W$ is given by the operator $S^BA \otimes \Lambda^FA$. Let now $\tilde{\varepsilon}$ be the operator on Σ^kW which acts by multiplication by $(-1)^F$ on $\Sigma^{B,F}W$. It is related to the parity operator in W by

$$\tilde{\varepsilon}(x_1 \dots x_k) = \varepsilon x_1 \dots \varepsilon x_k \qquad (3.18)$$

for $x_1,\dots,x_k$ in W.

After all these preparations, the identity (3.16) turns out as a *supertrace vanishing theorem*

$$\mathrm{Tr}(\tilde{\varepsilon}\tilde{A}) = 0 \qquad (\text{for } k \geq 1) \qquad (3.19)$$

(trace of operators acting on Σ^kW). A direct proof can be given using two new operators d and s acting on Σ^kW by

$$d(x_1 \dots x_B \cdot y_1 \dots y_F) = \sum_{i=1}^{B} x_1 \dots x_{i-1} \pi x_i x_{i+1} \dots x_B \cdot y_1 \dots y_F \qquad (3.20)$$

$$s(x_1 \dots x_B \cdot y_1 \dots y_F) = \sum_{j=1}^{F} (-1)^{j-1} x_1 \dots x_B \cdot y_1 \dots y_{j-1} \pi y_j y_{j+1} \dots y_F \qquad (3.21)$$

where x_i's are *even* and the y_j's are *odd*. Since d changes the fermion number F by +1 and s changes it by -1, one gets

$$d\tilde{\varepsilon} = -\tilde{\varepsilon}d \quad , \quad s\tilde{\varepsilon} = -\tilde{\varepsilon}s \quad . \qquad (3.22)$$

Moreover, from the previous definitions, one gets

$$\tilde{A}\tilde{\varepsilon} = \tilde{\varepsilon}\tilde{A} \quad , \quad \tilde{A}d = d\tilde{A} \quad , \quad \tilde{A}s = s\tilde{A} \quad . \qquad (3.23)$$

It can also be proved that sd + ds multiplies every element of Σ^kW by k (see for instance subsection
From these formulas, one derives

$$\begin{aligned}
\mathrm{Tr}(\tilde{\varepsilon}\tilde{A}sd) &= \mathrm{Tr}(\tilde{A}\tilde{\varepsilon}sd) && \text{because } \tilde{A}\tilde{\varepsilon} = \tilde{\varepsilon}\tilde{A} \\
&= -\mathrm{Tr}(\tilde{A}s\tilde{\varepsilon}d) && \text{because } \tilde{\varepsilon}s = -s\tilde{\varepsilon} \\
&= -\mathrm{Tr}(\tilde{\varepsilon}d\tilde{A}s) && \text{by cyclic invariance of the trace} \\
&= -\mathrm{Tr}(\tilde{\varepsilon}\tilde{A}ds) && \text{because } \tilde{A}d = d\tilde{A}
\end{aligned}$$

From $sd+ds = k$, one gets then $k\mathrm{Tr}(\tilde{\varepsilon}\tilde{A}) = \mathrm{Tr}(\tilde{\varepsilon}\tilde{A}(sd+ds)) = 0$ and $\mathrm{Tr}(\tilde{\varepsilon}\tilde{A}) = 0$ follows provided $k \geq 1$.

3.6. We derive as an application an important formula in group theory, known as *Cartan-Molien's formula*. Suppose G is a finite group, of order $|G|$, acting linearly on the vector space V. We let G act on the symmetric tensor space S^kV in such a way that $g.(x_1 \ldots x_k) = (gx_1) \ldots (gx_k)$ for the symmetric product of vectors $x_1, \ldots, x_k$ in V. Denote by I^k the subspace of S^kV consisting of the invariants of G. The dimension of these spaces is given by the following *generating series*

$$\sum_{k=0}^{\infty} t^k . \dim I^k = |G|^{-1} \sum_{g\in G} \det (1 - tg_V)^{-1} \quad . \tag{3.24}$$

For clarity, we denote by g_V the operator afforded by g on V. A similar formula with $|G|^{-1} \sum_{g\in G}$ replaced by $\int_G dg$ holds for a compact group G acting linearly on V.

The *proof* of formula (3.24) is as follows. Define an operator P_k in S^kV by $P_k u = |G|^{-1} \sum_{g\in G} g\,u$ for u in S^kV. Then P_k is a projection of S^kV onto I^k, hence its trace is equal to the dimension of I^k. But P_k is nothing else than $|G|^{-1} \sum_{g\in G} S^k g_V$, hence

$$\dim I^k = |G|^{-1} \sum_{g\in G} \mathrm{Tr}(S^k g_V) \quad . \tag{3.25}$$

One concludes using formula (3.13).

3.7. So far we have associated to an operator A acting on V the numbers $c_k(A) = \mathrm{Tr}(\Lambda^k A)$ and $h_k(A) = \mathrm{Tr}(S^k A)$. In terms of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, $c_k(A)$, is the *elementary symmetric function of order* k and $h_k(A)$ is the *complete symmetric function of order* k, namely the sum of all monomials of degree k in $\lambda_1,\ldots,\lambda_n$ each with coefficient one. We found the generating series

$$\sum_{k\geq 0} c_k(A)t^k = \det(1+tA) \quad , \tag{3.26}$$

$$\sum_{k\geq 0} h_k(A)t^k = \det(1-tA)^{-1} \quad . \tag{3.27}$$

We introduce now the *sum-of-powers* $\tau_k(A) = \lambda_1^k+\ldots+\lambda_n^k$, or what is the same $\tau_k(A) = \mathrm{Tr}(A^k)$ (put again A in triangular form). The corresponding generating series is expressed in the following various forms:

$$\sum_{k\geq 1} \tau_k(A)t^k = \frac{\lambda_1 t}{1-\lambda_1 t} + \ldots + \frac{\lambda_n t}{1-\lambda_n t} \quad , \tag{3.28}$$

$$\sum_{k\geq 1} \tau_k(A)t^k = t\,\mathrm{Tr}(A\cdot(1-tA)^{-1}) \quad , \tag{3.29}$$

$$\sum_{k\geq 1} \tau_k(A)t^k = -t\frac{d}{dt}\log\det(1-tA) \quad ; \tag{3.30}$$

for the last equality, use the product formula

$$\det(1-tA) = (1-\lambda_1 t)\ldots(1-\lambda_n t) \quad . \tag{3.31}$$

Formula (3.30) can easily be inverted to give

$$\det(1-tA) = \exp - \sum_{k\geq 1} \tau_k(A)t^k/k \quad . \tag{3.32}$$

We can now restate formulas (3.26) and (3.27) in the following form

$$\sum_{k\geq 0} c_k(A)t^k = \exp \sum_{k\geq 1} (-1)^{k-1}\tau_k(A)t^k/k \quad , \tag{3.33}$$

$$\sum_{k\geq 0} h_k(A)t^k = \exp \sum_{k\geq 1} \tau_k(A)t^k/k \quad . \tag{3.34}$$

If we expand the exponentials and compare the coefficients of the various powers of t, we get the following variants of *Waring's formula*:

$$c_k(A) = \Sigma(-1)^{k+\alpha_1+\ldots+\alpha_k} \frac{\tau_1(A)^{\alpha_1}\tau_2(A)^{\alpha_2}\ldots\tau_k(A)^{\alpha_k}}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}\ldots\alpha_k!k^{\alpha_k}} \quad , \tag{3.35}$$

$$h_k(A) = \Sigma \frac{\tau_1(A)^{\alpha_1}\tau_2(A)^{\alpha_2} \dots \tau_k(A)^{\alpha_k}}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2} \dots \alpha_k!k^{\alpha_k}} \quad . \tag{3.36}$$

In both cases, the summation is restricted to the systems $\alpha_1,\dots,\alpha_k$ of positive integers (including 0) such that $1\cdot\alpha_1 + 2\cdot\alpha_2 + \dots + k\cdot\alpha_k = k$. This latest restriction expresses the *homogeneity of the functions* c_k, h_k and τ_k: if we choose a basis of V and represent accordingly A by a matrix (a_{ij}), then $c_k(A)$, $h_k(A)$ and $\tau_k(A)$ are polynomials in the entries a_{ij}, homogeneous of degree k.

If U(t) and V(t) are power series such that U(t)=expV(t), one gets U'(t) = V'(t)U(t) for the derivatives. Using this remark in connection with formulas (3.33) and (3.34), we derive the following recursion formulas (due to Newton) (recall $c_0(A) = h_0(A) = 1$)

$$pc_p(A) = \sum_{j=1}^{p} (-1)^{j-1} \tau_j(A)\, c_{p-j}(A) \quad , \tag{3.37}$$

$$ph_p(A) = \sum_{j=1}^{p} \tau_j(A)\, h_{p-j}(A) \quad . \tag{3.38}$$

From these recursion formulas, one derives *determinantal formulas due to Plemelj,* namely

$$c_1 = \tau_1 \ , \quad 2!c_2 = \det\begin{pmatrix}\tau_1 & \tau_2\\ 1 & \tau_1\end{pmatrix}, \ 3!c_3 = \det\begin{pmatrix}\tau_1 & \tau_2 & \tau_3\\ 2 & \tau_1 & \tau_2\\ 0 & 1 & \tau_1\end{pmatrix}$$

and, in general, $p!c_p(A)$ is a p x p determinant whose nonzero entries are given by $u_{ij} = \tau_{j-i+1}(A)$ for $j \geq i$ and $u_{j+1,j} = p-j$ for j=1,...,p-1. To prove this statement, develop the p-th determinant according to its first row and get back to formula (3.37). To get a similar formula for $h_p(A)$, the best is to use a *duality principle*, which follows obviously from formulas (3.33) and (3.34): *in every formula, one can exchange* $c_1(A)$ *with* $h_1(A),\dots,c_k(A)$ *with* $h_k(A),\dots$ *provided that at the same time one multiplies* $\tau_k(A)$ *by* $(-1)^{k-1}$.

So far, we considered the generating series as formal power series. If we care about *convergence*, the following has to be said:

(a) If n is the dimension of V, then $c_k(A) = 0$ for $k > n$. Hence the series $\sum_{k\geq 0} c_k(A)t^k$ *is a polynomial of degree* n *in* t (and so does $\det(1 + tA)$).

(b) If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A, denote by $||A||_{sp}$ the maximum among the numbers $|\lambda_1|,\ldots,|\lambda_n|$ (the so-called *spectral radius* of A). Then the series $\sum_{k\geq 0} h_k(A)t^k$, $\sum_{k\geq 1} \tau_k(A)t^k$ and $\sum_{k\geq 1} \tau_k(A)t^k/k$ have the same radius of convergence equal to the inverse of $||A||_{sp}$.

3.8. In the preceding subsections, $c_k(A)$, $h_k(A)$ and $\tau_k(A)$ have been considered as symmetric functions of the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A and our relations are in fact theorems about symmetric functions. An operator in the finite-dimensional case can be transformed into a triangular form, and we made extensive use of this possibility. Nothing of this sort exists for operators acting on Hilbert or Banch spaces. It is therefore not without interest to sketch alternate proofs of some of the previous results. In the infinite-dimensional case, formula (3.26) shall be a definition of the determinant, with a proper interpretation of $c_p(A)$. One convinces easily oneself that everything can be deduced from the Waring formulas (3.35) and (3.36).

As Cauchy remarked already, any permutation σ in the group S_k can be decomposed into *cycles*. If there are α_1 cycles of length $1,\ldots,\alpha_k$ cycles of length k, obviously $1\cdot\alpha_1+2\cdot\alpha_2+\ldots+k\cdot\alpha_k = k$ and the signature of σ is $(-1)^{k+\alpha_1+\ldots\alpha_k}$, or $(-1)^{k+s}$ where s is the total number of cycles. Moreover, the number of permutations σ corresponding to given values of $\alpha_1,\ldots,\alpha_k$ is $k!/\alpha_1!1^{\alpha_1}\ldots\alpha_k!k^{\alpha_k}$. Introduce a function on the group S_k whose value $I(\sigma)$ is equal to $\tau_1(A)^{\alpha_1}\ldots\tau_k(A)^{\alpha_k}$.

It is characterized by the following two properties.

(a) I *is a class function, that is* $I(\sigma) = I(\tau\sigma\tau^{-1})$ *for* σ,τ *in* S_k.

(b) *If* σ *is decomposed into cycles* $(1\ldots a)(a+1\ldots a+b)(a+b+1\ldots a+b+c)\ldots$, *then* $I(\sigma)$ *is equal to* $\tau_a(A)\tau_b(A)\tau_c(A)\ldots$

Moreover, our formulas take the following form:

$$k!c_k(A) = \sum_{\sigma\in S_k} (\mathrm{sgn}\sigma)\cdot I(\sigma) \quad , \tag{3.35bis}$$

$$k!h_k(A) = \sum_{\sigma\in S_k} I(\sigma) \quad . \tag{3.36bis}$$

Here is *a direct proof*. For clarity, denote by π_σ the action of the permutation σ on the space $V^{\otimes k}$ defined by formula (2.3), namely

$$\pi_\sigma(x_1\otimes \dots \otimes x_k) = x_{\sigma^{-1}(1)}\otimes \dots \otimes x_{\sigma^{-1}(k)} \quad . \tag{3.39}$$

The operator $P_+ = (k!)^{-1}\sum_{\sigma\in S_k}\pi_\sigma$ is a projection of $V^{\otimes k}$ onto S^kV and similarly $P_-=(k!)^{-1}\sum_{\sigma\in S_k}(\mathrm{sgn}\sigma)\cdot\pi_\sigma$ is a projection of $V^{\otimes k}$ onto Λ^kV; they both commute with $A^{\otimes k}$. The operator $P_+\cdot A^{\otimes k}$ coincides with S^kA on S^kV and with 0 on a subspace of $V^{\otimes k}$ supplementary to S^kV; hence the trace $h_k(A)$ of S^kA is equal to the trace of the operator $P_+.A^{\otimes k}$ acting on $V^{\otimes k}$. Similarly $c_k(A)$ is equal to the trace of $P_-\cdot A^{\otimes k}$. Hence the formulas (3.35bis) and (3.36bis) are true with $I(\sigma)$ defined as $\mathrm{Tr}(\pi_\sigma\cdot A^{\otimes k})$. It remains to check that this function I enjoys properties (a) and (b) above.

As for (a), it follows from the cyclic invariance of the trace and the permutability of $A^{\otimes k}$ with π_τ, namely

$$I(\tau\sigma\tau^{-1}) = \mathrm{Tr}(\pi_\tau\pi_\sigma\pi_\tau^{-1}A^{\otimes k}) = \mathrm{Tr}(\pi_\sigma\pi_\tau^{-1}A^{\otimes k}\pi_\tau) =$$

$$= \mathrm{Tr}(\pi_\sigma A^{\otimes k}) = I(\sigma) \quad .$$

To prove (b), introduce a basis $e_1,\dots,e_n$ of V, and denote by $A(i,j)$ the matrix of A with respect to this basis. The tensor products $e_{i_1}\otimes \dots \otimes e_{i_k}$ make up a basis of $V^{\otimes k}$; the matrix of $A^{\otimes k}$ with respect to this basis is

$$A_k(i_1,\dots,i_k;\, j_1,\dots,j_k) = A(i_1;j_1)\dots A(i_k;j_k) \tag{3.40}$$

and moreover π_σ transforms $e_{i_1}\otimes \dots \otimes e_{i_k}$ into $e_{i_{\sigma^{-1}(1)}}\otimes\dots$ $e_{i_{\sigma^{-1}(k)}}$. It follows

$$I(\sigma) = \sum_{i_1 \dots i_k} A(i_1, i_{\sigma(1)}) \dots A(i_k, i_{\sigma(k)}) \quad . \quad (3.41)$$

If σ is decomposed into cycles $(1\dots a)(a+1\dots a+b)(a+b+1\dots a+b+c) \dots$, then the term under the summation sign can be written as the product of the factors

$$A(i_1, i_2)A(i_2, i_3)\dots A(i_{a-1}, i_a)A(i_a, i_1) = J_a$$

$$A(i_{a+1}, i_{a+2})A(i_{a+2}, i_{a+3})\dots A(i_{a+b-1}, i_{a+b})A(i_{a+b}, i_{a+1}) = J_b$$

. . . .

The sum in (3.41) breaks accordingly into a product. The sum of terms of the form J_a gives the trace of A^a, etc ... hence

$$I(\sigma) = Tr(A^a)Tr(A^b)Tr(A^c)\dots$$

$$= \tau_a(A)\tau_b(A)\tau_c(A) \dots$$

This ends our proof.

One particular case of the previous results is worth mentioning. Namely, consider the cyclic permutation γ_k acting on $V^{\otimes k}$ by

$$\gamma_k(x_1 \otimes x_2 \otimes \dots \otimes x_k) = x_k \otimes x_1 \otimes \dots \otimes x_{k-1} \quad . \quad (3.42)$$

Then we get

$$Tr(A^k) = Tr(A^{\otimes k} \cdot \gamma_k) \quad . \quad (3.43)$$

More generally, let $A_1, \dots, A_k$ be linear operators acting on V; define the operator $A_1 \otimes \dots \otimes A_k$ acting on $V^{\otimes k}$ in the following way

$$(A_1 \otimes \dots \otimes A_k) \cdot (x_1 \otimes \dots \otimes x_k) = A_1 x_1 \otimes \dots \otimes A_k x_k \quad . \quad (3.44)$$

Then

$$Tr(A_1 \dots A_k) = Tr((A_1 \otimes \dots \otimes A_k) \cdot \gamma_k) \quad . \quad (3.45)$$

4. Some Integration Formulas

4.1. The *basic formula* reads as follows, in the simplest form

$$\int_{-\infty}^{+\infty} e^{-\pi x^2}\, dx = 1 \quad . \tag{4.1}$$

The proof uses a well-known trick. Denoting by I the previous integral, we get

$$I^2 = \iint e^{-\pi(x^2+y^2)}\, dx\, dy$$

or, using polar coordinates,

$$I^2 = \int_0^\infty r dr \int_0^{2\pi} e^{-\pi r^2}\, d\theta = \int_0^\infty e^{-\pi r^2}\, 2\pi r dr = \int_0^\infty e^{-u} du = 1 \ . \qquad (u = \pi r^2)$$

Since I is obviously positive, we get I = 1.

4.2. Let E be a *real euclidean space* of dimension n. We denote by $x \cdot y$ the dot product of two vectors. It is bilinear and symmetrical, and $x \cdot x > 0$ unless $x = 0$. Let the basis $e_1, \ldots, e_n$ of E be orthonormal; hence $e_i \cdot e_i = 1$ and $e_i \cdot e_j = 0$ for $i \neq j$. Consider the volume form ω such that $\omega(e_1, \ldots, e_n) = 1$. A classical calculation gives the following formula (where $x_1, \ldots, x_n$ are arbitrary vectors in E)

$$\omega(x_1, \ldots, x_n)^2 = \det(x_i \cdot x_j) \quad . \tag{4.2}$$

In particular, one gets $\omega(e'_1, \ldots, e'_n)^2 = 1$ if $e'_1, \ldots, e'_n$ is another orthonormal basis of E. Up to a sign, ω is therefore independent of any orthonormal basis of E.

To the volume form ω is associated an integration element $d^n x$ in E. Explicitely, one gets

$$\int_E f(x) d^n x = \int \ldots \int f(x^1 e_1 + \ldots + x^n e_n) dx^1 \ldots dx^n \tag{4.3}$$

for every orthonormal basis $e_1, \ldots, e_n$ of E. More generally, from (4.2), one derives

$$\int_E f(x) d^n x = \int \ldots \int f(x^1 e_1 + \ldots + x^n e_n) g^{1/2}\, dx^1 \ldots dx^n \tag{4.4}$$

for an arbitrary basis (e_α) with $g_{\alpha\beta} = e_\alpha \cdot e_\beta$ and $g=\det(g_{\alpha\beta})$.

4.3. From the basic formula (4.1), one gets by multiplication

$$\int \ldots \int \exp\{-\pi \sum_{i=1}^{n} (x^i)^2\} \, dx^1 \ldots dx^n = 1 \quad . \tag{4.5}$$

To get an invariant formula, use (4.3) and get

$$\int_E e^{-\pi x \cdot x} \, d^n x = 1. \tag{4.6}$$

More generally, consider a symmetric positive-definite operator A in E. Hence $Ax \cdot x > 0$ unless $x = 0$, and $Ax \cdot y = x \cdot Ay$. It is legitimate to take as a new dot product of the vectors x and y the scalar $Ax \cdot y$. From subsection 4.2, one gets the existence of a volume form ω_A on E, associated to this dot product, and unique up to sign. From (4.2), one gets

$$\omega_A(x_1, \ldots, x_n)^2 = \det(Ax_i \cdot x_j) \tag{4.7}$$

for an arbitrary basis $x_1, \ldots, x_n$ of E. Specialize x_i to e_i and notice that the matrix (a_{ij}) of A with respect to the orthonormal basis $e_1, \ldots, e_n$ is given by $a_{ij} = Ae_i \cdot e_j$. We get

$$\omega_A(e_1, \ldots, e_n)^2 = \det(a_{ij}) = \det A. \tag{4.8}$$

Comparing with the definition of ω , we conclude

$$\omega_A = \pm (\det A)^{1/2} \, \omega \quad . \tag{4.9}$$

For the corresponding integration elements, we obtain $d^n_A x = (\det A)^{1/2} d^n x$. We can now replace $x \cdot x$ by $Ax \cdot x$ and $d^n x$ by $d^n_A x$ in formula (4.6). *Conclusion*

$$\int_E e^{-\pi Ax \cdot x} \, d^n x = (\det A)^{-1/2} \quad . \tag{4.10}$$

The most customary form is obtained by replacing A by $A/2\pi$ and reads as follows

$$\int_E \exp\{-\tfrac{1}{2} Ax \cdot x\} \, d^n x = (2\pi)^{n/2} (\det A)^{-1/2} \quad . \tag{4.11}$$

The normalization factor $(2\pi)^{n/2}$ may be troublesome when extending these formulas to the infinite-dimensional case.

4.4. We need a *complex form* of the previous formulas. We consider now a finite-dimensional Hilbert space H. That is, H is a complex vector space of dimension n, and there is given a scalar product $\langle x|y\rangle$ for vectors in H, with the following properties

$$\langle x|y_1 + y_2\rangle = \langle x|y_1\rangle + \langle x|y_2\rangle \quad , \tag{4.12}$$

$$\langle x|\lambda y\rangle = \lambda\langle x|y\rangle \quad , \tag{4.13}$$

$$\langle y|x\rangle = \langle x|y\rangle^* \quad , \quad , \tag{4.14}$$

$$\langle x|x\rangle > 0 \qquad \text{unless } x = 0 \quad , \tag{4.15}$$

where c^* is the complex-conjugate of a complex number c. We are following the physicist's convention (adopted also by Bourbaki!) that $\langle x|y\rangle$ is linear in y, and conjugate linear in x.

We denote by A^* the adjoint of any operator A in H, so

$$\langle Ax|y\rangle = \langle x|A^*y\rangle \quad . \tag{4.16}$$

Now assume that the operator A satisfies the inequality

$$\mathrm{Re}\langle z|Az\rangle > 0 \qquad \text{unless } z = 0 \quad , \tag{4.17}$$

that is $A = B + iC$ with $B^* = B$, $C^* = C$ and B is positive-definite $\langle z|Bz\rangle > 0$ for $z \neq 0$.

To define an integration element in H, let us remark that the dot product $x\cdot y = \mathrm{Re}\langle x|y\rangle$ enables one to consider the complex hilbertian space H of (complex) dimension n as a real euclidean space E of (real) dimension 2n. The dot product defines an integration element in E, (see section 4.2), which we denote by $d_H z$.

Let S be an invertible selfadjoint operator in H. It can be diagonalized, hence there exists an orthonormal basis $e_1,\dots,e_n$ in H and nonzero real numbers $s_1,\dots,s_n$ such that $Se_j = s_j e_j$ for $1 \leq j \leq n$. Put $e_{j+n} = ie_j$ and $s_{j+n} = s_j$

for $1 \leq j \leq n$. Hence $e_1, \ldots, e_{2n}$ is an orthonormal basis of the real euclidean space E, and $Se_j = s_j e_j$ for $1 \leq j \leq 2n$. Since the determinant of S is equal to $s_1 \ldots s_n$, we get the following change of variable formula

$$d_H(Sz) = (\det S)^2 d_H z \quad . \qquad (4.18)$$

4.5. After these preparations, we state the *complex version of formula* (4.10), namely

$$\int_H e^{-\pi \langle z | Az \rangle} \, d_H z = (\det A)^{-1} \quad . \qquad (4.19)$$

To prove it, we first remark that $A = B + iC$ where B is selfadjoint and positive-definite. Hence B can be diagonalized with strictly positive eigenvalues, and there exists a selfadjoint operator S in H such that $BS^2 = 1$. Set $D = SCS$; it is a selfadjoint operator. Using (4.18) we see that the integral $I = \int_H e^{-\pi \langle z | Az \rangle} \, d_H z$ is equal to

$$I = \int_H e^{-\pi \langle Sz | ASz \rangle} \, d_H(Sz)$$

$$= (\det S)^2 \int_H e^{-\pi \langle z | (1+iD)z \rangle} \, d_H z \quad .$$

Notice that $(\det S)^2 = (\det B)^{-1}$. Moreover, since D is selfadjoint, it is again diagonalizable with real eigenvalues $d_1 \ldots d_n$. We obtain

$$I = (\det B)^{-1} \prod_{j=1}^{n} \int_{\mathbb{C}} e^{-\pi z^*(1+id_j)z} \, dz \quad . \qquad (4.20)$$

Here dz is put for dxdy if $z = x + iy$. Now using polar coordinates in $\mathbb{C}$, one derives immediately

$$\int_{\mathbb{C}} e^{-\pi a |z|^2} \, dz = a^{-1} \qquad (4.21)$$

for any complex number a such that $\mathrm{Re}\, a > 0$. Hence

$$I = (\det B)^{-1} \cdot \prod_{j=1}^{n} (1 + id_j)^{-1}$$

$$= (\det B)^{-1} \det (1 + iD)^{-1} \quad .$$

But

$$\det(1 + iD) = \det(1 + iSCS) = \det(1 + iS^2C)$$

$$= \det(1 + iB^{-1}C) = (\det B)^{-1}\det(B + iC) \quad .$$

This finishes the proof of (4.19).

4.6. To conclude this section, we shall rewrite (4.19) as follows

$$\int_{C^n} \exp -\{\sum_{j=1}^{n} \sum_{k=1}^{n} z_j^* a_{jk} z_k\} \, dz_1^* \wedge dz_1 \wedge \ldots \wedge dz_n^* \wedge dz_n$$

$$= (2\pi i)^n (\det A)^{-1} \tag{4.22}$$

for every complex matrix $A = (a_{jk})$ whose selfadjoint part $B = \frac{1}{2}(A + A^*)$ is positive-definite. We remind the reader that for $z = x + iy$, with complex conjugate $z^* = x - iy$, one has

$$dz^* \wedge dz = 2i dx \wedge dy \quad . \tag{4.23}$$

PART TWO:

FREDHOLM DETERMINANTS

5. Fredholm Theory of Integral Equations

5.1. For orientation purposes, we record here *a few formulas in the finite-dimensional case*. We consider a vector space V with a finite basis $e_1,\ldots,e_n$ and a linear operator A acting on V, whose matrix with respect to the previous basis we denote by $(A(i,j))$. Let us denote by $x(i)$ (for $(1 \leq i \leq n)$ the coordinates of a vector x. Then the coordinates of the transformed vector Ax are given by

$$(Ax)(i) = \sum_{j=1}^{n} A(i,j)x(j) \qquad (1\leq i\leq n) \quad . \qquad (5.1)$$

Moreover, according to formula (3.10), the determinant of 1+A can be expanded as follows

$$\det(1+A) = \sum_{p\geq 0} (p!)^{-1} \sum_{i_1\ldots i_p} \Delta\begin{pmatrix} i_1 & \ldots & i_p \\ i_1 & \ldots & i_p \end{pmatrix} . \qquad (5.2)$$

The series breaks up after the term for $p = n$ and the *minors* of the matrix A are defined as follows

$$\Delta\begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_p \end{pmatrix} = \det_{\substack{1\leq k\leq p \\ 1\leq l\leq p}} (A(i_k,j_l)) \quad . \qquad (5.3)$$

After some routine calculations with determinants, *Cramer's formula* for inverting a matrix takes the following form: the operator inverse to $1 + A$ exists iff the determinant Δ of $1+A$ does not vanish, and it is then of the form $1 - \Delta^{-1}B$ where the matrix B is given by the following power series expansion:

$$B(i,j) = \sum_{p\geq 0} (p!)^{-1} \sum_{i_1\ldots i_p} \Delta\begin{pmatrix} i & i_1\ldots i_p \\ j & i_1\ldots i_p \end{pmatrix} \qquad (5.4).$$

Again, the series breaks up after the term for $p = n-1$ since the minors of order $p+1 > n$ are all zero.

5.2. We now replace finite sums by integrals using the well-known analogy. Let us denote by Ω a compact subset of some euclidean space $\mathbb{R}^m$ and by dx the integration element in $\mathbb{R}^m$. An *integral operator with kernel* K is defined by the formula

$$Kf(x) = \int_\Omega K(x,y)\, f(y)\, dy \; . \tag{5.5}$$

Let us dnote by $C(\Omega)$ the Banach space of complex-valued continuous functions on Ω, with the norm $||f|| = \sup_{x\in\Omega} |f(x)|$. One defines $C(\Omega \times \Omega)$ similarly. Then, if K belongs to $C(\Omega \times \Omega)$, the function Kf defined by (5.5) belongs to $C(\Omega)$ if f does, and the linear operator $f \mapsto Kf$ in $C(\Omega)$ is bounded with norm $C \leq ||K|| \cdot \mathrm{vol}\,(\Omega)$; here we denote by $\mathrm{vol}(\Omega)$ the volume $\int_\Omega dx$ of Ω.

Everything in the rest of this fifth section extends *verbatim* if one replaces Ω by a general Hausdorff compact space, integration being taken with respect to a regular Borel measure on Ω. This extension may be useful in some problems of classical statistical mechanics.

5.3. Let us denote by 1 the identity operator in $C(\Omega)$; it is not an integral operator of the sort previously defined, since we consider only *continuous kernels* $K(x,y)$ and not singular kernels like the Dirac $\delta(x-y)$.

Fredholm proposed (around 1900) to define the *determinant of the operator* 1+K (taking f into f + Kf) by the following formula, analogous to (5.2):

$$\det(1+K) = \sum_{p\geq 0} (p!)^{-1} \int_\Omega \ldots \int_\Omega \Delta\binom{x_1 \ldots x_p}{x_1 \ldots x_p}\, dx_1 \ldots dx_p . \tag{5.6}$$

Again the minors are defined by analogy

$$\Delta\binom{x_1 \ldots x_p}{y_1 \ldots y_p} = \det_{\substack{1\leq j\leq p\\ 1\leq k\leq p}} K(x_j, y_k) \quad ; \tag{5.7}$$

they are jointly continuous functions of their arguments. To prove the convergence of the series (5.6), one uses a *determinant inequality* discovered by Hadamard (and in a weaker form by Fredholm):

$$|\det A| \leq ||\vec{a}_1||_2 \ \ldots\ ||\vec{a}_p||_2 \quad . \tag{5.8}$$

Here A is any p x p matrix with complex entries, columns $\vec{a}_1,\ldots, \vec{a}_p$ in $\mathbb{C}^p$, and $||\vec{a}||_2$ is the hilbertian norm of a vector $\vec{a}$ in $\mathbb{C}^p$:

$$||\vec{a}|| = (|a_1|^2 + \ldots + |a_p|^2)^{1/2} \quad . \tag{5.9}$$

To prove (5.8), we may assume that det A ≠ 0, that is the columns $\vec{a}_1,\ldots,\vec{a}_p$ are linearly independent. By a well-known geometric construction, we may find new vectors $\vec{b}_1,\ldots,\vec{b}_p$ mutually orthogonal in $\mathbb{C}^p$, such that $\vec{b}_j - \vec{a}_j$ be a linear combination of the vectors $\vec{a}_1,\ldots,\vec{a}_{j-1}$. From the properties of volume forms explained at length in section 1, it follows that the matrices A with columns $\vec{a}_1,\ldots,\vec{a}_p$ and B with columns $\vec{b}_1,\ldots,\vec{b}_p$ have equal determinants. Moreover $\vec{a}_j - \vec{b}_j$ is also a linear combination of $\vec{b}_1,\ldots,\vec{b}_{j-1}$, hence it is orthogonal to $\vec{b}_j$; this implies the inequality $||\vec{b}_j|| \leq ||\vec{a}_j||$ by Pythagoras'theorem. Finally, the property that $\vec{b}_1,\ldots,\vec{b}_p$ are mutually orthogonal can be expressed by the fact that[1] B^*B is a diagonal matrix with diagonal entries $||\vec{b}_1||^2,\ldots,||\vec{b}_p||^2$. Hence

$$|\det A|^2 = |\det B|^2 = \det(B^*B) =$$

$$= ||\vec{b}_1||^2 \ldots ||\vec{b}_p||^2 \leq ||\vec{a}_1||^2 \ldots ||\vec{a}_p||^2$$

and Hadamard's inequality (5.8) follows.

Using Hadamard's inequality, we get the estimate

$$\left|\Delta\begin{pmatrix} x_1 \ldots x_p \\ y_1 \ldots y_p \end{pmatrix}\right|^2 \leq \prod_{j=1}^{p} \sum_{k=1}^{p} |K(x_j,y_k)|^2 \quad ,$$

hence

$$\left|\Delta\begin{pmatrix} x_1 \ldots x_p \\ y_1 \ldots y_p \end{pmatrix}\right| \leq p^{p/2}\, ||K||^p \quad . \tag{5.10}$$

[1]Here B^* is the matrix hermitian conjugate to B.

By definition, the determinant of $1+K$ is $\sum_{p\geq 0} c_p(K)$ with $c_0(K)=1$ and

$$c_p(K) = (p!)^{-1} \int_\Omega \cdots \int_\Omega \Delta\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p \qquad (5.11)$$
for $p \geq 1$.

Using (5.10), we obtain the inequality

$$|c_p(K)| \leq p^{p/2} \, ||K||^p \, \mathrm{vol}(\Omega)^p/p! \qquad (\text{for } p \geq 1). \quad (5.12)$$

By Stirling's formula, there exists a constant C_0 such that $0 < C_0 < 1$ and $p! \geq C_0 p^p e^{-p} \sqrt{2\pi p}$; therefore we get the estimate

$$|c_p(K)| \leq C_0^{-1}\{||K|| \, \mathrm{vol}(\Omega) \, p^{-1/2} e\}^p \qquad (5.13)$$

for $p \geq 1$, hence $\lim_{p\to\infty} |c_p(K)|^{1/p} = 0$. The convergence of the series $\sum_{p\geq 0} c_p(K)$ follows.

5.4. We can view the Fredholm determinant as a *functional* $K \mapsto \det(1 + K)$ *on the Banach space* $C(\Omega \times \Omega)$. From the basic estimate (5.13), one infers that the series $\sum_{p\geq 0} c_p(K)$ converges uniformly on the set $||K|| \leq R$, for every constant $R > 0$. Hence $\det(1 + K)$ is a *continuous functional* of K.

It follows also from (5.13) that $\det(1 + zK) = \sum_{p\geq 0} c_p(K) z^p$ *is an entire function of the complex variable* z. More can be said about analyticity. For $K_1,\ldots,K_p$ in $C(\Omega\times\Omega)$ let us define

$$c_p(K_1,\ldots,K_p) = (p!)^{-1} \int_\Omega \cdots \int_\Omega \{\det_{\substack{1\leq j\leq p \\ 1\leq k\leq p}} K_j(x_j,x_k)\} dx_1 \ldots dx_p. \qquad (5.14)$$

Obviously, $c_p(K_1,\ldots,K_p)$ is a multilinear functional of $K_1,\ldots,K_p$. Moreover, by construction,

$$\det(1 + K) = \sum_{p\geq 0} c_p(K,\ldots,K) \qquad (5.15)$$

(with p arguments equal to K). If we set $C_p' = C_0^{-1} \mathrm{vol}(\Omega)^p p^{-p/2} e^p$, one gets

$$\lim_{p\to\infty} (C_p')^{1/p} = 0 \quad . \qquad (5.16)$$

Moreover, from Hadamard's inequality (5.8), one gets the estimate

$$|c_p(K_1,\ldots,K_p)| \leq C'_p \, ||K_1|| \ldots ||K_p|| \tag{5.17}$$

by a proof completely analogous to the proof of (5.13). The last three formulas express that $K \mapsto \det(1 + K)$ *is an entire function on the Banach space* $C(\Omega)$ in the very strong sense used by Bourbaki [3. p.28].

As a corollary, suppose that $K_\lambda(x,y)$ is an *analytic family of kernels* in the following sense: D being a domain in the complex space $\mathbb{C}^r$, the function $(\lambda,x,y) \to K_\lambda(x,y)$ is continuous on $D \times \Omega \times \Omega$, and moreover $K_\lambda(x,y)$ is an holomorphic function of λ for x,y being fixed. Then *the determinant* $\det(1 + K_\lambda)$ *is an homomorphic function of* λ *in* D.

5.5. We come now to the *multiplicative property* of Fredholm determinants. Using the analogy between finite sums and integrals, the usual matrix product suggests the following product for continuous kernels

$$(KL)(x,z) = \int_\Omega K(x,y)L(y,z)\, dy \quad . \tag{5.18}$$

It is obviously linear in K and in L and possesses the expected associativity properties:

$$(KL)f = K(Lf) \tag{5.19}$$

$$(KL)M = K(LM) \tag{5.20}$$

where f is a continuous function on Ω and M another continuous kernel.

Let K and L be kernels, and consider the operators $U=1+K$ and $V=1+L$ acting on $C(\Omega)$. The product UV is of the form $1+M$ with a kernel $M = K + L + KL$. We claim:

$$\det(1+K) \det(1+L) = \det(1 + K + L + KL) \quad . \tag{5.21}$$

This could be proved by a brute force calculation. We prefer to resort to an *approximation procedure.*

A *decomposable kernel* is a function K in $C(\Omega \times \Omega)$ of the form $K(x,y) = \sum_{\alpha=1}^{r} f_\alpha(x)g_\alpha(y)$ for $f_1,\ldots,f_r$ and $g_1,\ldots,g_r$ in

$C(\Omega)$. Such functions are dense in the Banach space $C(\Omega \times \Omega)$ (a well-known lemma of Dieudonné [5, p.141],or an obvious corollary of Weierstrass' approximation theorem). Since the determinant is a continuous functional on $C(\Omega \times \Omega)$, it is enough to prove (5.21) for the case of decomposable kernels K and L.

Suppose now that K and L are decomposable kernels. Using *Schmidt's orthonormalization process*, we find a family of continous functions $f_1,\ldots,f_r$ on Ω, orthonormal in the following sense:

$$\int_\Omega \overline{f_\alpha(x)} f_\beta(x)dx = \delta_{\alpha\beta} \qquad (\text{for } 1 \leq \alpha \leq r,\ 1 \leq \beta \leq r), \tag{5.22}$$

and complex matrices $A = (a_{\alpha\beta})$ and $B = (b_{\alpha\beta})$ of size $r \times r$ such that

$$K(x,y) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} f_\alpha(x) a_{\alpha\beta} \overline{f_\beta(y)} \quad , \tag{5.23}$$

$$L(x,y) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} f_\alpha(x) b_{\alpha\beta} \overline{f_\beta(y)} \quad . \tag{5.24}$$

The kernel $M = K + L + KL$ admits of a similar description

$$M(x,y) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} f_\alpha(x)\ c_{\alpha\beta} \overline{f_\beta(y)} \tag{5.25}$$

with the matrix $C = (c_{\alpha\beta})$ given by $C = A + B + AB$. By the multiplication rule for finite determinants, we obtain therefore

$$\det\,(1 + A)\ \det\ 1 + B) = \det\,(1 + C) \quad . \tag{5.26}$$

To establish the multiplicative property (5.21) for kernels, it suffices to prove the equality

$$\det\,(1 + K) = \det\,(1 + A) \tag{5.27}$$

for a kernel K as in formula (5.23).

The proof of (5.27) rests on a *generalization of the multiplicative property of determinants known as Binet-Cauchy formula*. Namely, let $U = (u_{ij})$ be a matrix of size $m \times n$ and $V = (v_{jk})$ a matrix of size $n \times p$, so that the matrix $W = UV$ is of size $m \times p$. Define the minors of r-th order of U by

$$U\binom{i_1\ldots i_r}{j_1\ldots j_r} = \det_{\substack{1\leq\alpha\leq r\\ 1\leq\beta\leq r}} u_{i_\alpha j_\beta} \tag{5.28}$$

and similarly for the minors of V and W. Then

$$W\binom{i_1\ldots i_r}{k_1\ldots k_r} = \sum_{j_1<\ldots<j_r} U\binom{i_1\ldots i_r}{j_1\ldots j_r} V\binom{j_1\ldots j_r}{k_1\ldots k_r} \quad . \tag{5.29}$$

In invariant form, this can be expressed as follows. Set $E = \mathbb{C}^m$, $F = \mathbb{C}^n$, $G = \mathbb{C}^p$. Then the matrices U, V and W correspond to operators $u: F \to E$, $v: G \to F$, $w: G \to E$ such that $w = uv$. Moreover, there exists an operator $\Lambda^r u: \Lambda^r F \to \Lambda^r E$ characterized by $(\Lambda^r u)(x_1 \wedge \ldots \wedge x_r) = u(x_1)\wedge \ldots \wedge u(x_r)$ for vectors $x_1,\ldots,x_r$ in F. One defines similarly $\Lambda^r v$ and $\Lambda^r w$. From $w = uv$ one derives immediately $\Lambda^r w = \Lambda^r u \cdot \Lambda^r v$. In natural basis for $\Lambda^r E$ and $\Lambda^r F$, the entries of the matrix of $\Lambda^r u$ consist of the minors $U\binom{i_1\ldots i_r}{j_1\ldots j_r}$. Hence formula (5.29) is the matrix version of $\Lambda^r w = \Lambda^r u \cdot \Lambda^r v$.

Introduce now continuous functions on $\Omega^p = \Omega \times \ldots \times \Omega$ (p factors) as follows

$$f_{\alpha_1\ldots\alpha_p}(x_1,\ldots,x_p) = \det f_{\alpha_i}(x_j) \quad , \tag{5.30}$$

that is

$$f_{\alpha_1\ldots\alpha_p}(x_1,\ldots,x_p) = \sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\cdot f_{\alpha_{\sigma(1)}}(x_1)\ldots f_{\alpha_{\sigma(p)}}(x_p). \tag{5.31}$$

From the orthogonality property (5.22), one derives.

$$\int_\Omega\ldots\int_\Omega \overline{f_{\alpha_1\ldots\alpha_p}(x_1\ldots x_p)}\, f_{\beta_1\ldots\beta_p}(x_1,\ldots,x_p)dx_1\ldots dx_p =$$

$$= p!\ \delta_{\alpha_1\beta_1} \ldots \delta_{\alpha_p\beta_p} \tag{5.32}$$

for $\alpha_1< \ldots <\alpha_p$, $\beta_1< \ldots<\beta_p$. Moreover, from Cauchy's formula (5.29), one derives

$$\Delta\binom{x_1 \ldots x_p}{y_1 \ldots y_p} = \sum_{\substack{\alpha_1<\ldots<\alpha_p \\ \beta_1<\ldots<\beta_p}} A\binom{\alpha_1 \ldots \alpha_p}{\beta_1 \ldots \beta_p} f_{\alpha_1 \ldots \alpha_p}(x_1,\ldots,x_p) \cdot \overline{f_{\beta_1 \ldots \beta_p}(y_1,\ldots,y_p)} \qquad (5.33)$$

Using (5.32), one gets immediately

$$\int_\Omega \ldots \int_\Omega \Delta\binom{x_1 \ldots x_p}{x_1 \ldots x_p} dx_1 \ldots dx_p = p! \sum_{\alpha_1<\ldots<\alpha_p} A\binom{\alpha_1 \ldots \alpha_p}{\alpha_1 \ldots \alpha_p}, \qquad (5.34)$$

and by summing over p, one obtains finally

$$\det(1+K) = \sum_{p\geq 0} \sum_{\alpha_1<\ldots<\alpha_p} A\binom{\alpha_1 \ldots \alpha_p}{\alpha_1 \ldots \alpha_p} \quad . \qquad (5.35)$$

Our contention (5.27) follows by using formula (3.10) for matrices.

5.6. A crucial property of determinants in the finite-dimensional case is the following criterior: an operator A acting linearly on finite-dimensional vector space V is invertible iff its determinant is not zero. An analogous property holds for integral operators: *if* K *is a continuous kernel, the operator* 1+K *on* $C(\Omega)$ *possesses an inverse* (necessarily bounded by general results of functional analysis) *iff the Fredholm determinant* det (1+K) *is not zero.*

The proof consists of *three steps:*

(a) *Suppose that* T *is an inverse for* 1+K, hence T(1+K)=1, or T = 1 - TK. We claim that TK is an integral operator. Indeed, write $K_y(x)$ for K(x,y). Then $y \mapsto K_y$ is a continuous map from Ω into the metric space $C(\Omega)$ (by uniform continuity of K). For y in Ω, set $L_y = T(K_y)$ and let $L(x,y) = L_y(x)$. Then $y \mapsto L_y$ is a continuous map from Ω into $C(\Omega)$ or, equivalently, the function L belongs to $C(\Omega \times \Omega)$. We claim that *the integral operator with kernel* L *is equal to* TK. This is easily verified if K is a decomposable kernel of the form $K(x,y) = \sum_{\alpha=1}^{r} f_\alpha(x)\, g_\alpha(y)$ (with $f_1,\ldots,f_r,g_1,\ldots,g_r$ in $C(\Omega)$). The general case is obtained by using a sequence $(K_n)_{n\geq 0}$ of decomposable kernels such that $\lim_{n\to\infty} ||K - K_n|| = 0$.

(b) *Suppose that* 1+K *is invertible*. By step (a), there exists a kernel L such that (1+K)(1-L) = 1. By the multiplication property (5.21), one gets

$$\det(1+K)\det(1-L) = 1, \quad ,$$

hence $\det(1+K) \neq 0$.

(c) It remains to prove that *when the Fredholm determinant of* 1+K *is not zero, the operator* 1+K *is invertible*. This will be done by providing an explicit formula for the inverse. For every integer $p \geq 0$, one defines a continuous kernel L_p by

$$L_p(x,y) = \int_\Omega \cdots \int_\Omega \Delta\begin{pmatrix} x & x_1 \ldots x_p \\ y & x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p \quad . \tag{5.36}$$

From the basic estimate (5.10), one derives

$$|L_p(x,y)| \leq (p+1)^{(p+1)/2} \, ||K||^{p+1} \, \mathrm{vol}(\Omega)^p \quad . \tag{5.37}$$

Using Stirling's formula as in section 5.3, it follows that the series $\sum_{p\geq 0} (p!)^{-1} L_p(x,y)$ converges uniformly on $\Omega \times \Omega$, hence its sum $L(x,y)$ defines a continuous kernel.

From its definition, the Fredholm determinant $\Delta = \det(1+K)$ is given by the series $\sum_{p\geq 0} \gamma_p/p!$ where γ_p is defined as follows:

$$\gamma_p = \int_\Omega \cdots \int_\Omega \Delta\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p \quad . \tag{5.38}$$

Notice that $\Delta\begin{pmatrix} x & x_1 \ldots x_p \\ y & x_1 \ldots x_p \end{pmatrix}$ is the determinant of the matrix

$$\begin{pmatrix} K(x,y) & K(x,x_1) & \ldots\ldots & K(x,x_p) \\ K(x_1,y) & K(x_1,x_1) & \ldots\ldots & K(x_1,x_p) \\ \cdot & \cdot & \cdot & \cdot \\ K(x_p,y) & K(x_p,x_1) & \ldots\ldots & K(x_p,x_p) \end{pmatrix} .$$

Developing this determinant according to its first row, one obtains

$$\Delta\begin{pmatrix} x & x_1 \dots x_p \\ y & x_1 \dots x_p \end{pmatrix} = K(x,y)\Delta\begin{pmatrix} x_1 \dots x_p \\ x_1 \dots x_p \end{pmatrix} - \sum_{j=1}^{p} K(x,x_j)\Delta\begin{pmatrix} x_j & x_1 \dots \hat{x}_j \dots x_p \\ y & x_1 \dots \hat{x}_j \dots x_p \end{pmatrix} \quad . \tag{5.39}$$

We use the standard convention that a term with a carret $\hat{}$ has to be omitted. Integrating with respect to $x_1 \dots x_p$ gives then

$$L_p(x,y) = \gamma_p K(x,y) - p \int_\Omega K(x,z) L_{p-1}(z,y) dz \quad ; \tag{5.40}$$

notice that the labelling of the integration variables being irrelevant the p terms in the summation of (5.39) give the same integral. Notice also the limiting case $L_0(x,y) = \gamma_0 K(x,y)$ for $p = 0$ (and $\gamma_0 = 1!$). Since $L(x,y)$ is given by the uniformly convergent series $\sum_{p \geq 0} L_p(x,y)/p!$, integration term by term is legitimate, and from (5.40) one gets

$$L(x,y) = \Delta K(x,y) - \int_\Omega K(x,z) L(z,y)\, dz \tag{5.41}$$

In terms of kernels, this formula can be stated as follows (where Δ is a constant):

$$L = \Delta K - KL \quad . \tag{5.42}$$

By a similar proof, expanding the determinant $\Delta\begin{pmatrix} x & x_1 \dots x_p \\ y & x_1 \dots x_p \end{pmatrix}$ according to its first column, we get

$$L = \Delta K - LK \quad . \tag{5.43}$$

It is time to assume $\Delta \neq 0$. The previous formulas (5.42) and (5.43) just mean that *the operator* $1 - \Delta^{-1}L$ *is an inverse for* $1 + K$. The reader will undoubtedly notice the analogy of this statement with formula (5.4).

Let us just add one *remark*. An inverse of $1+K$ is given by the geometris series $(1+K)^{-1} = 1-K+K^2 \dots = 1 - \sum_{p \geq 0} (-1)^p K^{p+1}$.

By definition, K^{p+1} is given by a p-fold integral

$$K^{p+1}(x,y) = \int_\Omega \dots \int_\Omega K(x,x_1)K(x_1,x_2) \dots K(x_p,y)dx_1 \dots dx_p \ , \tag{5.44}$$

hence the estimate $|K^{p+1}(x,y)| \leq ||K||^{p+1} \operatorname{vol}(\Omega)^p$. *The convergence of the series* $M = \sum_{p\geq 0} (-1)^p K^{p+1}$ is therefore guaranteed if $||K|| < \operatorname{vol}(\Omega)^{-1}$, *but may fail in general.* Multiply M by the convergent series $\Delta = \sum_{p\geq 0} \gamma_p/p!$, and rearrange the terms according to Cauchy's rule for multiplying series. We get $\Delta M = \sum_{p\geq 0} L'_p/p!$ with

$$L'_0 = \gamma_0 K \ , \quad L'_1 = \gamma_1 K - \gamma_0 K^2, \quad L'_2 = \gamma_2 K - 2\gamma_1 K^2 + 2\gamma_0 K^3, \dots$$

Formula (5.40) provides a recursive definition of the kernel L_p, and the equality $L'_p = L_p$ follows easily. Using Hadamard's inequality (5.10) as above then shows that the series $\sum_{p\geq 0} L'_p/p!$ converges uniformly on $\Omega \times \Omega$. Hence, *after rearranging, the product* ΔM *is given by a convergent series, even if* M *does not.*

5.7. Using once more the analogy between sums and integrals, we are led to define *the trace of a kernel* K as the scalar

$$\operatorname{Tr}(K) = \int_\Omega K(x,x)\, dx \quad . \tag{5.45}$$

The trace of a product of p kernels is given by the multiple integral

$$\begin{aligned}\operatorname{Tr}(K_1 \dots K_p) = \int_\Omega \dots \int_\Omega & K_1(x_1,x_2)K_2(x_2,x_3)\dots \\ & K_{p-1}(x_{p-1},x_p)K_p(x_p,x_1)dx_1 \dots dx_p \ .\end{aligned} \tag{5.46}$$

It is then obvious that *the trace* $\operatorname{Tr}(K_1 \dots K_p)$ *is invariant under cyclic permutations of* $K_1,\dots,K_p$.

We proceed now to the proof of an analogous to Waring's formula, namely

$$\det(1 - zK) = \exp\{-\sum_{n\geq 1} \operatorname{Tr}(K^n)z^n/n\} \quad . \tag{5.47}$$

Notice the estimate $|Tr(K^n)| \leq ||K||^n vol(\Omega)^n$, which follows from (5.46); the series $\sum_{n\geq 1} Tr(K^n)z^n/n$ is then guaranteed to converge when $|z|<||K||^{-1} vol(\Omega)^{-1}$ and the identity (5.47) holds under this assumption.

Denote by $I(\sigma)$ (for σ in the symmetric group S_p) the following integral

$$I(\sigma) = \int_\Omega \dots \int_\Omega K(x_1,x_{\sigma(1)}) \dots K(x_p,x_{\sigma(p)}) dx_1 \dots dx_p \tag{5.48}$$

By expanding completely the determinant $\Delta\begin{pmatrix} x_1 \dots x_p \\ x_1 \dots x_p \end{pmatrix}$, one gets

$$\gamma_p = \sum_{\sigma\in S_p} (\mathrm{sgn}\sigma) \cdot I(\sigma) \quad . \tag{5.49}$$

Moreover, one does not change the integral (5.48) by relabelling the integration variables x_j as $y_{\tau(j)}$, for τ in S_p. It then follows easily that

$$I(\sigma) = I(\tau\sigma\tau^{-1}) \quad , \tag{5.50}$$

that is, $I(\sigma)$ is a *class function* of σ in S_p. Finally, if σ is decomposed into cycles $(1...a)(a+1...a+b)(a+b+1...a+b+c)...$ the integrand in (5.48) can be written as a product of the factors

$$K(x_1,x_2) \dots K(x_{a-1},x_a)\, K(x_a,x_1) = J_a$$

$$K(x_{a+1},x_{a+2}) \dots K(x_{a+b-1},x_{a+b})\, K(x_{a+b},x_{a+1}) = J_b$$

. . . .

The integral (5.48) splits accordingly, and by (5.46), we get

$$I(\sigma) = Tr(K^a)\, Tr(K^b)\, Tr(K^c) \dots \quad . \tag{5.51}$$

The rest of the proof of formula (5.47) is now completely similar to the proof in the finite-dimensional case (see subsection 3.8).

The consequences of formula (5.47) are derived as in subsection 3.7. Taking the logarithm, we get

$$\log \det(1 - zK) = - \sum_{n\geq 1} Tr(K^n)z^n/n \tag{5.52}$$

for the principal branch, in the domain $|z| < ||K||^{-1} \mathrm{vol}(\Omega)^{-1}$. By derivation, this implies

$$\frac{d}{dz} \log \det (1-zK) = - \sum_{n\geq 1} \mathrm{Tr}(K^n) z^{n-1} = -\mathrm{Tr}(K(1-Kz)^{-1}) \qquad (5.53)$$

in the same domain. Recall the series expansion

$$\det(1 + zK) = \sum_{p\geq 0} c_p(K)\, z^p \quad , \qquad (5.54)$$

where $c_p(K)$ is defined by formula (5.11). We get an inductive definition of these coefficients

$$p c_p(K) = \sum_{j=1}^{p} (-1)^{j+1} \mathrm{Tr}(K^j) c_{p-j}(K) \qquad (\text{for } p\geq 1). \quad (5.55)$$

Finally, $c_p(K)$ can be given in *determinantal form* (with $\tau_j = \mathrm{Tr}(K^j)$):

$$p!\, c_p(K) = \det \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 & \cdots & \tau_{p-2} & \tau_{p-1} & \tau_p \\ p-1 & \tau_1 & \tau_2 & \cdots & \tau_{p-3} & \tau_{p-2} & \tau_{p-1} \\ 0 & p-2 & \tau_1 & \cdots & \tau_{p-4} & \tau_{p-3} & \tau_{p-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & \tau_1 & \tau_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \tau_1 \end{pmatrix} . \qquad (5.56)$$

5.8. *Fredholm's alternative* (or at least the main statement in in) reads as follows:

- either the operator 1+K is invertible,
- or there exists a nonzero function f in $C(\Omega)$ such that $(1+K)f = 0$.

According to the theorem proved in subsection 5.6, the first case occurs if $\det(1+K) \neq 0$ and the second if $\det(1+K) = 0$. Otherwise stated, if the Fredholm determinant of 1+K is 0, then there exists a nonzero function f in $C(\Omega)$ such that $(1+K)f = 0$.

Consider the entire function $D(z) = \det(1 - zK)$, with $D(0) = 1$. From the previous statement, one concludes that a nonzero complex number λ is a zero of this entire function, namely $D(\lambda) = 0$, iff there exists a nonzero function f in $C(\Omega)$ such that $(1-\lambda K)f = 0$, or equivalently $Kf = \lambda^{-1}f$. Hence *the zeroes of the entire function* $D(z)$ *are the inverses of the nonzero eigenvalues of the operator* K. In typical applications of Fredholm theory, the integral operator with continuous kernel K shall be the inverse of some differential operator P, and the equations $Kf = \lambda^{-1}f$ and $Pf = \lambda f$ are equivalent. In this case, *the zeroes of the entire function* $D(z)$ *are the eigenvalues of the operator* P.

The Fredholm alternative was established by Fredholm using computational methods. The method used in modern textbooks is due to F.Riesz and relies on a *compactness property* of the integral operators, namely: *the closure of the set of functions* Kf*, for* f *in* $C(\Omega)$ *of norm* ≤ 1*, is compact in the metric space* $C(\Omega)$. This is proved using a sequence of decomposable kernel K_n, converging uniformly to K on $\Omega \times \Omega$; then the integral operator with kernels K_n is of finite rank and converges in operator norm to the integral operator with kernel K.

5.9. Integral operators with continuous kernels are special cases of Hilbert-Schmidt operators (see subsection 7.16). The eigenvalues of these operators satisfy therefore the following properties:

(a) *the nonzero eigenvalues of the operator* K *can be arranged into a sequence* $(\lambda_j)_{j\geq 1}$ *with* $|\lambda_1| \leq |\lambda_2| \geq \ldots$ *and* $\lim_{j\to\infty} \lambda_j = 0$, *each eigenvalue being repeated according to a well-specified multiplicity.*

(b) *The sum* $\sum_{j\geq 1} |\lambda_j|^2$ *if finite, but in general* $\sum_{j\geq 1} |\lambda_j|$ *is not finite.*

(c) *For the traces of powers of* **K***, one gets*

$$\mathrm{Tr}(K^n) = \sum_{j\geq 1} \lambda_j^n \quad \textit{for every integer } n\geq 2 \ . \tag{5.57}$$

It may occur that there is only a finite number of nonzero eigenvalues for K, possibly none of them. In this case, formula (5.57) remains valid with an obvious interpretation, namely $Tr(K^n) = 0$ (for every integer $n \geq 2$) if there is no nonzero eigenvalue of K.

We can now derive a *product formula* for the Fredholm determinant, namely

$$\det(1 - zK) = e^{-zTr(K)} \prod_{j\geq 1} (1 - \lambda_j z)e^{\lambda_j z} \qquad (5.58)$$

for every complex number z. If there is no nonzero eigenvalue of K, the product in (5.58) has to be interpreted as 1, that is $\det(1 - zK) = e^{-zTr(K)}$ in this case. We mention also that the infinite product in (5.58) converges uniformly on every bounded domain of $\mathbb{C}$, since $\sum_{j\geq 1} |\lambda_j|^2$ is finite.

To prove formula (5.58) define two entire functions by the formulas

$$D_1(z) = e^{zTr(K)}\det(1-zK), \quad D_2(z) = \prod_{j\geq 1}(1-\lambda_j z)e^{\lambda_j z} \quad . \qquad (5.59)$$

We want to prove the equality $D_1(z) = D_2(z)$. By analytic continuation, it suffices to prove that these functions, with values $D_1(0) = D_2(0) = 1$ at the origin, have equal logarithms around the origin. By formula (5.52) one gets

$$-\log D_1(z) = \sum_{n\geq 2} Tr(K^n)\, z^n/n \qquad (5.60)$$

whenever $|z| < ||K||^{-1} vol\,(\Omega)^{-1}$. Moreover, using the usual Taylor series for the logarithm , one gets

$$-\log D_2(z) = \sum_{j\geq 1} \sum_{n\geq 2} (\lambda_j z)^n/n \qquad (5.61)$$

whenever $|z| < (\sum_{j\geq 1} |\lambda_j|^2)^{-1/2}$. This double series is then absolutely convergent, hence can be rearranged as

$$-\log D_2(z) = \sum_{n\geq 2} (\sum_{j\geq 1} \lambda_j^n) z^n/n \qquad (5.62)$$

Using now the equality of $Tr(K^n)$ and $\sum_{j\geq 1} \lambda_j^n$ (formula (5.57)), one deduces $-\log D_1(z) = -\log D_2(z)$, hence $D_1(z)=D_2(z)$ for $|z|$ small.

From the formula (5.58), one deduces that *the multiplicity of* λ *as a zero of the entire function* $D(z) = \det(1 - zK)$ *is equal to the multiplicity of* λ^{-1} *as an eigenvalue of* K.

5.10. *One central difficulty* in Fredholm's theory is that *the identity* (5.57) *is valid for* $n \geq 2$, *but not in general for* $n = 1$. When the series $\sum_{j\geq 1} \lambda_j$ converges to the sum $\mathrm{Tr}(K)$, formula (5.58) can be simplified, namely

$$\det(1 - zK) = \prod_{j\geq 1} (1 - \lambda_j z) \quad . \tag{5.63}$$

That this can not be true in general can be seen using Fourier series. Namely, assume that our space Ω is the closed interval $[0,1]$ with the endpoints 0 and 1 identified and consider a kernel of the form $K(x,y) = k(x-y)$, where k is a continuous function on the real line with period one: $k(x+1)=k(x)$. Introduce the exponential functions e_n by $e_n(x) = e^{2\pi i n x}$. Any continuous function f on Ω with $f(0) = f(1)$ can be uniformly approximated by finite linear combinations of the e_n's. Moreover, one gets $Ke_n = c_n e_n$ where c_n is the usual Fourier coefficient $\int_0^1 k(x)e_n(-x)dx$ of k. The eigenvalues of the integral operator with kernel K are therefore the Fourier coefficients c_n. One gets

$$\mathrm{Tr}(K) = k(0) \quad , \qquad \mathrm{Tr}(K^2) = \int_0^1 k(x)k(-x)dx \, . \tag{5.64}$$

The identity $\mathrm{Tr}(K^2) = \sum_{n=-\infty}^{+\infty} c_n^2$ *is just another form of Parseval's identity. But there are well-known examples* (see Titchmarsh [14, p.416]) *of continuous periodic functions* $k(x)$ *whose Fourier series* $\sum_{n=-\infty}^{+\infty} c_n e_n(x)$ *fails to converge and represent* $k(x)$, *for* $x = 0$ *say.*

6. A Review of Operator Theory in Hilbert Spaces

6.1. We consider a Hilbert space H, and we assume that H is both infinite-dimensional and separable. Then there exists an orthonormal basis $(\psi_n)_{n\geq 0}$ in H. Every vector ψ in H is determined by its components $c_n = \langle\psi_n|\psi\rangle$, restricted only by the convergence of the series $\sum_{n\geq 0} |c_n|^2$ which represents $\langle\psi|\psi\rangle$, that is the square of the norm $||\psi||$. For scalar products, we follow the conventions introduced in subsection 4.4.

Let A be a bounded operator in H. We shall write $\langle\psi|A|\psi'\rangle$ for the scalar product of ψ with $A\psi'$. We say that A is *selfadjoint* in case $\langle\psi|A|\psi\rangle$ is real for every ψ, and that it is *positive* in case $\langle\psi|A|\psi\rangle \geq 0$ for every ψ in H.

Assume now A to be positive. We claim that *the sum* $\sum_{n\geq 0} \langle\psi_n|A|\psi_n\rangle$ (a positive number or $+\infty$) *is independent of the orthonormal basis* (ψ_n). Indeed, A possesses a square root B (also denoted by $A^{\frac{1}{2}}$) which is the unique positive operator B such that $B^2 = A$. Choose another orthonormal basis $(\theta_m)_{m\geq 0}$. Our contention follows from the following calculation:

$$\sum_n \langle\psi_n|A|\psi_n\rangle = \sum_n \langle\psi_n|B^*B|\psi_n\rangle = \sum_n \langle B\psi_n|B\psi_n\rangle =$$

$$= \sum_{n,m} |\langle\theta_m|B\psi_n\rangle|^2 = \sum_{n,m} |\langle\theta_m|B|\psi_n\rangle|^2 .$$

Indeed, since B is selfadjoint $\langle\theta_m|B|\psi_n\rangle$ is the complex-conjugate of $\langle\psi_n|B|\theta_m\rangle$ and a symmetrical calculation gives the result

$$\sum_{m\geq 0} \langle\theta_m|A|\theta_m\rangle = |\sum_{m,n} \langle\psi_n|B|\theta_m\rangle|^2 .$$

We define the *trace of the positive operator* A as the number $\mathrm{Tr}(A) = \sum_{n\geq 0} \langle\psi_n|A|\psi_n\rangle$ in $[0,+\infty]$.

6.2. A fundamental theorem asserts that *a positive operator with a finite trace can be diagonalized.* More precisely, we can find an orthonormal set $(\psi_0,\psi_1,\ldots,\psi_n,\ldots)$ in H and a non-increasing sequence $(\lambda_0,\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots)$ of strictly positive numbers such that $A\psi_n = \lambda_n\psi_n$ for every n, and $A\psi = 0$ for every vector ψ in H orthogonal to all ψ_n's. Both sequences can consist of finitely many terms $(\psi_0,\dots,\psi_{N-1})$ and $(\lambda_0,\dots,\lambda_{N-1})$ if A is of finite rank N, or be infinite. In the last case, completing the sequence $(\psi_n)_{n\geq 0}$ to an orthonormal basis of H, one concludes to the equality

$$\mathrm{Tr}(A) = \sum_{n=0}^{\infty} \lambda_n \quad . \tag{6.1}$$

A similar statement holds in the finite rank case.

The vectors ψ_n are not uniquely defined by the previous conditions. The scalars λ_n are unique because of the following peoperty: for any number $\lambda > 0$, the number of times λ occurs in the sequence $(\lambda_0,\lambda_1,\dots)$ is equal to the dimension of the eigenspace corresponding to λ, that is the vectors ψ in H such that $A\psi = \lambda\psi$.

6.3. For the mathematically inclined reader, we sketch a proof of the diagonalization theorem based on *a compactness argument* (due essentially to Hilbert). Let A, B and θ_m as in subsection 6.1. Denote by H_1 the set of vectors ψ in H such that $||\psi|| \leq 1$. Associating to a vector ψ its components $c_m = \langle\theta_m|\psi\rangle$, we map bijectively H_1 onto the subset Σ of the sequence space $\mathbb{C}^{\mathbb{N}}$ consisting of the sequences $\vec{c} = (c_m)_{m\geq 0}$ such that $\sum_{m=0}^{\infty} |c_m|^2 \leq 1$. Endow $\mathbb{C}^{\mathbb{N}}$ with the product topology, which may be defined via the distance $\sum_{m\geq 0} 2^{-m}\inf(|c_m-c'_m|,1)$ between two sequences $\vec{c} = (c_m)$ and $\vec{c}\,' = (c'_m)$. If $\vec{c} = (c_m)$ belongs to Σ, then $|c_m| \leq 1$ for every m; moreover $\Sigma = \bigcap_{m\geq 0} \Sigma_m$ where Σ_m is defined by the inequality $|c_0|^2+\dots+|c_m|^2 \leq 1$. It follows that Σ is bounded and closed in $\mathbb{C}^{\mathbb{N}}$, hence compact by Tychonov's theorem. Transporting the topology to H_1, we conclude that H_1 *is a compact Hausdorff space for what is the weak topology.*

Let θ be a fixed vector in H, with components d_m. Since a uniform limit of continuous functions in continuous, the inequality

$$|\langle\theta|\psi\rangle - (\bar{d}_0c_0 + \dots + \bar{d}_mc_m)| \leq \sum_{p=m+1}^{\infty} |d_p|^2 \quad .$$

for ψ in H_1, together with $\lim_{m\to\infty} \sum_{p=m+1}^{\infty} |d_p|^2 = 0$, shows that the function $\psi \mapsto \langle\theta|\psi\rangle$ is continuous on H_1.

For ψ in H_1, put $F(\psi) = \langle\psi|A|\psi\rangle$. A calculation similar to the one in subsection 6.1 gives $F(\psi) = \sum_{m=0}^{\infty} |\langle B\theta_m|\psi\rangle|^2$. Since $|\langle B\theta_m|\psi\rangle|^2$ is majorized by $||B\theta_m||^2$ and the series $\sum_{m=0}^{\infty} ||B\theta_m||^2$ converges to the finite limit $\sum_{m=0}^{\infty} \langle\theta_m|A|\theta_m\rangle = \mathrm{Tr}(A)$, it follows,again by uniform convergence, that *the functional* F *is continuous on* H_1. Since the space H_1 is compact, F achieves its maximum at some point ψ_0 of H_1. If $\psi_0 = 0$, then F is identically 0 on H_1, hence A = 0. Otherwise, $||\psi_0||^{-1}\psi_0$ belongs to H_1, hence $F(\psi_0)$ majorizes $F(||\psi_0||^{-1}\psi_0) = F(\psi_0)/||\psi_0||^2$, hence $||\psi_0|| = 1$. Let ψ in H be orthogonal to ψ_0; comparing the values of F at ψ_0 and at the point $\psi(t) = (\psi_0 + t\psi)/\sqrt{1+t^2}$ for real t, one gets $\mathrm{Re}\langle\psi|A|\psi_0\rangle = 0$. Replacing ψ by $i\psi$ we prove that $\mathrm{Im}\langle\psi|A|\psi_0\rangle$ is also 0. *Conclusion:* $A\psi_0$ is orthogonal to any vector orthogonal to ψ_0, that is $A\psi_0 = \lambda_0\psi_0$ for some scalar λ_0. Notice that $F(\psi_0)=\langle\psi_0|A\psi_0\rangle= \lambda_0$, hence $\lambda_0 > 0$.

We repeat the same reasoning in the space $H(1)$ of vectors orthogonal to ψ_0. We get a vector ψ_1 in $H(1)$, of norm 1, such that ψ_1 attains on $H_1 \cap H(1)$ its maximum λ_1 at ψ_1; moreover $\lambda_1 \geq 0$ and $A\psi_1 = \lambda_1\psi_1$. Continuing in this way, we get an orthonormal sequence $(\psi_n)_{n\geq 0}$ of vectors and scalars $\lambda_n > 0$ such that $A\psi_n = \lambda_n\psi_n$, $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \lambda_{n+1} \geq \ldots$ The process may break after a finite number of steps, or give an infinite sequence. Anyhow, if a vector in H_1 is orthogonal to $\psi_0, \psi_1, \ldots$, one gets $F(\psi) = 0$, that is $B\psi = 0$, that is $A\psi = 0$ (*Hint:* use the definition of the λ_n as successive maxima !).

6.4. For any bounded operator A on H, the operator A^*A is (selfadjoint) positive, hence its square root $|A| = (A^*A)^{1/2}$ is defined. We set

$$||A||_1 = \mathrm{Tr}(|A|) \tag{6.2}$$

and denote it the *trace-norm*. The basic property

$$||A + B||_1 \leq ||A||_1 + ||B||_1 \tag{6.3}$$

is not at all obvious. The main difficulty is to prove that if $||A||_1$ and $||B||_1$ are finite, then $|A + B|$ can be diagonalized; one can again rely on compactness arguments.

One denotes by $L^1(H)$ the set of bounded operators A in H for which $||A||_1$ is finite. According to (6.3), this is a subspace of the vector space $L(H)$ of all bounded operators. Moreover, any element A in $L^1(H)$ is a finite linear combination of positive operators A_j, with $Tr(A_j)$ finite. It follows that the trace of A can be defined by

$$Tr(A) = \sum_{n\geq 0} \langle\psi_n|A|\psi_n\rangle \quad ; \qquad (6.4)$$

the series converges absolutely for every orthonormal basis $(\psi_n)_{n\geq 0}$ of H, and its sum does not depend on the particular basis chosen. Moreover, one proves the inequality

$$|Tr(A)| \leq ||A||_1 \quad . \qquad (6.5)$$

By formula (6.3), $A \mapsto ||A||_1$ is a norm on the space $L^1(H)$, *which is a Banach space*, that is satisfies Cauchy's convergence criterion.

Let A in $L^1(H)$. As we saw in subsection 6.2 and 6.3, the operator $|A|$ can be diagonalized. From the definition of $|A|$ by $|A|^2 = A^*A$, it follows that $A\psi$ and $|A|\psi$ have the same norm for every vector ψ in H. Suppose that $|A|$ is not of finite rank, and diagonalize it with eigenvectors ψ_n and eigenvalues λ_n as in subsection 6.2. Then

$$||A\psi_n|| = || \; |A|\psi_n|| = ||\lambda_n\psi_n|| = \lambda_n$$

and, for $m \neq n$

$$\langle A\psi_n|A\psi_m\rangle = \langle\psi_n|A^*A\psi_m\rangle = \langle\psi_n| \; |A|^2\psi_m\rangle =$$
$$= \langle\psi_n|\lambda_m^2\psi_m\rangle = 0 \quad .$$

It follows that there exists an orthonormal sequence $(\theta_n)_{n\geq 0}$ such that $A\psi_n = \lambda_n\theta_n$ for every n, hence the following representation for the operator A

$$A\psi = \sum_{n=0}^{\infty} \lambda_n \theta_n \langle \psi_n | \psi \rangle \qquad (\text{for } \psi \text{ in } H) \quad . \tag{6.6}$$

Notice that the series converges in norm since $\sum_n \lambda_n$ is finite. In Dirac's notation, this can be expressed as

$$A = \sum_{n=0}^{\infty} \lambda_n |\theta_n\rangle\langle\psi_n| \quad . \tag{6.7}$$

When A is of finite rank N, just replace the summation symbol by $\sum_{n=0}^{N-1}$.

Conversely, if ψ_n and θ_n are vectors of norm 1 in H, and the scalars λ_n form an absolutely convergent series, then formula (6.7) defines an operator A in $L^1(H)$; we do not assume any orthogonality property of the vectors ψ_n and θ_n. In a series of operators like (6.7), the following estimate holds

$$\left|\left|A - \sum_{n=0}^{N-1} \lambda_n |\theta_n\rangle\langle\psi_n| \right|\right|_1 \leq \sum_{n=N}^{\infty} |\lambda_n|$$

and since $\sum_{n=N}^{\infty} |\lambda_n|$ tends to 0 with 1/N, it follows that *the operators of finite rank are dense in the Banach space* $L^1(H)$.

Define $\mu_n(A)$ as the eigenvalue of rank n of $|A|$. Hence

$$\mu_0(A) \geq \mu_1(A) \geq \ldots \geq \mu_n(A) \geq \mu_{n+1}(A) \geq \ldots \geq 0 \quad ,$$

and put $\mu_n(A) = 0$ for $n \geq N$ if A is of finite rank N. Moreover by definition

$$||A||_1 = \sum_{n=0}^{\infty} \mu_n(A) \quad . \tag{6.8}$$

The inequality (6.3) can be strengthened to the sequence of inequalities

$$\mu_{n+m}(A+B) \leq \mu_n(A) + \mu_m(B) \tag{6.9}$$

for $n > 0$, $m \geq 0$ and A,B in $L^1(H)$. For $n = 0$, $\mu_0(A)$ is equal to the *operator norm of* A, that is the smallest constant $||A||$ such that $||A\psi|| \leq ||A||\ ||\psi||$ for all vectors ψ in H.

In general, $\mu_n(A)$ can be calculated using the *minimax principle*. Let V be a vector subspace of H , of finite dimension n; let us denote by $||A||_V$ the smallest constant such that $||A\psi|| \leq ||A||_V||\psi||$ for each vector ψ in H orthogonal to V. Then the following inequality holds

$$||A||_V \geq \mu_n(A) \quad , \tag{6.10}$$

with equality when V is spanned by the vectors $\psi_0,\ldots,\psi_{n-1}$, where $|A|\psi_k = \mu_k(A)\psi_k$ for any $k \geq 0$.

6.5. The *trace class* of operators $L^1(H)$ is very important in theory, but there exists no easy criterion to decide whether a concrete operator is in $L^1(H)$. According to the connection between Fourier series and integral operators described in subsection 5.10, such a criterion would settle the question of characterizing the continuous functions with absolutely convergent Fourier series, a notably difficult question.

Contrasting with this situation, *Hilbert-Schmidt operators* are plentiful and easy to characterize. Define $L^2(H)$ as the class of operators A for which $\mathrm{Tr}(A^*A) = \mathrm{Tr}(|A|^2)$ is finite. This means that the positive operator $|A|$ can be diagonalized with eigenvectors $\psi_0,\psi_1,\ldots,\psi_n,\ldots$ and eigenvalues $\lambda_0,\lambda_1,\ldots,\lambda_n,\ldots$ such that $\sum_{n=0}^{\infty}|\lambda_n|^2$ be finite. As before, set $\mu_n(A) = \lambda_n$ and define the Hilbert-Schmidt norm by

$$||A||_2 = \mathrm{Tr}(A^*A)^{1/2} = \Big(\sum_{n=0}^{\infty} \mu_n(A)^2\Big)^{1/2} \quad . \tag{6.11}$$

The minimax principle holds again, as well as the representation (6.7) with two orthonormal sequences (ψ_n) and (θ_n).

Let $(e_n)_{n\geq 0}$ be an arbitrary orthonormal basis of H . One gets, for any bounded operator A in $L(H)$

$$\mathrm{Tr}(A^*A) = \sum_n \langle e_n|A^*A\, e_n\rangle = \sum_n ||Ae_n||^2 =$$
$$= \sum_{m,n} |\langle e_m|A|e_n\rangle|^2 \quad .$$

If we associate to any operator A its matrix with entries $a_{mn} = \langle e_m|A|e_n\rangle$, we get an isomorphism of $L^2(H)$ with the set of matrices (a_{mn}) such that $\sum_{m,n} |a_{mn}|^2$ be finite. Otherwise stated $L^2(H)$ is a Hilbert space, and the operators $|e_m\rangle\langle e_n|$ form an orthonormal basis in $L^2(H)$.

Let A,B be operators in $L^2(H)$. From the *polarization formula*

$$4A^*B = (A+B)^*(A+B) - (A-B)^*(A-B) - i(A+iB)^*(A+iB) + i(A-iB)^*(A-iB) \quad ,$$

and the definition of $L^2(H)$, it follows that A^*B is in $L^1(H)$, and the trace of A^*B is defined. By repeating the calculation of $\mathrm{Tr}(A^*A)$, one gets

$$\mathrm{Tr}(A^*B) = \sum_{m,n} \overline{\langle e_m|A|e_n\rangle}\langle e_m|B|e_n\rangle \quad . \tag{6.12}$$

Otherwise stated, the scalar product in the Hilbert space $L^2(H)$ is given by

$$\langle A|B\rangle = \mathrm{Tr}(A^*B) \quad . \tag{6.13}$$

Cauchy-Schwarz inequality then holds:

$$|\mathrm{Tr}(A^*B)| \leq ||A||_2||B||_2 \quad . \tag{6.14}$$

It can be strengthened to

$$||AB||_1 \leq ||A||_2||B||_2 \tag{6.15}$$

(compare with formula (6.5) and notice that $L^2(H)$ is stable under $A \mapsto A^*$). It can be shown that conversely, any operator in $L^1(H)$ can be factored as the product of two operators in $L^2(H)$.

6.6. We conclude by two remarks:

(a) Suppose that Ω is any (measurable) subset of some euclidean space $\mathbb{R}^m$. Consider the Hilbert space $L^2(\Omega)$ of square-integrable functions on Ω with scalar product

$$\langle f|g\rangle = \int_\Omega \overline{f(x)}\, g(x)\, dx \quad . \tag{6.16}$$

The Hilbert-Schmidt operators in $L^2(\Omega)$ are then the operators given by a kernel K in $L^2(\Omega \times \Omega)$. More precisely, for f in $L^2(\Omega)$, and K in $L^2(\Omega \times \Omega)$, the integral

$$Kf(x) = \int_\Omega K(x,y)\, f(y)\, dy \tag{6.17}$$

converges for almost all x in Ω, the function Kf is in $L^2(\Omega)$, the operator $f \mapsto Kf$ is in $\mathcal{L}^2(L^2(\Omega))$ and

$$\mathrm{Tr}(K^*K) = \int_\Omega \int_\Omega |K(x,y)|^2\, dx\, dy \quad . \tag{6.18}$$

The proofs are easy consequences of Fubini's theorem about double integrals.

(b) For any bounded operator A acting on $\mathcal{H}$, we have

$$||A|| \leq ||A||_2 \leq ||A||_1 \quad , \tag{6.19}$$

hence hence in the opposite direction

$$\mathcal{L}(\mathcal{H}) \supset \mathcal{L}^2(\mathcal{H}) \supset \mathcal{L}^1(\mathcal{H}) \quad . \tag{6.20}$$

One can interpolate with the spaces $\mathcal{L}^p(\mathcal{H})$, introduced by Schatten around 1940, that is the set of operators A for which $|A|$ can be diagonalized with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ such that $\sum_n |\lambda_n|^p$ be finite. One can mimic the basic properties of classical Lebesgue spaces $L^p(\Omega)$, for instance Minkowski and Hölder inequalities.

7. Fredholm Determinants in Hilbert Space

7.1. Suppose one wants to define the determinant of an operator B acting on some Hilbert space $\mathcal{H}$. Choose an orthonormal basis $(\psi_n)_{n\geq 0}$ of $\mathcal{H}$ and represent the operator by its matrix $b = (b_{mn})$ where $b_{mn} = \langle\psi_m|B|\psi_n\rangle$. To define the determinant Δ of this infinite matrix, a natural procedure is to truncate it to a finite determinant

$$\Delta_N = \det \begin{pmatrix} b_{oo} & \cdots\cdots & b_{oN} \\ b_{1o} & \cdots\cdots & b_{1N} \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ b_{No} & \cdots\cdots & b_{NN} \end{pmatrix} \tag{7.1}$$

and to look for the limit $\Delta = \lim_{N\to\infty} \Delta_N$. If the matrix b is diagonal, we get $\Delta_N = b_{oo} \ldots b_{NN}$ and Δ is the infinite product $\prod_{n\geq 0} b_{nn}$. It is known that such an infinite product converges absolutely iff b_{nn} can be put in the form $b_{nn} = 1 + a_n$ where $\sum_{n\geq 0} |a_n|$ if finite. This remark led Poincaré and von Koch to assume that the operator B is of the form $1 + A$, where A is "small" in a suitable sense. Denoting as before by $\Delta\binom{i_1 \ldots i_p}{j_1 \ldots j_p}$ the minors of the matrix $a = (a_{mn})$ associated to the operator A, we get

$$\Delta_N = \sum_{p\geq 0} \; \sum_{0\leq i_1<\ldots<i_p\leq N} \Delta\binom{i_1 \ldots i_p}{i_1 \ldots i_p} \tag{7.2}$$

by formula (5.2) (first noticed by von Koch). Hence, at least formally, we get

$$\Delta = \sum_{p\geq 0} \; \sum_{i_1<\ldots<i_p} \Delta\binom{i_1 \ldots i_p}{i_1 \ldots i_p} \tag{7.3}$$

for the determinant of $1 + A$. It can be shown that the previous series converges absolutely provided the double sum $\sum_{m,n} |a_{mn}|$ is finite (Poincaré's criterion)[1]

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[1] For a lively account of the prehistory of infinite-determinants, the reader may consult Dieudonné's book [4].

We shall follow this method, but to obtain a theory independent of the orthonormal basis chosen, we shall suppose that A is of trace class, a more general assumption than the mere convergence of $\sum_{m,n} |a_{mn}|$. There are a number of problems connected with such a definition:

- convergence of the expansion (7.3)
- the multiplicative property of determinants
- relations between the eigenvalues of A and the zeroes of the characteristic function $\det(1 - zA)$, where z is a complex variable.

7.2. Our method will be based on the construction of the *fermionic Fock space*. Let us extend to the Hilbert space set up the constructions of tensor spaces given in subsection 2. Let H_1 and H_2 be two Hilbert spaces. We propose to associate to H_1 and H_2 a new hilbert space H, to be denoted by $H_1 \otimes_2 H_2$, together with a map associating to a vector x_1 in H_1 and a vector x_2 in H_2 a vector $x_1 \otimes x_2$ in H and assume the following properties to hold:

(a) The vector $x_1 \otimes x_2$ depends linearly on x_1 for a fixed x_2, and symmetrically in x_2 for a fixed x_1.

(b) For the scalar products, one gets

$$\langle x_1 \otimes x_2 | y_1 \otimes y_2 \rangle = \langle x_1 | y_1 \rangle \langle x_2 | y_2 \rangle \quad . \tag{7.4}$$

(c) Any vector in H is a limit of finite linear combinations $\sum_{i=1}^{N} x_i \otimes y_i$. Equivalently, if z is any nonzero vector in H, there exist x_1 and x_2 such that $\langle x_1 \otimes x_2 | z \rangle \neq 0$.

As for the *existence of* H, choose an orthonormal basis $(\psi_n^{(i)})_{n\geq 0}$ in H_i (for i = 1 or 2), take any Hilbert space H with an orthonormal basis indexed as a double sequence $(e_{mn})_{m\geq 0, n\geq 0}$ and define the map $(x_1, x_2) \to x_1 \otimes x_2$ by the formula

$$x_1 \otimes x_2 = \sum_{m,n} \langle \psi_m^{(1)} | x_1 \rangle \langle \psi_n^{(2)} | x_2 \rangle e_{mn} \quad . \tag{7.5}$$

The properties (a), (b), (c) are easily checked. Moreover, according to this definition, one gets in particular $e_{mn} = \psi_m^{(1)} \otimes \psi_n^{(2)}$.

To prove the *uniqueness* of this construction, one first proves using (a), (b) and (c) that the tensor product $(\psi_m^{(1)} \otimes \psi_n^{(2)})$ of the orthonormal basis in H_1 and H_2 is an orthonormal basis in $H_1 \otimes_2 H_2$. Moreover by specializing formula (7.4), one gets the values $\langle\psi_m^{(1)}|x_1\rangle\langle\psi_n^{(2)}|x_2\rangle$ for the components of $x_1 \otimes x_2$ in the previous basis of $H_1 \otimes_2 H_2$, hence formula (7.5)is forced upon us.

By an obvious generalization, one defines the tensor product $H_1 \otimes_2 \ldots \otimes_2 H_k$ of any finite family of Hilbert spaces $H_1,\ldots,H_k$. For the record, notice the formula for the scalar product

$$\langle x_1 \otimes \ldots \otimes x_k | y_1 \otimes \ldots \otimes y_k\rangle = \prod_{i=1}^{k} \langle x_i | y_i\rangle \quad ; \tag{7.6}$$

moreover, if $(\psi_n^{(i)})_{n\geq 0}$ is any orthonormal basis in H_i (for $i=1,\ldots,k$) then the multiple sequence of tensor products $\psi_{n_1}^{(1)} \otimes \ldots \otimes \psi_{n_k}^{(k)}$ is an orthonormal basis in $H_1 \otimes_2 \ldots \otimes_2 H_k$. In the sequel, we restrict our attention to the tensor product $H^{\otimes k}$ of k identical spaces $H_1 = \ldots = H_k = H$. Whenever convenient, we assume that an orthonormal basis $(\psi_n)_{n\geq 0}$ has been chosen for H , hence the tensors $\psi_{n_1} \otimes \ldots \otimes \psi_{n_k}$ form an orthonormal basis in $H^{\otimes k}$.

7.3. The symmetry operators in $H^{\otimes k}$ are defined as follows. Given any permutation σ in S_k, there exists a unitary operator U_σ in $H^{\otimes k}$ permuting the basic tensors as follows

$$U_\sigma(\psi_{n_1} \otimes \ldots \otimes \psi_{n_k}) = \psi_{n'_1} \otimes \ldots \otimes \psi_{n'_k} \tag{7.7}$$

where $n_i = n'_{\sigma(i)}$ for $1\leq i\leq k$. By linearity and continuity, one deduces the relation

$$U_\sigma(x_1 \otimes \ldots \otimes x_k) = x_{\sigma^{-1}(1)} \otimes \ldots \; x_{\sigma^{-1}(k)} \tag{7.8}$$

for $x_1,\ldots,x_k$ in H. Hence U_σ is invariantly defined, and the product rule $U_\sigma U_\tau = U_{\sigma\tau}$ holds. Otherwise stated, one obtains a unitary representation of the group S_k in the Hilbert space $H^{\otimes k}$.

As in the finite dimensional case, one introduces the subspace $\Lambda^k H$ of the antisymmetric tensors, elements t in $H^{\otimes k}$

such that $U_\sigma t = (\mathrm{sgn}\sigma).t$ for all σ in S_k. It is customary to modify the definition of the wedge product as follows

$$x_1\wedge\ldots\wedge x_k = (k!)^{-1/2} \sum_{\sigma\in S_k} (\mathrm{sgn}\sigma)\, U_\sigma(x_1\otimes\ldots\otimes x_k) \qquad (7.9)$$

With this convention, the scalar product is given by

$$\langle x_1\wedge\ldots\wedge x_k | y_1\wedge\ldots\wedge y_k\rangle = \det_{\substack{1\leq i\leq k\\ 1\leq j\leq k}} \langle x_i|y_j\rangle \qquad (7.10)$$

without normalization constant. It follows that *the wedge products* $\psi_{n_1}\wedge\ldots\wedge\psi_{n_k}$ *where* $0 \leq n_1<\ldots<n_k$ *form an orthonormal basis of* $\Lambda^k H$.

By the antisymmetry properties of the wedge product, it follows that $x_1\wedge\ldots\wedge x_k$ is equal to $x_1'\wedge\ldots\wedge x_k'$ where $x_1',\ldots,x_k'$ are deduced from $x_1,\ldots,x_k$ by the orthonormalisation process: $x_j' - x_j$ is a linear combination of $x_1,\ldots,x_{j-1}$ and $x_1',\ldots,x_k'$ are mutually orthogonal. By Pythagoras'theorem, one gets $||x_i'|| \leq ||x_i||$, and from (7.10) one gets

$$||x_1\wedge\ldots\wedge x_k||^2 = ||x_1'\wedge\ldots\wedge x_k'||^2 = ||x_1'||^2\ldots||x_k'||^2$$

hence

$$||x_1\wedge\ldots\wedge x_k|| \leq ||x_1||\ldots||x_k|| \quad . \qquad (7.11)$$

Using Cauchy-Schwarz inequality, one derives the corollary

$$|\langle x_1\wedge\ldots\wedge x_k|y_1\wedge\ldots\wedge y_k\rangle| \leq \prod_{i=1}^{k} ||x_i||\,||y_i|| \quad . \qquad (7.12)$$

These formulas are just reincarnations of Hadamard's inequality for determinants.

In many applications, H is a Hilbert space $L^2(\Omega)$ of square-integrable functions. One identifies $H^{\otimes k}$ to the space $L^2(\Omega\times\ldots\times\Omega)$ (k factors) of square- integrable functions $f(t_1,\ldots,t_k)$ where the variables t_i run over Ω. The tensor product $f_1\otimes\ldots\otimes f_k$ of one variable functions in $L^2(\Omega)$ is then given by

$$(f_1\otimes\ldots\otimes f_k)(t_1,\ldots,t_k) = \prod_{i=1}^{k} f_i(t_i) \qquad (7.13)$$

and the wedge product is a determinant

$$(f_1\wedge\ldots\wedge f_k)(t_1,\ldots,t_k) \;=\; (k!)^{-1/2} \det_{\substack{1\leq i\leq k\\ 1\leq j\leq k}} f_i(t_j)\;. \qquad (7.14)$$

7.4. Let us insert a few remarks.

(a) We remind the reader of the so-called *"reconstruction theorem"* for Hilbert spaces. Let I be any index set and let a_{ij} be complex numbers, for i and j in I. In order that there exist a Hilbert space H and vectors ψ_i in H (for i in I) such that $\langle\psi_i|\psi_j\rangle = a_{ij}$, it is necessary and sufficient that the following inequalities hold

$$\sum_{r=1}^{N}\sum_{s=1}^{N} \bar{z}_r a_{i_r i_s} z_s \geq 0 \qquad (7.15)$$

for any set $\{i_1,\ldots,i_N\}$ of indices and any set $\{z_1,\ldots,z_N\}$ of complex numbers. We can always assume that the set of vectors ψ_i generates H , that is no vector $\psi \neq 0$ in H is orthogonal to all ψ_i's. If this is the case, the Hilbert space H is unique; namely if H' is any Hilbert space, generated by vectors ψ_i' such that $\langle\psi_i'|\psi_j'\rangle = a_{ij}$, there exists a unique unitary operator $U: H \to H'$ such that $U(\psi_i) = \psi_i'$ for all indices i.

The Hilbert spaces $H^{\otimes k}$ and $\Lambda^k H$ are respectively generated by vectors of the form $x_1\otimes\ldots\otimes x_k$ and $x_1\wedge\ldots\wedge x_k$. The scalar products are given by the formulas (7.6) and (7.10) respectively. One could therefore appeal to the reconstruction theorem to establish the existence of the spaces $H^{\otimes k}$ and $\Lambda^k H$. For this it would be necessary to establish the inequalities (7.15) directly.

(b) The concept of tensor product of Hilbert spaces is closely connected to the notion of Hilbert-Schmidt operators. Namely, let H be any Hilbert space. We associate to it a new Hilbert space H^* and a map $\psi \mapsto \psi^*$ from H onto H^* as follows: H^* consists of the bounded linear forms on H (i.e. linear mappings $h: H \to \mathbb{C}$ for which there exists a bound $C \geq 0$ with $|h(\psi)| \leq C||\psi||$). For ψ in H, we denote by ψ^* the linear form $\psi' \mapsto \langle\psi|\psi'\rangle$. It is a well-known result by F.Riesz that $\psi \mapsto \psi^*$ is a bijection of H onto H^*. The Hilbert space structure of H^*

is defined in such a way that $(c\psi)^* = \bar{c}\psi^*$ and $\langle\psi^*|\psi'^*\rangle = \langle\psi'|\psi\rangle$ for ψ , ψ' in H and a complex scalar c. In Dirac's notation, denoting ta vector ψ in H as a ket $|\psi\rangle$, the element ψ^* of H^* is the corresponding bra $\langle\psi|$.

There is a natural isomorphism of the Hilbert space $H \otimes_2 H^*$ onto the space $L^2(H)$ of Hilbert-Schmidt operators. It associates to a generator $\psi_1 \otimes \psi_2^*$ the operator $|\psi_1\rangle\langle\psi_2|$ (taking ψ into $\psi_1\langle\psi_2|\psi\rangle$ for ψ in H). This isomorphism is implicitely used in the Dirac notation of bras and kets.

7.5. We show how to extend a bounded operator A acting on H to the spaces $H^{\otimes k}$ and $\Lambda^k H$. First of all consider two Hilbert spaces H_1 and H_2, with orthonormal basis $(\psi_n^{(1)})$ and $(\psi_n^{(2)})$ respectively and bounded operators A_1 in $L(H_1)$ and A_2 in $L(H_2)$. Any vector in $H_1 \otimes_2 H_2$ can be uniquely expanded as t $t = \sum_n t_n \otimes \psi_n^{(2)}$ where $||t||^2 = \sum_n ||t_n||^2$ is finite; for instance, if $t = x_1 \otimes x_2$ then $t_n = \langle\psi_n^{(2)}|x_2\rangle \cdot x_1$. Extend A_1 to $H_1 \otimes_2 H_2$ by the rule

$$A_1^{(1)}t = \sum_n At_n \otimes \psi_n^{(2)} \qquad \text{for } t = \sum_n t_n \otimes \psi_n^{(2)} \quad . \tag{7.16}$$

This defines a bounded operator $A_1^{(1)}$ in $L(H_1 \otimes_2 H_2)$ according to the estimate

$$||A_1^{(1)}t||^2 = \sum_n ||At_n||^2 \leq ||A||^2 \sum_n ||t_n||^2 = ||A||^2 ||t||^2 ,$$

hence $||A_1^{(1)}|| = ||A_1||$. Moreover, if $t = x_1 \otimes x_2$ we get $A_1^{(1)}t = A_1x_1 \otimes x_2$, thereby the definition of $A_1^{(1)}$ is independent of the basis $(\psi_n^{(2)})$. Similarly, there exists a bounded operator $A_2^{(2)}$ acting on $H_1 \otimes_2 H_2$ in such a way that $A_2^{(2)}(x_1 \otimes x_2) = x_1 \otimes A_2x_2$ with a bound $||A_2^{(2)}|| = ||A_2||$. Define the operator $A_1 \otimes A_2$ as the product $A_1^{(1)}A_2^{(2)}$, hence the rules

$$\begin{aligned} (A_1 \otimes A_2)(x_1 \otimes x_2) &= A_1x_1 \otimes A_2x_2 \quad , \\ ||A_1 \otimes A_2|| &= ||A_1|| \cdot ||A_2|| \quad . \end{aligned} \tag{7.17}$$

The extension to the case of k spaces $H_1,\ldots,H_k$ is obvious.

Coming back to H and A, there exists therefore a bounded operator $A^{\otimes k}$ acting on $H^{\otimes k}$ in such a way that

$$A^{\otimes k}(x_1 \otimes \ldots \otimes x_k) = Ax_1 \otimes \ldots \otimes Ax_k \tag{7.18}$$

and $||A^{\otimes k}|| = ||A||^k$. From formula (7.8), it follows that $A^{\otimes k}$ and U_σ commute for σ in S_k, hence $A^{\otimes k}$ induces a bounded operator $\Lambda^k A$ in the closed subspace $\Lambda^k H$ of $H^{\otimes k}$. It is characterized by its action on the wedge products

$$\Lambda^k A\,(x_1 \wedge \ldots \wedge x_k) = Ax_1 \wedge \ldots \wedge Ax_k \quad . \tag{7.19}$$

7.6. THEOREM. *Suppose* A *is a trace-class operator on* H. *Then* $\Lambda^k A$ *is a trace-class operator on* $\Lambda^k H$*; its bound is given by* $||\Lambda^k A|| = \mu_0(A) \ldots \mu_{k-1}(A)$ *and moreover* $||\Lambda^k A||_1 \leq ||A||_1^k / k!$.

Proof: It follows from the construction of $\Lambda^k A$ that $\Lambda^k(A^*)$ is equal to $(\Lambda^k A)^*$ and $\Lambda^k(AB)$ to $\Lambda^k A \cdot \Lambda^k B$ for another operator B in $L(H)$. Since the positive operators are the operators of the form C^*C and $|C|$ is the unique positive operator such that $|C|^2 = C^*C$, we get $|\Lambda^k A| = \Lambda^k |A|$. Since A is trace-class, the operator $|A|$ can be diagonalized with eigenvectors ψ_n and eigenvalues $\nu_n \geq 0$. Then the wedge products $\psi_{n_1} \wedge \ldots \wedge \psi_{n_k}$ for $n_1 < n_2 < \ldots < n_k$ form an orthonormal basis of $\Lambda^k H$, and $\Lambda^k |A|$ multiplies it by $\nu_{n_1} \ldots \nu_{n_k}$. By definition, the sequence

$$\mu_0(A) \geq \mu_1(A) \geq \ldots \geq \mu_{k-1}(A) \geq \mu_k(A) \geq \ldots$$

is obtained by rearranging in descending order the nonzero eigenvalues ν_n. Therefore $\mu_0(A) \ldots \mu_{k-1}(A)$ is the largest among the products $\nu_{n_1} \ldots \nu_{n_k}$, hence is equal to $||\Lambda^k A||$. Finally $||\Lambda^k A||_1$ is the trace of $|\Lambda^k A|$, that is

$$||\Lambda^k A||_1 = \sum_{n_1 < \ldots < n_k} \nu_{n_1} \ldots \nu_{n_k} \tag{7.20}$$

In the same vein

$$||A||_1 = \sum_n \nu_n \tag{7.21}$$

hence the inequality $||\Lambda^k A||_1 \leq ||A||_1^k/k!$.

Q.E.D.

Let us add a few comments:

(a) Suppose the operator A is of finite rank N. Then the formula $||\Lambda^k A|| = \mu_0(A) \ldots \mu_{k-1}(A)$ holds for $0 \leq k \leq N$, moreover $\Lambda^k A = 0$ for $k > N$. This justifies the convention $\mu_k(A)=0$ for $k \geq N$.

(b) For a while, we do not assume that A is trace-class. It can be shown that the norm $||\Lambda^k A||$ can be defined as

$$||\Lambda^k A|| = \sup||Ax_1 \wedge \ldots \wedge Ax_k|| \tag{7.22}$$

or

$$||\Lambda^k A|| = \sup\left| \det_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} \langle x_i | A | y_j \rangle \right| \tag{7.23}$$

where $x_1, \ldots, x_k$ run over the vectors of norm 1 in H(for (7.22)) and similarly for $y_1, \ldots, y_k$ in (7.23). Define the numbers

$$\mu_k(A) = ||\Lambda^{k+1} A|| \cdot ||\Lambda^k A||^{-1} \quad . \tag{7.24}$$

By reduction to the finite-dimensional case (hint: use (7.22)), it can be shown that the sequence is non-increasing

$$\mu_0(A) \geq \mu_1(A) \geq \ldots$$

with $\mu_0(A) = ||A||$. It may very well be that $\mu_k(A) = ||A||$ for all k's.

(c) By a proof similar to that of theorem 7.6, one gets that $A^{\otimes k}$ is trace-class if A is. Moreover

$$||A^{\otimes k}|| = ||A||^k \quad , \quad ||A^{\otimes k}||_1 \leq ||A||_1^k \quad , \tag{7.25}$$

and the inequality is optimal. The antisymmetric part $\Lambda^k H$ of $H^{\otimes k}$ is the *fermionic Fock space*. One can define in a similar way the symmetric part $S^k H$ of $H^{\otimes k}$, consisting of the element t such that $U_\sigma t = t$ for σ in S_k. It is the *bosonic Fock space*.

The operator $A^{\otimes k}$ induces an operator S^kA in S^kH , and we get the estimates

$$||S^kA|| \leq ||A||^k \quad , \quad ||S^kA||_1 \leq ||A||_1^k \tag{7.26}$$

which are again optimal. The constant 1/k! in the estimate $||\Lambda^kA||_1 \leq (1/k!)||A||_1^k$ has therefore no counterpart in the spaces $H^{\otimes k}$ and S^kH . It is one of the deepest manifestations of *Pauli's exclusion principle.*

7.7. We are ready to define the determinant det(1+A) in case A is trace-class. Indeed, define $c_k(A)$ as the trace of Λ^kA. By theorem 7.6 one gets the estimate

$$|c_k(A)| = |\mathrm{Tr}\Lambda^kA| \leq ||\Lambda^kA||_1 \leq ||A||_1^k/k! \quad .$$

The series $\sum_{k\geq 0} c_k(A)$ is therefore absolutely convergent and we put

$$\det(1+A) = \sum_{k\geq 0} c_k(A) \quad . \tag{7.27}$$

Replacing A by zA, where z is a complex variable, one defines the *characteristic determinant*

$$\det(1-zA) = \sum_{k\geq 0} (-1)^k c_k(A) z^k \quad , \tag{7.28}$$

which is an entire function of z by the basic estimate

$$|c_k(A)| \leq ||A||_1^k/k! \quad . \tag{7.29}$$

Put in another form, introduce the *total Fock space* ΛH , orthogonal sum of the Hilbert spaces $\Lambda^0 H$, $\Lambda^1 H$, ... It is generated by the wedge products $x_1\wedge\ldots\wedge x_k$ of varying order k, with scalar product given by

$$\langle x_1\wedge\ldots\wedge x_k|y_1\wedge\ldots\wedge y_k\rangle = \det\langle x_i|y_j\rangle \tag{7.30}$$

$$\langle x_1\wedge\ldots\wedge x_k|y_1\wedge\ldots\wedge y_l\rangle = 0 \qquad \text{if } k \neq l \ . \tag{7.31}$$

The various operators Λ^kA extend to an operator ΛA acting on ΛH and mapping $x_1\wedge\ldots\wedge x_k$ into $Ax_1\wedge\ldots\wedge Ax_k$. From theorem 7.6, it follows that $\wedge A$ is a trace-class operator such that $||\wedge A||_1 = \sum_{k\geq 0} ||\wedge^k A||_1$ hence

$$||\wedge A||_1 \leq \exp||A||_1 \quad . \tag{7.32}$$

From (7.27) we get the *compact definition*

$$\det(1+A) = \mathrm{Tr}(\wedge A) \quad . \tag{7.33}$$

Introduce an orthonormal basis (ψ_n) for H and the matrix a corresponding to A with elements $a_{mn} = \langle\psi_m|A|\psi_n\rangle$. Then the wedge products $\psi_{i_1}\wedge\ldots\wedge\psi_{i_k}$ for $i_1<\ldots<i_k$ (k variable) form an orthonormal basis of $\wedge H$. Moreover the minors $\Delta\binom{i_1\ldots i_k}{j_1\ldots j_k}$ are simply scalar products, namely

$$\Delta\binom{i_1\ldots i_k}{j_1\ldots j_k} = \langle\psi_{i_1}\wedge\ldots\wedge\psi_{i_k}|\wedge A|\psi_{j_1}\wedge\ldots\wedge\psi_{j_k}\rangle \quad . \tag{7.34}$$

The trace of an operator can be calculated using any orthonormal basis, hence the *absolutely convergent expansions*

$$c_k(A) = \mathrm{Tr}(\wedge^k A) = \sum_{i_1<\ldots<i_k} \Delta\binom{i_1\ldots i_k}{i_1\ldots i_k} \tag{7.35}$$

and

$$\det(1-zA) = \sum_{k\geq 0}(-1)^k z^k \sum_{i_1<.<i_k} \Delta\binom{i_1\ldots i_k}{i_1\ldots i_k} \quad . \tag{7.36}$$

Von Koch' definition (7.3) is fully justified !

7.8. The previous definitions can be illustrated with the help of some considerations of *quantum statistical mechanics*. So suppose H corresponds to a quantum-mechanical particle, with hamiltonian operator $H^{(1)}$. Then the space $\wedge H$ corresponds to an assembly of particles obeying Fermi-Dirac statistics. The total hamiltonian H_F acts on $\wedge H$ in such a way that

$$H_F(x_1\wedge\ldots\wedge x_k) = \sum_{i=1}^{k} x_1\wedge\ldots\wedge H\, x_i\wedge\ldots\wedge x_k \tag{7.37}$$

(non-interacting system, the total energy is the sum of the energies of the k constituents). We do not discuss the question of domains, the self-adjoint operator $H^{(1)}$ being in general unbounded. At least formally, one gets

$$e^{-\beta H_F}(x_1\wedge\ldots\wedge x_k) = e^{-\beta H^{(1)}}x_1\wedge\ldots\wedge e^{-\beta H^{(1)}}x_k \qquad (7.38)$$

for $\beta > 0$. So the last definition of the total hamiltonian H_F is as the infinitesimal generator of the semi-group of operators $e^{-\beta H_F} = \Lambda(e^{-\beta H^{(1)}})$. In the cases of physical interest, the spectrum of $H^{(1)}$ has a lower bound, hence the operator $e^{-\beta H^{(1)}}$ is bounded.

The particle number operator N is given by $Nt = kt$ for t in $\Lambda^k H$. In statistical mechanics, one introduces the inverse temperature $\beta = 1/kT$ where k is Boltzmann's constant, and the chemical potential μ. According to Gibbs and Boltzmann, the thermodynamical quantities can be calculated using the so-called (fermionic) *partition function* $Z_F(\beta,\mu)=\mathrm{Tr}(e^{-\beta(H_F+\mu N)})$. From our definitions, one gets

$$Z_F(\beta,\mu) = \det(1 + e^{-\beta(H^{(1)}+\mu)}) \quad . \qquad (7.39)$$

These definitions make sense provided the operator $e^{-\beta H^{(1)}}$ be of trace-class for $\beta > 0$. In this case, the operator $H^{(1)}$ can be diagonalized with eigenvalues $E_0 \leq E_1 \leq E_2 \leq \ldots$ and eigenvectors $\psi_0, \psi_1, \psi_2, \ldots$. For k fixed, the state with lowest energy is $\psi_0\wedge\ldots\wedge\psi_{k-1}$ corresponding to the eigenvalue $E_0 + \ldots + E_{k-1}$ of H_F. Put $A = e^{-\beta(H^{(1)}+\mu)}$, hence $\Lambda A = e^{-\beta(H_F+\mu N)}$. Then $\psi_0\wedge\ldots\wedge\psi_{k-1}$ corresponds to the largest eigenvalue $\exp -\beta(E_0+\ldots+E_{k-1}+k\mu)$ of $\Lambda^k A$, hence we get a "physical" interpretation of the formula

$$||\Lambda^k A|| = \mu_0(A)\ \ldots\ \mu_{k-1}(A)$$

where $\mu_i(A) = \exp -\beta(E_i+\mu)$. Since the operator ΛA is now in diagonal form, it is easy to calculate its trace, that is $\det(1+A)$, and we get the well-known formula

$$Z_F(\beta,\mu) = \prod_{i\geq 0}(1 + e^{-\beta(E_i+\mu)}) \quad . \qquad (7.40)$$

7.9. We can repeat almost *verbatim* the considerations in section 3. For instance, the operator

$$P_-^k = (k!)^{-1} \sum_{\sigma \in S_k} (\mathrm{sgn}\,\sigma) \cdot U_\sigma \tag{7.41}$$

is the orthogonal projection of $H^{\otimes k}$ onto the closed subspace $\Lambda^k H$. For σ in S_k, put

$$I(\sigma) = \mathrm{Tr}(U_\sigma \cdot A^{\otimes k}) \quad . \tag{7.42}$$

The trace of $\Lambda^k A$ is equal to $\mathrm{Tr}(P_-^k \cdot A^{\otimes k})$ since $A^{\otimes k}$ restricts to $\Lambda^k A$ on $\Lambda^k H$; hence we get

$$k!\ c_k(A) = \sum_{\sigma \in S_k} (\mathrm{sgn}\,\sigma)\ I(\sigma) \quad . \tag{7.43}$$

We have now to check the following properties:

(a) for σ, τ *in* S_k, *we get* $I(\tau\sigma\tau^{-1}) = I(\sigma)$;

(b) *if* σ *is decomposed into cycles* $(1 \ldots a)(a+1 \ldots a+b)$ $(a+b+1 \ldots a+b+c) \ldots$, *then* $I(\sigma)$ *is equal to* $\mathrm{Tr}(A^a).\mathrm{Tr}(A^b).\mathrm{Tr}(A^c) \ldots$

The proof of (a) rests on the fact that U_τ commutes to $A^{\otimes k}$ and on the *unitary invariance of the trace*

$$\mathrm{Tr}(UBU^{-1}) = \mathrm{Tr}(B) \quad , \tag{7.44}$$

where U is any unitary operator (notice that the trace can be calculated using *any* orthonormal basis). For the proof of (b), introduce an orthonormal basis (ψ_n) in H; since the tensors $\psi_{n_1} \otimes \ldots \otimes \psi_{n_k}$ form an orthonormal basis of $H^{\otimes k}$, we get the *absolutely convergent expansion.*

$$I(\sigma) = \sum_{n_1 \ldots n_k} \prod_{i=1}^{k} \langle \psi_{n_i} | A | \psi_{n_{\sigma(i)}} \rangle \quad . \tag{7.45}$$

If σ is decomposed into cycles as in (b), this summation breaks into a product of sums like

$$J_a = \sum_{n_1 \ldots n_a} \langle \psi_{n_1} | A | \psi_{n_2} \rangle \langle \psi_{n_2} | A | \psi_{n_3} \rangle \ldots \langle \psi_{n_a} | A | \psi_{n_1} \rangle .$$

From Parseval theorem, one gets in general the matrix rule

$$\sum_s \langle\psi_r|B|\psi_s\rangle\langle\psi_s|C|\psi_t\rangle = \langle\psi_r|BC|\psi_t\rangle \tag{7.46}$$

hence $J_a = \sum_{n_1} \langle\psi_{n_1}|A^a|\psi_{n_1}\rangle = \mathrm{Tr}(A^a)$. Hence (b) follows. A similar proof would give

$$\mathrm{Tr}((A_1 \otimes \ldots \otimes A_k)\gamma_k) = \mathrm{Tr}(A_1 \ldots A_k) \tag{7.47}$$

for trace-class operators $A_1,\ldots,A_k$ in H and the cyclic permutation γ_k.

In a similar vein, introduce the trace $h_k(A)$ of S^kA acting on the bosonic Fock space S^kH. Since the orthogonal projection of $H^{\otimes k}$ onto S^kH is given by

$$P_+^k = (k!)^{-1} \sum_{\sigma\in S_k} U_\sigma \quad , \tag{7.48}$$

one gets

$$k!h_k(A) = \sum_{\sigma\in S_k} I(\sigma) \quad . \tag{7.49}$$

Putting $\tau_k(A) = \mathrm{Tr}(A^k)$, the formulas (3.33) to (3.38) as well as Plemelj's determinantal formulas (5.56) can be taken *verbatim* in our new context. Let us also mention the analogue of formula (3.27)

$$\sum_{k\geq 0} h_k(A)\, z^k = \det(1-zA)^{-1} \tag{7.50}$$

as well as the logarithmic derivative

$$\frac{d}{dz} \log \det(1 + zA) = \mathrm{Tr}(A(1 + zA)^{-1}) \ . \tag{7.51}$$

Notice that formula (7.50) holds for $|z|$ small enough and that both sides in (7.51) are meromorphic functions of z.

Here is a "physical" interpretation of formula (7.50). Suppose again that H is the one-particle state space with hamiltonian operator $H^{(1)}$. Introduce the total bosonic Fock space SH as the orthogonal sum of the spaces S^oH, S^1H, $S^2H,\ldots$ The total hamiltonian $\mathbb{H}$ acts on SH in such a way that

$$H_B(x_1 \ldots x_k) = \sum_{i=1}^{k} x_1 \ldots H^{(1)}x_i \ldots x_k \tag{7.52}$$

(symmetric product of vectors !). The bosonic partition function $Z_B(\beta,\mu) = \mathrm{Tr}(e^{-\beta(H_B+\mu N)})$ is then given in invariant form as

$$Z_B(\beta,\mu) = \det(1 - e^{-\beta(H^{(1)}+\mu)})^{-1} \tag{7.53}$$

provided μ is large enough. This corresponds to the customary relation of Planck-Einstein

$$Z_B(\beta,\mu) = \prod_{i\geq 0} \frac{1}{1 - e^{-\beta(E_i+\mu)}} \tag{7.54}$$

(valid whenever $E_0+\mu > 0$) in terms of the energy levels $E_0 \leq E_1 \leq \ldots$

7.10. Recall that the space of all trace-class operators is a Banach space $L^1(H)$ with norm $||A||_1 = \mathrm{Tr}(|A|)$, and that the operators of finite rank are dense in $L^1(H)$.

By definition, $\det(1+A)$ is given by the series $\sum_{k\geq 0} c_k(A)$ with the estimate $|c_k(A)| \leq ||A||_1^k/k!$. It follows that this series converges uniformly on any set of operators with $||A||_1 \leq R$, where R is a fixed bound. Hence the functional $A \mapsto \det(1+A)$ is *continuous* on the Banach space $L^1(H)$. It is even *holomorphic*. Indeed as in subsection 5.4, introduce a multilinear form on $L^1(H)$ by

$$c_k(A_1,\ldots,A_k) = \mathrm{Tr}(P_-^k\,(A_1 \otimes \ldots \otimes A_k)) \tag{7.55}$$

hence

$$\det(1+A) = \sum_{k\geq 0} c_k(A, \ldots, A) \quad . \tag{7.56}$$

Use now the *polarization formula*

$$2^k k!\, c_k(A_1,\ldots,A_k) = \sum_{\varepsilon_1,\ldots,\varepsilon_k} c_k(\varepsilon_1 A_1+\ldots+\varepsilon_k A_k)\varepsilon_1\ldots\varepsilon_k \tag{7.57}$$

where $\varepsilon_1,\ldots,\varepsilon_k$ take independently the values 1 and -1. By the estimate $|c_k(A)| \leq ||A||_1^k/k!$, one obtains easily

$$|c_k(A_1,\dots,A_k)| \leq \gamma_k \,||A_1||\dots||A_k|| \tag{7.58}$$

with a constant

$$\gamma_k = 2^{-k}k^k/(k!)^2 \quad . \tag{7.59}$$

From Stirling's formula,one gets $\lim_{k\to\infty} \gamma_k^{1/k} = 0$ and one concludes as in subsection 5.4.

As a corollary, suppose $(A_\lambda)_{\lambda\in D}$ is a family of operators in $L^1(H)$, bounded in norm $||A_\lambda||_1 \leq R$ for a fixed constant R, and holomorphic in λ (D is a domain in a complex space C^r) in the sense that the matrix elements $\langle\psi|A|\psi'\rangle$ are holomorphic in λ for fixed vectors ψ and ψ'. Then the determinant $\det(1+A_\lambda)$ is a holomorphic function of λ in D.

7.11. The *multiplicative property* is now easy to prove. Let A and B be trace-class operators acting on H. Then $(1+A)(1+B)$ is equal to $1+C$ with $C = A + B + AB$. The product of a trace-class operator and a bounded operator is again trace-class, hence C is trace-class. We state

$$\det(1+A)\det(1+B) = \det(1 + A + B + AB) \quad . \tag{7.60}$$

Any trace-class operator can be approximated by finite rank operators in the Banach space $L^1(H)$ and the determinant is a continuous functional on $L^1(H)$. Hence it suffices to consider the case where A and B are of finite rank. We may then choose an orthonormal basis (ψ_n) of H such that A and B map H into the subspace K generated by $\psi_1,\dots,\psi_N$ for a suitable N. Let the operators A_0 and B_0 in the finite-dimensional space K be obtained by restricting A and B respectively. Since the multiplicative rule holds for determinants of operators in K, it suffices to check that $1+A$ and $1+A_0$ have the same determinant (and similarly for $1+B$ and $1+B_0$). The matrix elements $a_{mn} = \langle\psi_m|A|\psi_n\rangle$ of A agree with those of A_0 for $1 \leq m \leq N$, $1 \leq n \leq N$ and are zero otherwise. The minors $\Delta\binom{i_1\dots i_k}{i_1\dots i_k}$ of A agree therefore with those of A_0 when

$i_1, \ldots, i_k$ lie between 1 and N, and are 0 otherwise. By using formula (7.36) for both A and A_0, we get $\det(1+A) = \det(1+A_0)$ as asserted.

7.12. We derive a few consequences of the multiplicativity of determinants. Let A be again a trace-class operator in H. According to formula (6.6), the operator A can be approximated in operator norm by finite rank operators, hence is compact. By F.Riesz'theory, *Fredholm's alternative* is valid:

(a) *either the operator* 1+A *has a bounded inverse,*
(b) *or there exists a nonzero vector* ψ *in* H *such that* $(1+A)\psi=0$.

In case (a), let T be a bounded inverse to 1+A. Then T=1-AT and AT is trace-class again. By the multiplicativity of determinants, we get

$$\det(1 + A) \ \det(1 - AT) = \det(1) = 1$$

hence $\det(1+A) \neq 0$.

In case (b) choose an orthonormal basis $\psi_1, \psi_2, \ldots$ of H with $\psi = \psi_1$ and represent A by its matrix (a_{mn}). Then the determinant of 1+A is the limit of the truncated tdeterminants

$$D_N = \det \begin{pmatrix} 1+a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & 1+a_{22} & \cdots & a_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ a_{N1} & a_{N2} & \cdots & 1+a_{NN} \end{pmatrix} ;$$

the hypothesis $(1+A)\psi_1 = 0$ means that the first columns of the previous determinant consists of 0's, hence $D_N = 0$ and in the limit $\det(1+A) = 0$.

Hence, the cases (a) and (b) of the alternative correspond to $\det(1+A)\neq 0$ and $\det(1+A)=0$ respectively.

7.13. Put in another way, the *inverse eigenvalues* $1/\lambda_n$ corresponding to the nonzero eigenvalues λ_n of A are the roots of the equation $\det(1 - zA) = 0$. We want now to express the determinant itself in terms of the eigenvalues; the question is very easy when A is selfadjoint and lies much deeper in the general case.

Assume first that A is trace class and *selfadjoint*. There exists an orthonormal basis (ψ_n) diagonalizing A, that is $A\psi_n = \lambda_n\psi_n$. The matrix elementsof A are given by

$$<\psi_m|A|\psi_n> = \begin{cases} \lambda_n & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} . \tag{7.61}$$

The minors are those of a diagonal matrix, hence

$$\Delta\begin{pmatrix} i_1 \dots i_k \\ i_1 \dots i_k \end{pmatrix} = \lambda_{i_1} \dots \lambda_{i_k} \tag{7.62}$$

for $i_1 < \dots < i_k$. Since A is trace-class, the series of diagonal elements $\sum_n <\psi_n|A|\psi_n> = \sum_n \lambda_n$ converges absolutely, hence the infinite product $\prod_n (1-\lambda_n z)$ converges absolutely and can be expanded as follows

$$\prod_n (1 - \lambda_n z) = \sum_{k\geq 0} (-z)^k \sum_{i_1<\dots<i_k} \lambda_{i_1} \dots \lambda_{i_k} . \tag{7.63}$$

Comparing with (7.36), one concludes

$$\det(1 - zA) = \prod_{n\geq 1} (1 - \lambda_n z) . \tag{7.64}$$

We come to the general case and denote by D(z) the entire function det(1-zA) of the complex variable z. As it is customary, let us introduce the *growth indicator*

$$M(R) = \sup_{|z|=R} |D(z)| . \tag{7.65}$$

To estimate M(R), *the strategy consists of comparing the eigenvalues of* A *and* $|A|$. Indeed, one gets

$$|c_k(A)| = |\mathrm{Tr}(\Lambda^k A)| \leq ||\Lambda^k A||_1 = \mathrm{Tr}(\Lambda^k |A|) = c_k(|A|).$$

Since $D(z) = \sum_{k\geq 0} (-1)^k c_k(A)\, z^k$, using a term-by-term estimate, one gets

$$M(R) \leq \sum_{k\geq 0} |c_k(A)| R^k \leq \sum_{k\geq 0} c_k(|A|) R^k = \det(1+R|A|).$$

Introduce now the eigenvalues $\mu_0, \mu_1, \ldots$ of $|A|$; they are positive and $\sum_{n \geq 0} \mu_n$ is finite. Hence

$$M(R) \leq \prod_{n \geq 0} (1 + \mu_n R) \quad . \tag{7.66}$$

Taking the logarithm, one gets

$$\frac{1}{R} \log M(R) \leq \sum_{n \geq 0} \frac{1}{R} \log(1 + \mu_n R) \quad . \tag{7.67}$$

Each term in the left-hand side converges to 0 with $1/R$, and is majorized by μ_n. Since $\sum_n \mu_n$ is finite, one gets by dominated convergence

$$\lim_{R \to \infty} \frac{1}{R} \log M(R) = 0 \quad .$$

(Notice that $M(R)$ tends to $+\infty$ with R, unless $D(z)$ is a constant).

We can now appeal to *Hadamard's factorization theorem.* (see Titchmarsh [14], th.8.24). We know already that a number z_0 satisfies $D(z_0) = 0$ iff there exists an eigenvalue $\lambda \neq 0$ of A with $z_0 = 1/\lambda$. Hence there are two possible cases:

(a) *The characteristic determinant* $\det(1 - zA)$ *is a polynomial in* z, *and can be expanded as* $(1 - \lambda_1 z) \ldots (1 - \lambda_N z)$ *where* $\lambda_1, \ldots, \lambda_N$ *are the nonzero eigenvalues of* A (with possible repetition). The limiting case $D(z) = 1$, that is $\mathrm{Tr}(A^k) = 0$ for each $k \geq 1$, occurs iff there is no eigenvalue of A, except possibly 0. In the self-adjoint case, this would mean $A = 0$, but not necessarily here.

(b) *There is an infinite sequence* $(\lambda_n)_{n \geq 1}$ *tending to* 0 *such that* $\sum_n |\lambda_n|$ *converges and*

$$\det(1 - zA) = \prod_{n \geq 1} (1 - \lambda_n z) \quad . \tag{7.69}$$

7.14. We know that the λ_n's are the eigenvalues of A. There remains to settle the *question of multiplicities*. Let $\lambda \neq 0$ be any eigenvalue of A and let m be the number of times λ occurs among the λ_n's, that is the multiplicity of $1/\lambda$ as a zero of the entire function $D(z) = \det(1 - zA)$. We need a *refinement of Fredholm's alternative*. For every integer $p \geq 0$, let L_p be the set of vectors ψ in H such that $(1-\lambda^{-1}A)^p\psi = 0$. Then L_p is *finite-dimensional*, and there exists an integer N such that $L_p = L_N$ for $p \geq N$ (notice that $L_0 \subset L_1 \subset \ldots \subset L_p \subset L_{p+1} \subset \ldots$). Moreover, let M_N be the set of vectors of the form $(1-\lambda^{-1}A)^N\psi$. Then M_N *is a closed vector subspace of* H *, the space* H *is the direct sum of* L_N *and* M_N *and* $1-\lambda^{-1}A$ *induces an operator in* M_N *with a bounded inverse*. There is a slight difficulty, namely L_N and M_N are not necessarily orthogonal. So let us introduce the orthogonal complement L_N' of L_N, so that the operator A can be expressed in block form

$$A = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix} ,$$

where E is an operator acting on the finite-dimensional space L_N and G is a trace-class operator acting on the Hilbert space L_N'. Since $1-\lambda^{-1}A$ has a bounded inverse, so does $1-\lambda^{-1}G$. Calculating the determinant of $1 - zA$ as a limit of finite determinants, and using an orthonormal basis $(\psi_n)_{n\geq 1}$ of H such that $(\psi_1,\ldots,\psi_d)$ forms a basis of L_N, one gets

$$\det(1 - zA) = \det(1 - zE)\det(1 - zG) .$$

From our previous description, one gets $(1-\lambda^{-1}E)^N = 0$, hence λ is the only eigenvalue of E and therefore $\det(1 - zE)=(1-\lambda z)^d$. Define $D_0(z)$ as $\det(1-zG)$, so that $D(z) = (1-\lambda z)^d D_0(z)$. Then $D_0(z)$ is an entire function of z, and $D_0(\lambda^{-1}) = \det(1-\lambda^{-1}G)$ is not zero, since the operator $1-\lambda^{-1}G$ has a bounded inverse. Hence $1/\lambda$ is a zero of $D(z)$ of multiplicity d.

Summing up: *an eigenvalue* λ *of* A *occurs in the list* $\lambda_1,\lambda_2,\ldots$ *a number of times equal to the* (finite) *dimension of the subspace* H_λ *of* H *consisting of the vectors* ψ *for which there exists an integer* $p>0$ *with* $(A-\lambda)^p\psi = 0$.

The previous subtleties are connected with the so-called *Jordan normal form* of a matrix, when it can not be put in diagonal form. In most cases the space H_λ defined above consists of the eigenvectors, that is the vectors ψ such that $(A-\lambda)\psi = 0$.

We know that for small $|z|$, the logarithmic derivative $D'(z)/D(z)$ can be expanded as $\sum_{k\geq 1}(-1)^k \mathrm{Tr}(A^k)z^{k-1}$. Since $D(z)$

is equal to the product $\prod_{n\geq 1}(1-\lambda_n z)$ where $\sum_n |\lambda_n|$ is finite, $D'(z)/D(z)$, can be expanded as an absolutely convergent double series $\sum_{n\geq 1}\sum_{k\geq 1}(-1)^k \lambda_n^k z^{k-1}$. Comparing the two expansions of $D'(z)/D(z)$, one gets

$$\mathrm{Tr}(A^k) = \sum_{n\geq 1} \lambda_n^k \tag{7.70}$$

and in particular

$$\mathrm{Tr}(A) = \sum_{n\geq 1} \lambda_n \quad . \tag{7.71}$$

These relations are obvious in the selfadjoint case and were established by Dikii in 1957 for the general case.

7.15. We consider now the case of a *Hilbert-Schmidt operator* A in $L^2(H)$. In general we can define the traces of $A^2, A^3, \ldots$ but not of A itself, and so we need a modification of the determinant; this way introduced by Carleman around 1930, and the method simplified by Seiler in 1972.

We introduce a map Φ from $L^2(H)$ into $L^1(H)$ by $\Phi(A) = 1-(1+A)e^{-A}$. Indeed, expanding the exponential into the familiar power series, we get

$$\Phi(A) = \sum_{k\geq 2} (-1)^k (k-1)\, A^k/k!$$

Notice that for A in $L^2(H)$, its square is in $L^1(H)$ and $||A^2||_1 = ||A||_2^2$. For $k \geq 2$, one gets therefore

$$||A^k||_1 \leq ||A^2||_1 ||A^{k-2}|| \leq ||A||_2^2 ||A||^{k-2} \leq ||A||_2^k$$

since $||A|| \leq ||A||_2$. Hence the general term in the series for $\Phi(A)$ is bounded in L^1-norm by $||A||_2^k/(k-2)!$ This is enough to show that this series is absolutely convergent in the Banach space $L^1(H)$, and that Φ is a continuous map from $L^2(H)$ into $L^1(H)$. We set

$$\det{}_2(1+A) = \det(1-\Phi(A)) = \det(e^{-A}(1+A)) \tag{7.72}$$

where the determinant of $1-\Phi(A)$ is the one considered before. We state the main properties of the modified determinant:

(a) *The functional* $A \mapsto \det_2(1+A)$ *is continuous on the Hilbert space* $L^2(H)$, as a composition of continuous maps

(b) *The operator* $1+A$ *is invertible iff* $\det_2(1+A) \neq 0$; indeed e^{-A} has a bounded inverse e^A hence $1+A$ is invertible iff $1 - \Phi(A)$ is, that is iff $\det(1 - \Phi(A)) \neq 0$.

(c) *For a trace-class operator* A, *one gets*

$$\det_2(1+A) = e^{-\mathrm{Tr}(A)} \det(1+A) \qquad . \tag{7.73}$$

Indeed $L^1(H)$ is a subspace of $L^2(H)$ and the embedding is continuous since $||A||_2 \leq ||A||_1$. Therefore in formula (7.73) both sides depend continuously on A in $L^1(H)$ and it suffices to check it for a finite rank operator A. But then we can split H as $F \oplus G$ where F is finite-dimensional, A induces an operator A_o in F, and vanishes on the orthogonal complement G of F. We are left with the verification of $\det(e^{-A_o}(1+A_o) = e^{-\mathrm{Tr}(A_o)} \det(1+A_o)$, and using the multiplicativity of determinants, everything is reduced to the proof of $\det(e^{-A_o}) = e^{-\mathrm{Tr}(A_o)}$. Putting A_o in triangular form, we find

$$\det(e^{-A_o}) = e^{-\lambda_1} \ldots e^{-\lambda_N}$$

and $\mathrm{Tr}(A_o) = \lambda_1 + \ldots + \lambda_N$ where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of A_o. Done !

(d) *If* A *and* B *are Hilbert-Schmidt operators, then*

$$\det_2((1+A)(1+B)) = \det_2(1+A)\det_2(1+B)\, e^{-\mathrm{Tr}(AB)} \quad . \tag{7.74}$$

Notice that AB is trace-class, hence Tr(AB) is defined. Arguing by continuity, we need only to prove (7.74) for finite rank operators, and this case follows from (7.73) by an easy calculation.

7.16. A Hilbert-Schmidt operator is compact, hence its eigenvalues are described qualitatively by F.Riesz'theory. So let Σ be the set of nonzero eigenvalues of A, and for λ in Σ its multiplicity $m(\lambda)$ be defined as the (finite) dimension of the space H_λ of vectors annihilated by some power of the operator $A-\lambda$. Alternatively, we can rearrange the eigenvalues in a sequence $\lambda_1, \lambda_2, \ldots$ such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$ and each eigenvalue $\lambda \neq 0$ occurs $m(\lambda)$ times in the sequence (λ_n). If A is of finite rank, there are only a finite number of nonzero eigenvalues; otherwise, $|\lambda_n|$ tends to 0 with 1/n.

Fix an integer $k \geq 2$, so that the operator A^k is trace-class. Let $\Sigma(k)$ be the set of nonzero eigenvalues of A^k, and $m_k(\lambda)$ be the multiplicity of λ. By an algebraic reasoning, it can be shown that a complex number $\lambda \neq 0$ belongs to $\Sigma(k)$ iff it is the k-th power of an element of Σ. Moreover, the set of vectors annihilated by some power of $A^k - \lambda$ is the direct sum $\bigoplus_\mu H_\mu$ where μ runs over the k-th roots of λ belonging to Σ. Hence $m_k(\lambda) = \sum_{\mu^k = \lambda} m(\mu)$, and this implies

$$\sum_{\lambda \in \Sigma(k)} \lambda m_k(\lambda) = \sum_{\mu \in \Sigma} \mu^k m(\mu) \quad . \tag{7.75}$$

By Dikii's theorem (7.71), the left-hand side represents the trace of A^k. Hence we get

$$\mathrm{Tr}(A^k) = \sum_{\mu \in \Sigma} \mu^k m(\mu) = \sum_{n \geq 1} \lambda_n^k \quad (\text{for } k \geq 2) \quad . \tag{7.76}$$

Of course, when A itself is trace-class, the same formula remains true for k = 1. Notice that the series in (7.75) are absolutely convergent and in particular $\sum_{n \geq 1} |\lambda_n|^2$ *is finite*.

Let us introduce the characteristic determinant

$$D_2(z) = \det_2(1 - zA) \tag{7.77}$$

where z is a complex variable. Notice the series expansion

$$e^{zA}(1 - zA) = 1 - \sum_{k \geq 2} (k-1)(k!)^{-1} A^k z^k \tag{7.78}$$

which converges absolutely in the Banach space $L^1(H)$. Since the map $B \mapsto \det(1+B)$ from $L^1(H)$ to $\mathbb{C}$ is holomorphic, it follows by composition that $D_2(z)$ is an entire function of z. By property (b) of subsection 7.15, *the roots of the equation* $D_2(z) = 0$ *are the inverses of the eigenvalues* $\lambda \neq 0$ *of* A. More precisely, we claim

$$D_2(z) = \prod_{n \geq 1} (1 - \lambda_n z)\, e^{\lambda_n z} \quad , \tag{7.79}$$

where the infinite product converges absolutely since $\sum_n |\lambda_n|^2$ is finite. To prove this formula, let us remark that both sides are entire functions of z, hence it suffices to prove it for small $|z|$. Moreover, both sides take the value 1 for z = 0, hence it suffices to consider the logarithmic derivatives, that is to prove

$$D_2'(z)/D_2(z) = - \sum_{n\geq 1} \frac{\lambda_n^2 z}{1-\lambda_n z} \quad . \tag{7.80}$$

The right-hand side can be expanded into a double series, and taking (7.76) into account we get

$$- \sum_{n\geq 1} \frac{\lambda_n^2 z}{1-\lambda_n z} = - \sum_{n\geq 1} \sum_{k\geq 2} \lambda_n^k z^{k-1} = - \sum_{k\geq 2} z^{k-1} Tr(A^k)$$

$$= -z\, Tr(\sum_{k\geq 2} z^{k-2} A^{k-2} A^2) = -zTr(A^2(1-zA)^{-1}).$$

The formal manipulations are easily justified for $|z|$ small. On the other hand, using the modified multiplicative property (7.74) and putting $R(z) = A\,(1-zA)^{-1}$, one gets

$$D_2'(z)/D_2(z) = \lim_{\varepsilon\to 0} \{det_2(1-zA-\varepsilon A)det_2(1-zA)^{-1} - 1\}$$

$$= \lim_{\varepsilon\to 0} \{det(e^{\varepsilon R(z)}(1-\varepsilon R(z)))\ e^{-\varepsilon zTr(A\cdot R(z))} - 1\}$$

Since the expansion of $e^{\varepsilon R(z)}(1 - \varepsilon R(z))$ into powers of ε has no term of degree 1, one gets $det_2(e^{\varepsilon R(z)}(1-\varepsilon R(z)) = 1 + O(\varepsilon^2)$ hence $D_2'(z)/D_2(z) = - zTr(A\cdot R(z))$ and we are done.

7.17. We noticedalready that the determinant $det_2(1 + zA)$ is an entire function of z when A is in $L^2(H)$. Introduce the power series expansion

$$det_2(1 + zA) = \sum_{k\geq 0} b_k(A)\, z^k \quad . \tag{7.81}$$

The coefficients $b_k(A)$ can be calculated as follows

$$b_k(A) = (2\pi)^{-1} R^{-k} \int_0^{2\pi} det_2(1 + Re^{i\theta}A)e^{-ik\theta}\, d\theta \ . \tag{7.82}$$

It is now easy to estimate $b_k(A)$. Indeed using the product expansion (7.79) and the elementary inequality $|(1+\xi)e^{-\xi}| \leq e^{|\xi|^2/2}$ (for a complex number ξ), one gets

$$\det_2(1 + zA) \leq \exp \frac{1}{2} \sum_n |\lambda_n|^2 |z|^2 \quad . \tag{7.83}$$

Using (7.82) and choosing R as the square root of $k/\sum_n|\lambda_n|^2$, one gets

$$|b_k(A)| \leq (e/k)^{k/2} (\sum_n |\lambda_n|^2)^{k/2} \quad . \tag{7.84}$$

A *crucial estimate, due to H.Weyl,* asserts

$$\sum_n |\lambda_n|^2 \leq ||A||_2^2 \quad . \tag{7.85}$$

It is in turn derived from a similar statement for trace-class operators, which can be proved using the comparison of eigenvalues of $\Lambda^k B$ and $\Lambda^k|B|$ as in subsection 7.6. Hence we can derive from (7.84) the final estimate

$$|b_k(A)| \leq (e/k)^{k/2} ||A||_2^k \quad . \tag{7.86}$$

We offer now a method to calculate the coefficients $b_k(A)$. First of all, use polarization as in subsection 7.10 and define $b_k(A_1,\ldots,A_k)$ for $A_1,\ldots,A_k$ in $L^2(H)$ by

$$2^k k! b_k(A_1,\ldots,A_k) = \sum_{\varepsilon_1\ldots\varepsilon_k} b_k(\varepsilon_1 A_1+\ldots+\varepsilon_k A_k)\varepsilon_1\ldots\varepsilon_k \tag{7.87}$$

($\varepsilon_1,\ldots,\varepsilon_k$ take independently the values 1 and -1). From (7.86) we get the estimate

$$|b_k(A_1,\ldots,A_k)| \leq \gamma'_k ||A_1||_2 \cdots ||A_k||_2 \quad , \tag{7.88}$$

with a constant

$$\gamma'_k = (ek/4)^{k/2}/k! \quad . \tag{7.89}$$

Since the functional $A \mapsto \det_2(1+A)$ is continuous on the Hilbert space $L^2(H)$, it follows from (7.82) and (7.87) that the functional $b_k(A_1,\dots,A_k)$ is jointly continuous on $L^2(H) \times \dots \times L^2(H)$. By continuity, it suffices to consider the case where $A_1,\dots,A_k$ are of finite rank. When A is of finite rank, we know that $\det_2(1+zA)$ is equal to $e^{-z\mathrm{Tr}(A)}\det(1+zA)$, hence we get

$$b_k(A) = \sum_{j=0}^{k} (-1)^j \mathrm{Tr}(A)^j \, c_{k-j}(A,\dots,A)/j! \qquad (7.90)$$

using the multilinear form

$$c_r(A_1,\dots,A_r) = (r!)^{-1} \sum_{\sigma\in S_r} (\mathrm{sgn}\,\sigma)\, \mathrm{Tr}((A_1\otimes\dots\otimes A_r)U_\sigma) \qquad (7.91)$$

as in subsection 7.10. It follows immediately that $b_k(A_1,\dots,A_k)$ is a continuous multilinear form on $L^2(H) \times \dots \times L^2(H)$. Moreover, by some group-theoretical calculations, one derives

$$k!\,b_k(|\psi_1\rangle\langle\theta_1|,\dots,|\psi_k\rangle\langle\theta_k|) = \det_{\substack{1\le i\le k\\1\le j\le k}} a_{ij} \quad , \qquad (7.92)$$

with

$$a_{ij} = \begin{cases} \langle\theta_i|\psi_j\rangle & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \qquad (7.93)$$

Problem: derive the estimate (7.88) from the definition of $b_k(A_1,\dots,A_k)$ by the previous formulas.

To conclude this subsection, let us remark that the formula for the logarithmic derivative of $D_2(z)$ given at the end of subsection 7.16 can be rewritten as follows

$$\det_2(1 + zA) = \exp \sum_{k\ge2} (-1)^{k-1} z^k \mathrm{Tr}(A^k)/k \quad . \qquad (7.94)$$

Notice the similarity with the formula for trace-class operators

$$\det(1 + zA) = \exp \sum_{k\ge1} (-1)^{k-1} z^k \mathrm{Tr}(A^k)/k \quad . \qquad (7.95)$$

Since Tr *makes in general no sense for* $k = 1$, *when* A *is Hilbert-Schmidt, just omit it !* We leave it to the reader to modify accordingly Waring's formula.

7.18. Let us go back to integral operators. So let Ω and the continuous kernel $K(x,y)$ be as in section 5. Since a continuous kernel $K(x,y)$ is a square-integrable function on $\Omega \times \Omega$, the operator $f \mapsto Kf$ acting on the space $C(\Omega)$ of continuous functions is the restriction of a Hilbert-Schmidt operator A_k acting in $L^2(\Omega)$. The main results in subsection 5.9 can be expressed as follows:

$$\det(1 + zK) = e^{z\tau} \det_2(1 + zA_K) \tag{7.96}$$

where the left-hand side if Fredholm's determinant and $\tau = \int_\Omega K(x,x)\, dx$.

The question about the trace can be reformulated as follows: *suppose the continuous kernel* K *on* $\Omega \times \Omega$ *is such that* A_k *is a trace-class operator in* $L^2(\Omega)$. *Is it true that Fredholm's determinant* $\det(1+zK)$ *agrees with the determinant* $\det(1 + zA_K)$ *of Hilbert space operators ?* Or, according to formula (7.96), do we have in this case

$$\operatorname{Tr}(A_k) = \int_\Omega K(x,x)\, dx. \tag{7.97}$$

The answer is yes, by Mercer's theorem, if A_k is a positive operator. I do not know the answer in general.

From (7.96), one can derive a power series expansion for $\det_2(1 + zA_k)$. Indeed recall the definition of Fredholm's determinant

$$\det(1+zK) = \sum_{k\geq 0} (z^k/k!) \int_\Omega \ldots \int_\Omega \Delta\begin{pmatrix} x_1 \ldots x_k \\ x_1 \ldots x_k \end{pmatrix} dx_1 \ldots dx_k \tag{7.98}$$

where $\Delta\begin{pmatrix} x_1 \ldots x_k \\ x_1 \ldots x_k \end{pmatrix}$ is the determinant of the matrix with elements $K(x_i,x_j)$ for $1 \leq i \leq k$, $1 \leq j \leq k$. Define similarly $\Delta'\begin{pmatrix} x_1 \ldots x_k \\ x_1 \ldots x_k \end{pmatrix}$ as the modified determinant where we replace the diagonal elements $K(x_i,x_i)$ by 0. Then we get

$$\det_2(1+zA_k) = \sum_{k\geq 0} (z^k/k!) \int_\Omega \ldots \int_\Omega \Delta'\begin{pmatrix} x_1 \ldots x_k \\ x_1 \ldots x_k \end{pmatrix} dx_1 \ldots dx_k \quad . \tag{7.99}$$

This formula remains valid if we suppose only that K belongs to $L^2(\Omega \times \Omega)$, hence A_K is Hilbert-Schmidt. This modification of Fredholm's definition was first proposed by Hilbert and Harnack in 1906.

8. Fredholm Determinants: the Case of Banach Spaces

8.1. Let E be a Banach space, with norm $||x||$ for vectors. If A is a bounded operator in E, its norm $||A||$ is the smallest constant $C \geqq 0$ such that $||Ax|| \leqq C\,||x||$ for all x in E. We owe to Grothendieck the following definition of a *nuclear* (or trace-class) operator in E: any operator which can be represented as a series $A = \sum_n h_n$ where each h_n is of rank 1, and $\sum_n ||h_n||$ is finite(such a series converges in operator norm). *The nuclear norm* $||A||_1$ is the infimum of the set of numbers $\sum_n ||h_n||$ for all such decompositions $A = \sum_n h_n$. Using the fact that a normed space is complete (hence a Banach space) iff any absolutely convergent series has a sum, it follows easily that the set $L^1(E)$ of nuclear operators is a Banach space for the nuclear norm.

Introduce the *dual space* E' of E and denote by $\langle x'|x\rangle$ the natural pairing between E' and E (for x in E and x' in E') More explicitely, a nuclear operator can be represented as follows

$$Ax = \sum_n \lambda_n \langle x'_n|x\rangle\, x_n \tag{8.1}$$

where x_n are vectors in E and x'_n in E', λ_n are complex numbers and $||x_n|| = ||x'_n|| = 1$, $\sum_n |\lambda_n|$ finite. In Dirac's notation, where the elements of E' are bras and those of E are kets, this is expressed as

$$A = \sum_n \lambda_n\, |x_n\rangle\langle x'_n| \quad . \tag{8.2}$$

According to subsection 6.4, when E *is a Hilbert space*, the nuclear operators are exactly the trace-class operators. Moreover, in this case, we can compute the trace of A using any orthonormal basis (ψ_m), hence

$$Tr(A) = \sum_m \langle\psi_m|A|\psi_m\rangle = \sum_{m,n} \lambda_n \langle\psi_m|x_n\rangle\langle x'_n|\psi_m\rangle$$

and using Parseval's equality, we conclude

$$Tr(A) = \sum_n \lambda_n \langle x'_n|x_n\rangle \quad . \tag{8.3}$$

8.2. There arises the question whether a similar definition of trace works for nuclear operators in a Banach space. The series (8.3) converges absolutely since $\sum_n |\lambda_n|$ is finite and $|\langle x'_n|x_n\rangle| \leq ||x'_n|| \, ||x_n|| \leq 1$. The question is whether the value given by (8.3) is independent of the chosen decomposition (8.2) for A. First of all, the trace is well-defined for *finite rank operators*. Indeed, let B be such an operator, represented as a sum $\sum_{n=1}^{N} |x_n\rangle\langle x'_n|$. Let us choose a set of linearly independent vectors $e_1,\dots,e_M$ such that every x_n is a linear combination of the e_m's. Now any vector of the form Bx is a unique linear combination of $e_1,\dots,e_M$, hence there exist uniquely defined elements $e'_1,\dots,e'_M$ of E' such that $B = \sum_{m=1}^{M} |e_m\rangle\langle e'_m|$. Write $x_n = \sum_{m=1}^{M} u_{nm} e_m$ with complex numbers u_{nm}. Then

$$B = \sum_n |x_n\rangle\langle x'_n| = \sum_{n,m} u_{nm} \, |e_m\rangle\langle x'_n|$$

and, by the uniqueness of the expansion $B = \sum_m |e_m\rangle\langle e'_m|$, one gets $e'_m = \sum_n u_{nm} x'_n$. Therefore

$$\sum_n \langle x'_n|x_n\rangle = \sum_{n,m} u_{nm} \langle x'_n|e_m\rangle = \sum_m \langle e'_m|e_m\rangle \; .$$

If we start from two decompositions

$$B = \sum_n |x_n\rangle\langle x'_n| = \sum_r |y_r\rangle\langle y'_r| \quad ,$$

we can use the same e_m's for both, hence $\sum_n \langle x'_n|x_n\rangle$ and $\sum_r \langle y'_r|y_r\rangle$ are equal. *Conclusion: The finite rank operators form a dense linear subspace* $L_f(E)$ *of* $L^1(E)$ *and a trace is defined on* $L_f(E)$ *which is linear and takes* $|x\rangle\langle x'|$ *into* $\langle x'|x\rangle$.

We can extend the trace to the whole of $L^1(E)$ when E satisfies *Banach approximation property:* given any compact set K in E, there exists a sequence of finite rank operators p_k such that $\sup_k ||p_k||$ be finite and $\lim_{k\to\infty} ||p_k(x) - x|| = 0$ uniformly for x in K. The standard Banach spaces (continuous functions, Hilbert spaces, L^p spaces, Sobolev spaces) satisfy the property. At the time Grothendieck considered these problems (around 1950), it was still conjectured that every Banach space does. Counter examples were discovered much later the most remarkable being the space $L(H)$ of all bounded operators in a Hilbert space H, with the operator norm $||A||$. Suppose that E satisfies the approximation property and let $A = \sum_n \lambda_n |x_n><x'_n|$ be a nuclear operator. Since $\sum_n |\lambda_n|$ is finite, we can write $\lambda_n = \mu_n \cdot \nu_n$ with $\sum_n |\mu_n|$ finite and $\lim_{n\to\infty} \nu_n = 0$. Choose a compact set K in E containing the vectors $\nu_n x_n$, and operators p_k adapted to K. Then $p_k A$ are finite rank operators, hence their traces are defined; the formula

$$\mathrm{Tr}(p_k A) = \sum_n \mu_n <x'_n | p_k(\nu_n x_n)> \tag{8.4}$$

is easily checked. By dominated convergence for series, one deduces

$$\lim_{k\to\infty} \mathrm{Tr}(p_k A) = \sum_n \mu_n <x'_n | \nu_n x_n> = \sum_n \lambda_n <x'_n | x_n> \quad . \tag{8.5}$$

As in the finite rank case, the same set K and the same p_k's can be used simultaneously for two decompositions

$$A = \sum_n \lambda_n |x_n><x'_n| = \sum_r \pi_r |y_r><y'_r| \quad , \tag{8.6}$$

hence

$$\sum_n \lambda_n <x'_n | x_n> = \sum_r \pi_r <y'_r | y_r> \quad . \tag{8.7}$$

Conclusion: when E *satisfies the approximation property, there exists a linear form* Tr *on* $L^1(E)$ *such that* $|\mathrm{Tr}(A)| \leq ||A||_1$ *and taking* $|x><x'|$ *into* $<x'|x>$.

8.3. We come now to the general case. It turns out that nuclear operators in Banach spaces are better compared to Hilbert-Schmidt operators than to trace-class operators in Hilbert spaces. We proceed to define *the trace of the square of a nuclear operator.*

Let B and C be nuclear operators in E, with decompositions

$$B = \sum_n \lambda_n |x_n\rangle\langle x_n'| \ , \qquad C = \sum_m \mu_m |y_m\rangle\langle y_m'| \ . \tag{8.8}$$

Then one gets absolutely convergent series

$$\sum_{n,m} \lambda_n \mu_m \langle x_n'|y_m\rangle\langle y_m'|x_n\rangle = \sum_n \lambda_n \langle x_n'|Cx_n\rangle = \sum_m \mu_m \langle y_m'|By_m\rangle \ .$$

The second expression does not depend on the decomposition chosen for C and the third one does the same for B. Hence these expressions depend solely on B and C. It follows that *there exists a bilinear form* Tr(B;C) *on* $L^1(E) \times L^1(E)$ *with the following properties:*

(a) *the inequality* $|\mathrm{Tr}(B;C)| \leq ||B||_1 \ ||C||_1$;

(b) *when* $B = |x\rangle\langle x'|$ *and* $C = |y\rangle\langle y'|$ *are decomposable, then* $\mathrm{Tr}(B;C) = \langle x'|y\rangle\langle y'|x\rangle$;

(c) *symmetry* $\mathrm{Tr}(B;C) = \mathrm{Tr}(C;B)$.

We prove now that Tr(B;C) *depends solely on the product* A = BC. Let us introduce an auxiliary Hilbert space H with an orthonormal basis (ψ_r). We factor every λ_n as $\beta_n\beta_n'$ with $|\beta_n| = |\beta_n'|$, hence $\sum_n |\beta_n|^2 = \sum_n |\beta_n'|^2$ is equal to $\sum_n |\lambda_n|$, hence finite. Define operators $\beta: H \to E$ and $\beta': E \to H$ by

$$\beta = \sum_n \beta_n |x_n\rangle\langle\psi_n| \tag{8.9}$$

$$\beta' = \sum_n \beta_n' |\psi_n\rangle\langle x_n'| \tag{8.10}$$

(by convention, our indices run over the integers 1,2,3...). More explicitely, for ψ in H, one has

$$\beta(\psi) = \sum_n \beta_n \langle\psi_n|\psi\rangle \ x_n \tag{8.11}$$

and the series converges absolutely in E since $||x_n|| = 1$ and

$\sum_n |\beta_n|^2$, $\sum_n |\langle\psi_n|\psi\rangle|^2$ are finite (*Hint:* use Cauchy-Schwarz inequality). Similarly

$$\beta'(x) = \sum_n \beta_n' \langle x_n'|x\rangle \, \psi_n \qquad (8.12)$$

and $\sum_n |\beta_n' \langle x_n'|x\rangle|^2$ is bounded by $||x||^2 \sum_n |\beta_n'|^2$. According to these calculations, we get the norm estimates

$$||\beta||^2 \leq \sum_n |\beta_n|^2 \quad , \quad ||\beta'||^2 \leq \sum_n |\beta_n'|^2 \quad . \qquad (8.13)$$

From the construction of β and β' , we get $B = \beta\beta'$ since $\beta(\psi_n) = \beta_n x_n$ by (8.11). On the other hand, $\beta'\beta$ is a bounded operator in the Hilbert space H, with matrix elements

$$\langle\psi_m|\beta'\beta|\psi_n\rangle = \beta_n \beta_m' \langle x_m'|x_n\rangle \quad . \qquad (8.14)$$

Since $|\langle x_m'|x_n\rangle|$ is bounded by 1, and $\sum_n |\beta_n|^2$, $\sum_m |\beta_m'|^2$ are finite, it follows that $\sum_{m,n} |\langle\psi_m|\beta'\beta|\psi_n\rangle|^2$ is finite. Taking into account the definition of the nuclear norm, we conclude: *given the nuclear operator* B *and any* $\varepsilon > 0$ *, there exists a decomposition* $B = \beta\beta'$ *with bounded operators* $\beta: H \to E$ *and* $\beta': E \to H$, *while* $\beta'\beta$ *is a Hilbert-Schmidt operator in* H *with* $||\beta'\beta||_2 \leq ||B||_1 + \varepsilon$.

Introduce a similar decomposition $C = \gamma\gamma'$ where $\gamma'\gamma$ is a Hilbert-Schmidt operator in H with $||\gamma'\gamma||_2 \leq ||C||_1 + \varepsilon$. By calculations similar to the previous ones, one shows that $\beta'\gamma$ and $\gamma'\beta$ are Hilbert-Schmidt operators in H. Putting $\delta = \beta'\gamma\gamma'$ and noticing that $\delta\beta = (\beta'\gamma)(\gamma'\beta)$, we conclude: *the operator* $A = BC$ *in* E *can be factored as* VU *with bounded operators* $U: E \to H$ *and* $V: H \to E$ *, in such a way that* UV *be a trace-class operator in* H *with trace given by* $\mathrm{Tr}(UV)=\mathrm{Tr}(B;C)$.

We can calculate the trace of UV in terms of its eigenvalues. By easy calculations, one shows that for any $\lambda \neq 0$, and any integer $N > 0$, U maps the set of solutions ψ in H of $(UV - \lambda)^N \psi = 0$ isomorphically onto the set of solutions x in E of $(VU - \lambda)^N x = 0$. A similar statement holds with U, V interchanged and E,H interchanged. Hence the operators UV in H and VU in E have the same eigenvalues $\lambda \neq 0$, with a common multiplicity $m(\lambda)$. Borrowing from the spectral theory of trace-class operators in Hilbert spaces (see subsection 7.14), we can conclude:

Let the operator A *in* E *be factored as* BC , *where* B *and* C *are nuclear. Let* Σ *be the set of nonzero eigenvalues of* A , *and* $m(\lambda)$ *the multiplicity of* λ *in* Σ. *Then the series* $\sum_{\lambda\in\Sigma} m(\lambda)\lambda$ *is absolutely convergent and its sum is equal to* Tr(B;C). In particular Tr(B;C) depends only on BC, as asserted.

8.4. From now on, it is very easy to transfer the properties of Hilbert-Schmidt operators into properties of nuclear operators. Taking B = C in the previous result, we get:

(a) *Let* B *be a nuclear operator, not of finite rank. Then its eigenvalues (multiplicities included) can be arranged as a sequence* $(\lambda_n)_{n\geq 1}$, *tending to zero, with* $|\lambda_1| \geq |\lambda_2| \geq \ldots$ *Moreover* $\sum_n |\lambda_n|^2$ *is finite and bounded by* $||B||_1^2$.

Then taking $C = B^{k-1}$, with $k \geq 2$:

(b) *For every integer* $k \geq 2$, *one gets*

$$\mathrm{Tr}(B;B^{k-1}) = \sum_{n\geq 1} \lambda_n^k \quad . \tag{8.15}$$

The (modified) determinant of 1 + B can be defined as follows

$$\det_2(1+B) = \prod_{n\geq 1} (1 + \lambda_n)\, e^{-\lambda_n} \quad ; \tag{8.16}$$

the convergence of the infinite product is guaranteed since $\sum_n |\lambda_n|^2$ is finite. The *characteristic determinant of* B is the function $D_2(z) = \det_2(1 - zB)$, that is

$$D_2(z) = \prod_{n\geq 1} (1 - \lambda_n z)\, e^{\lambda_n z} \quad . \tag{8.17}$$

Hence $D_2(z)$ is an entire function of the complex variable z. We can expand it as a power series

$$D_2(z) = \sum_{k\geq 0} (-1)^k\, b_k(B)\, z^k \quad . \tag{8.18}$$

Recall the existence of a factorization $E \overset{\beta'}{\to} H \overset{\beta}{\to} E$ of B such that $\beta'\beta$ be a Hilbert-Schmidt operator in H. By a previous remark, the operators $B = \beta\beta'$ and $T = \beta'\beta$ have the same eigenvavalues, with equal multiplicities. Hence one gets

$$\mathrm{Tr}(B;B^{k-1}) = \mathrm{Tr}(T^k) \tag{8.19}$$

$$\det_2(1 - zB) = \det_2(1 - zT) \quad . \tag{8.20}$$

Notice also that, given $\varepsilon > 0$, we can choose β and β' such that $||T||_2 \leq ||B||_1 + \varepsilon$. We can derive the properties of $D_2(z)$ from those of T without any new calculation. For instance, we get

$$D_2(z) = \exp - \sum_{k\geq 2} \mathrm{Tr}(B;B^{k-1})\, z^k/k \tag{8.21}$$

for $|z| < 1/||B||_1$ as well as the estimate

$$|b_k(B)| \leq (e/k)^{k/2}\, ||B||_1^k \quad . \tag{8.22}$$

From (8.21) flow the usual corollaries, such as Waring formulas and Plemelj determinants. From (8.21) and the bilinearity of Tr(B;C) it follows that there exists continuous multilinear forms $b_k(B_1,\ldots,B_k)$ on $L^1(E)\times \ldots \times L^1(E)$ such that $b_k(B) = b_k(B, \ldots, B)$. We can take $b_k(B_1,\ldots,B_k)$ as symmetrical, hence given by the polarization formula (7.87). This provides the following bound, implying that $B \mapsto \det_2(1+B)$ is *holomorphic* on the Banach space $L^1(E)$:

$$|b_k(B_1,\ldots,B_k)| \leq (ek/4)^{k/2}(k!)^{-1}||B_1||_1 \ldots ||B_k||_1 \quad . \tag{8.23}$$

Let us mention the following analogue of formula (7.92)

$$k!\,b_k(B_1,\ldots,B_k) = \det \begin{pmatrix} 0 & \langle x_1'|x_2\rangle & \ldots & \langle x_1'|x_k\rangle \\ \langle x_2'|x_1\rangle & 0 & & \langle x_2'|x_k\rangle \\ \cdot & \cdot & \cdot & \cdot \\ \langle x_k'|x_1\rangle & \langle x_k'|x_2\rangle & \ldots & 0 \end{pmatrix} \tag{8.24}$$

for $B_1 = |x_1\rangle\langle x_1'|, \ldots, B_k = |x_k\rangle\langle x_k'|$. We could use this formula to estimate $b_k(B_1,\ldots,B_k)$; using Hadamard's estimate on determinants, we recover (8.23) with a slightly larger constant, namely $k^{k/2}/k!$.

To conclude this subsection, let us mention the *modified multiplicative rule*

$$\det_2((1+B)(1+C)) = \det_2(1+B)\det_2(1+C)e^{-\mathrm{Tr}(B;C)} \qquad (8.25)$$

which is analogous to (7.74). The simplest way to derive it, is by noticing that both sides are continuous functions of B and C in the Banach space $L^1(E)$ and that finite rank operators are dense in $L^1(E)$. This reduces the proof to the finite dimensional case where $\det_2(1+B) = e^{-\mathrm{Tr}(B)}\det(1+B)$ and $\mathrm{Tr}(B;C) = \mathrm{Tr}(BC)$; the rest of the calculation is easy.

8.5. The theory behaves in a much smoother way when the Banach space E enjoys the approximation property. In this case, the trace of a nuclear operator is defined and Tr(B;C) is the trace of the product BC. We then define the determinant by

$$\det(1+B) = e^{\mathrm{Tr}(B)} \det_2(1+B) \quad . \qquad (8.26)$$

For the characteristic determinant we get

$$D(z) = \det(1 - zB) = e^{z\tau} \prod_{n\geq 1} (1 - \lambda_n z)\, e^{\lambda_n z} \qquad (8.27)$$

with $\tau = \mathrm{Tr}(B)$. In general, the series $\sum_n \lambda_n$ is not convergent, and even if it converges, may fail to sum to the trace τ of B.

The formulas of the previous subsections admit of the following variants:

$$D(z) = \exp - \sum_{k\geq 1} \mathrm{Tr}(B^k)\, z^k/k \qquad (8.28)$$

(for $|z| < 1/||B||_1$) and

$$D(z) = \sum_{k\geq 0} (-1)^k c_k(B)\, z^k \quad . \qquad (8.29)$$

Here again $c_k(B)$ is obtained by putting $B_1 = \ldots = B_k = B$ in a continuous symmetrical multilinear form $c_k(B_1,\ldots,B_k)$. It is characterized by the formula

$$k!\,c_k(B_1,\ldots,B_k) = \det_{\substack{1\leq i\leq k\\ 1\leq j\leq k}} \langle x_i' | x_j \rangle \qquad (8.30$$

for decomposable operators $B_1 = |x_1\rangle\langle x_1'|, \ldots, B_k = |x_k\rangle\langle x_k'|$. Using Hadamard's estimate for determinants, one gets the estimate

$$|c_k(B_1,\ldots,B_k)| \leq k^{k/2}(k!)^{-1}||B_1||_1\ldots||B_k||_1 \quad . \qquad (8.31)$$

Grothendieck originally developed his theory of Fredholm determinants in Banach space by using formulas (8.29) and (8.30) as a starting point.

As we expect, the operator $1+B$ is invertible iff $\det(1+B) \neq 0$ and from (8.25) one derives immediately the multiplicative rule

$$\det((1+B)(1+C)) = \det(1+B)\det(1+C) \quad . \qquad (8.32)$$

8.6. To make the story complete, let us specialize our theory to the Banach space $E = C(\Omega)$ of continuous functions. For every integer $r \geq 1$, choose a finite open covering $(U_\alpha)_{\alpha\in I(r)}$ of Ω by sets of diameter $< 1/r$. For each α, choose a point x_α in U_α and a continuous function φ_α, taking positive values and vanishing outside U_α, in such a way that $\sum_{\alpha\in I(r)} \varphi_\alpha = 1$ ("partition of unity"). Define the finite rank linear operator p_r in $C(\Omega)$ by

$$(p_r f)(x) = \sum_{\alpha\in I(r)} f(x_\alpha)\varphi_\alpha(x) \quad . \qquad (8.33)$$

Any continuous function f being uniformly continuous on the compact space Ω, the sequence of functions $p_r f$ converges uniformly to f on Ω. Moreover, by Ascoli theorem, the convergence is uniform in f when f runs over a compact subset of $C(\Omega)$. The space $C(\Omega)$ enjoys therefore the approximation property. It can be shown that an integral operator A_K with continuous kernel K *acts on $C(\Omega)$ as a nuclear operator,* although *it does not act on $L^2(\Omega)$ as a trace-class operator,* generally speaking. With the previous notations, the finite rank operator $p_r A_K$ transforms a function f in $C(\Omega)$ into $\sum_\alpha \varphi_\alpha\, \mu_\alpha(f)$ where $\mu_\alpha(f) = \int_\Omega K(x_\alpha,y)f(y)\,dy$. Therefore $\mathrm{Tr}(p_k A_K)$ is equal to $\sum_\alpha \mu_\alpha(\varphi_\alpha) =$

$\sum_\alpha \int_\Omega K(x_\alpha, x)\varphi_\alpha(x)\, dx$. This converges to $\int_\Omega K(x,x)\, dx$ hence *the trace of* A_K *as a nuclear operator in* $C(\Omega)$ *is equal to the "naive" trace* $\int_\Omega K(x,x)\, dx$.

From this fact and the results given in subsection 7.18, it follows that *the Fredholm determinant* $\det(1+K)$ *is equal to the Grothendieck determinant* $\det(1+A_K)$ *associated to the nuclear operator* A_K.

PART THREE:

OVERVIEW OF RECENT DEVELOPMENTS

9. Grassmann Calculus and Berezin Determinants

9.1. Let V be a complex vector space of finite dimension n, and choose a basis $e_1, \ldots, e_n$ of V. We introduced in subsection 3.4 the symmetric algebra SV and remarked that its elements can be put in bijective correspondence with the polynomials in n variables $x_1, \ldots, x_n$. In this correspondence, the vector e_i corresponds to the variable x_i. The multiplication obeys the *commutative law* $x_i x_j = + x_j x_i$. From the variables we built the monomials, products of variables, which can be put in the normal form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ because of commutativity. From the monomials, one builds the polynomials by using linear combinations with complex coefficients.

Consider now the exterior algebra ΛV built on V. In a similar way, it can be considered as the algebra of Grassmann variables $\xi_1, \ldots, \xi_n$, the vector e_i corresponding to ξ_i. These variables obey the *anticommutative law* $\xi_i \xi_j = - \xi_j \xi_i$. This implies $\xi_i \xi_i = -\xi_i \xi_i$, hence $\xi_i^2 = 0$. The monomials in $\xi_1, \ldots, \xi_n$ can therefore be normalized as $\xi_{i_1} \ldots \xi_{i_k}$ with indices $i_1, \ldots, i_k$ in strictly increasing order. A Grassmann polynomial in the variable $\xi_1, \ldots, \xi_n$ can be expressed in a unique way as a linear combination of the 2^n monomials with complex coefficients.

$\sum_\alpha \int_\Omega K(x_\alpha, x)\varphi_\alpha(x)\, dx$. This converges to $\int_\Omega K(x,x)\, dx$ hence *the trace of* A_K *as a nuclear operator in* $C(\Omega)$ *is equal to the "naive" trace* $\int_\Omega K(x,x)\, dx$.

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9. Grassmann Calculus and Berezin Determinants

9.1. Let V be a complex vector space of finite dimension n, and choose a basis $e_1,\ldots,e_n$ of V. We introduced in subsection 3.4 the symmetric algebra SV and remarked that its elements can be put in bijective correspondence with the polynomials in n variables $x_1,\ldots,x_n$. In this correspondence, the vector e_i corresponds to the variable x_i. The multiplication obeys the *commutative law* $x_i x_j = + x_j x_i$. From the variables we built the monomials, products of variables, which can be put in the normal form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ because of commutativity. From the monomials, one builds the polynomials by using linear combinations with complex coefficients.

Consider now the exterior algebra ΛV built on V. In a similar way, it can be considered as the algebra of Grassmann variables $\xi_1,\ldots,\xi_n$, the vector e_i corresponding to ξ_i. These variables obey the *anticommutative law* $\xi_i\xi_j = - \xi_j\xi_i$. This implies $\xi_i\xi_i = -\xi_i\xi_i$, hence $\xi_i^2 = 0$. The monomials in $\xi_1,\ldots,\xi_n$ can therefore be normalized as $\xi_{i_1}\ldots\xi_{i_k}$ with indices $i_1,\ldots,i_k$ in strictly increasing order. A Grassmann polynomial in the variable $\xi_1,\ldots,\xi_n$ can be expressed in a unique way as a linear combination of the 2^n monomials with complex coefficients.

A monomial $\xi_{i_1}\ldots\xi_{i_k}$ is called *even* or *odd* if k is respectively even or odd. An even (odd) polynomial is a linear combination containing only even (odd) monomials.

There is only one monomial of degree n, namely $\xi_1 \ldots \xi_n$. Let $A = (a_{ij})$ be any matrix of size n x n, and introduce the Grassmann polynomials $\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j$ of degree 1. Then the anticommutativity rule $\eta_i\eta_j = -\eta_j\eta_i$ holds and the product $\eta_1\ldots\eta_n$ is homogeneous of degree n, hence a scalar multiple of $\xi_1 \ldots\xi_n$. Since the Grassmann polynomials are just another way of denoting the elements of ΛV, their product corresponds to the wedge product,hence formula (2.16) can be rewritten as

$$\eta_1\ldots\eta_n = (\det A)\, \xi_1 \ldots \xi_n \quad . \tag{9.1}$$

We can develop the product $\eta_1\ldots\eta_n$ as the sum of the n^n products $a_{1j_1}\ldots a_{nj_n}\, \xi_{j_1}\ldots\xi_{j_n}$. The monomial $\xi_{j_1}\ldots\xi_{j_n}$ is 0 unless the indices $j_1\ldots j_n$ form a permutation σ of 1 ... n, and in this case is equal to $(\mathrm{sgn}\sigma)\xi_1\ldots\xi_n$. Hence we get the familiar complete expansion of the determinant

$$\det A = \sum_{\sigma\in S_n} (\mathrm{sgn}\sigma)\, a_{1\,\sigma(1)}\cdots a_{n\sigma(n)} \quad . \tag{9.2}$$

9.2. Berezin's very original idea was to define a differential and integral calculus of Grassmann polynomials. Consider first the derivatives. Choose an index i between 1 and n. Then any given Grassmann polynomial $P(\xi_1,\ldots,\xi_n)$ can be written uniquely as $A + \xi_i B$, where A and B are Grassmann polynomials in the variables different from ξ_i. We define the partial derivative $\delta_i P = \delta P/\delta\xi_i$ as the coefficient B of ξ_i in P. Notice that, due to anticommutativity we have to distinguish $\xi_i B$ from $B\xi_i$. For this reason, $\delta_i P$ is called the *forward derivative* with respect to ξ_i.

We record here a few basic formulas, which except for the minus signs, are similar to familiar formulas

$$\delta_i\delta_j = -\delta_j\delta_i \quad , \quad \xi_i\xi_j = -\xi_j\xi_i \tag{9.3}$$

$$\delta_i\xi_j + \xi_j\delta_i = \delta_{ij} \tag{9.4}$$

$$\delta_i(PQ) = \delta_i P\cdot Q \pm P\cdot\delta_i Q \quad . \tag{9.5}$$

In formula (9.4), ξ_i is interpreted as the operator mapping a Grassman polynomial P into $\xi_i P$ ("forward multiplication" by ξ_i); hence, we get more explicitely

$$\delta_i(\xi_j P) + \xi_j\cdot(\delta_i P) = \delta_{ij} P \quad . \tag{9.6}$$

In formula (9.5), the sign is + or - if P is even or odd respectively. Notice that $\delta_i P$ is odd (even) when P is even(odd), hence δ_i changes the parity. According to the sign rule for the parity of products

even x even = even ; even x odd = odd ;

odd x even = odd ; odd x odd = even ,

(see formula (3.3)), the operators δ_i (as well as ξ_i) are to be considered as *odd*. Formula (9.4) can be rewritten as $\delta_i\xi_j = -\xi_j\delta_i$ for $i \neq j$ in analogy with (9.3). Moreover in formula (9.5), a minus sign occurs only at the place where δ_i and P are interchanged and only when P is odd. All this agrees fully with Koszul's sign rule (see subsection 2.9)).

If I is any subset of the set $\{1,2,\ldots,n\}$ with elements $i_1,\ldots,i_k$ arranged in increasing order, we set

$$\xi_I = \xi_{i_1}\cdots\xi_{i_k} \quad , \qquad \delta_I = \delta_{i_1}\cdots\delta_{i_k} \quad . \tag{9.7}$$

According to the rules (9.3) and (9.4), we can shift in any product of factors ξ_i and δ_j the factors δ_j to the right and the factors ξ_i to the left. Hence a differential operator acting on the Grassman polynomials can be written in the normal form

$$D = \sum_{I,J} a_{I,J}\, \xi_I \delta_J \quad . \tag{9.8}$$

Moreover the monomials ξ_K form a basis of the Grassman algebra $\Lambda(\xi_1,\ldots,\xi_n)$. The action of the operator $\xi_I\delta_J$ is given as follows

$$(\xi_I\delta_J)(\xi_K) = \begin{cases} \pm\,\xi_L & \text{if } J \subset K \text{ and } I \cap (K\setminus J) = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{9.9}$$

(where $L = I \cup (K \setminus J)$). It is then easy to prove that any linear operator acting on $\Lambda(\xi_1,\ldots,\xi_n)$ can be uniquely written in the form (9.8). In particular, *any operator is a differential operator.*

Let us add two remarks. When $n = 1$, write ξ_1 as ξ. Hence the Grassmann algebra $\Lambda(\xi)$ has a basis $1, \xi$ in which the operators $\xi = \xi_1$ and $\delta = \delta_1$ are expressed by the matrices

$$\xi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad . \tag{9.10}$$

The relations (9.3) and (9.4) take the form

$$\xi\xi = \delta\delta = 0, \qquad \xi\delta + \delta\xi = 1 \quad , \tag{9.11}$$

which are easy to check on (9.10). Notice that the Pauli matrices are $\sigma_+ = \delta, \sigma_- = \xi$, $\sigma_3 = \delta\xi - \xi\delta$.

Moreover, the relations (9.3) and (9.4) define a Clifford algebra with generators $\xi_1,\ldots,\xi_n$, $\delta_1,\ldots,\delta_n$ and our result about differential operators corresponds to the well-known fact that such an algebra is an algebra $M_{2^n}(\mathbb{C})$ of complex matrices of size $2^n \times 2^n$.

9.3. We come now to *Berezin integral*. Write a Grassmann polynomial in $\Lambda(\xi_1,\ldots,\xi_n)$ as

$$P = c_0 + \sum_{i=1}^{n} c_i \xi_i + \ldots + c_{12\ldots n} \xi_1 \ldots \xi_n \quad .$$

The coefficient $c_{12\ldots n}$ is called the *Berezin integral* of P, to be denoted by $\int P \cdot \delta^n \xi$, or $\int P \cdot \delta\xi_1 \ldots \delta\xi_n$. This integral can be calculated as a repeated integral

$$\int \delta\xi_1 \int \delta\xi_2 \ldots \int \delta\xi_n \; P(\xi_1,\ldots,\xi_n) \quad .$$

In this kind of calculation, we assume that the $\delta\xi_i$ anticommute with each other, and *commute* with the ξ_j (there is no unanimous agreement on this last point). The basic rules are as follows:

$$\int P(\xi_1,\ldots,\xi_n)\delta\xi_1 \ldots \delta\xi_n = \int Q(\xi_1,\ldots,\xi_q)\delta\xi_1 \ldots \delta\xi_q \cdot \int R(\xi_{q+1},\ldots,\xi_n)\delta\xi_{q+1} \ldots \delta\xi_n \tag{9.12}$$

if $P(\xi_1,\ldots,\xi_n)$ splits as $Q(\xi_1,\ldots,\xi_q)R(\xi_{q+1},\ldots,\xi_n)$

$$\int \delta\xi_i = 0 \quad , \qquad \int \xi_i \delta\xi_i = 1 \quad . \tag{9.13}$$

In the derivative $\delta_i P$, we have a Grassmann polynomial of $\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n$ hence there can be no term proportional to $\xi_1,\ldots,\xi_n$. Hence we get

$$\int \delta_i P \cdot \delta^n \xi = 0 \quad , \tag{9.14}$$

and from (9.5) one derives the rule of *integration by parts*

$$\int \delta_i P \cdot Q \delta^n \xi = \mp \int P \cdot \delta_i Q \delta^n \xi \tag{9.15}$$

where the sign + (-) holds for P odd(even).

9.4. Let us mention the formula for a *linear change of variables in a Berezin integral*. Recall first that given a real function $f(x)$ on $\mathbb{R}^n$ and an invertible linear map A from $\mathbb{R}^n$ to $\mathbb{R}^n$, one gets

$$\int_{\mathbb{R}^n} f(x) d^n x = \det A \int_{\mathbb{R}^n} f(Ay) d^n y \quad . \tag{9.16}$$

Hence the rule: the integral is unchanged under the simultaneous substitutions $x \mapsto Ay$, $d^n x \mapsto (\det A) \cdot d^n y$.

Consider now a Grassmann polynomial P in $\Lambda(\xi_1,\ldots,\xi_n)$ and express the ξ_i as linear combinations of new Grassmann variables

$$\xi_i = \sum_{j=1}^{n} a_{ij}\eta_j \quad , \text{ or in shorter form } \xi = A\eta, \tag{9.17}$$

where the matrix $A = (a_{ij})$ is invertible. Then we get

$$\int P(\xi)\delta^n \xi = (\det A)^{-1} \int P(A\eta)\delta^n \eta \tag{9.18}$$

or in symbolic form

$$\delta^n \xi = (\det A)^{-1} \delta^n \eta \qquad \text{for } \xi = A\eta \quad . \tag{9.19}$$

To prove (9.18), let us remark that for $P(\xi)$ homogeneous of degree $k < n$ in ξ, the Grassmann polynomial $P(A\eta)$ is homogeneous of degree $k < n$ in η, hence both integrals in (9.18) are 0.

It remains the case $P(\xi) = \xi_1 \cdots \xi_n$, hence $P(A\eta) = (\det A)\eta_1 \cdots \eta_n$ and our formula follows from

$$\int \xi_1 \ldots \xi_n \,\delta\xi_1 \ldots \delta\xi_n = \int \eta_1 \ldots \eta_n \delta\eta_1 \ldots \delta\eta_n = 1 .$$

9.5. We come to the *exponential*. If P and Q are even Grassmann polynomials, we get the commutativity rule PQ = QP. We define the exponential of P by the familiar power series

$$\exp P = \sum_{m=0}^{\infty} P^m/m! \tag{9.20}$$

provided it converges. The convergence can be proved as follows. Write P as c + Q where c is the constant term of P. Then Q having no constant term begins with terms of degree ≥ 2, hence $Q^m = 0$ for $m > \frac{n}{2}$. The series for exp Q breaks down, namely

$$\exp Q = \sum_{m=0}^{N} Q^m/m! \quad , \tag{9.21}$$

where N is the integral part of $\frac{n}{2}$. Moreover, by the binomial theorem, one obtains

$$P^m/m! = \sum_{r=0}^{N} (c^{m-r}/(m-r)!)\cdot(Q^r/r!) \quad . \tag{9.22}$$

By the convergence of the ordinary exponential series for exp c, we get, after rearranging, the convergence of the series for exp P, and the formula

$$\exp P = \exp c \cdot \exp Q \quad . \tag{9.23}$$

The functional equation

$$\exp(P + P') = \exp P \cdot \exp P' \tag{9.24}$$

can now be proved by expanding the exponentials in power series and using the binomial theorem to calculate $(P+P')^m/m!$. The algebra works because P and P' commute, and the calculus goes on because of the convergence of the series. One could also use formula (9.23).

Notice that the square of any monomial in $\xi_1,\ldots,\xi_n$ is 0. Hence, any even element Q of $\Lambda(\xi_1,\ldots,\xi_n)$ without constant term, can be written as $Q = \lambda_1 + \ldots + \lambda_r$ where

$\lambda_1^2 = \ldots = \lambda_r^2 = 0$ and of course $\lambda_i\lambda_j = \lambda_j\lambda_i$. If $\lambda^2 = 0$ then $\exp\lambda = 1 + \lambda$, and from the functional equation (9.24), one concludes

$$\exp Q = \prod_{i=1}^{r} (1 + \lambda_i) \quad . \tag{9.25}$$

9.6. We derive now the Grassmann analogues of the gaussian integrals (see section 4). Consider an element Q of $\Lambda(\xi_1,\ldots,\xi_n)$ homogeneous of degree 2, hence $Q = \sum_{i,j} q_{ij}\xi_i\xi_j$, with a skew-symmetric matrix (q_{ij}). We can write also $\frac{1}{2}Q = \sum_{i<j} q_{ij}\xi_i\xi_j$ and from (9.25),one derives

$$\exp \frac{1}{2}Q = \prod_{i<j} (1 + q_{ij}\xi_i\xi_j) \quad . \tag{9.26}$$

We have to calculate the coefficient c of $\xi_1 \ldots\xi_n$ in exp Q. It is obviously 0 if n is odd. Suppose n even, n = 2m say. Then c is obtained as follows: one considers all possible partitions of the set $\{1,2,\ldots,2m\}$ into m pairs $\{i_1,j_1\},\ldots,$ $\{i_m,j_m\}$, denotes by ε the sign of the permutation sending $12 \ldots 2m$ into $i_1j_1i_2j_2 \ldots i_mj_m$, and multiplies it by $q_{i_1j_1}\ldots$ $\ldots q_{i_mj_m}$. Then make the sum of all such contributions, two partitions into m pairs differing by the order of the pairs being considered as identical. This is the so-called *Pfaffian* of the matrix (q_{ij})

Examples: a) for m = 1 , $c = q_{12}$

b) for m = 2 , $c = q_{12}q_{34}-q_{13}q_{24}+q_{14}q_{23}$.

This Pfaffian, to be denoted Pf(Q) or $\mathrm{Pf}(q_{ij})$ is a Berezin integral

$$\mathrm{Pf}(Q) = \int \exp \tfrac{1}{2}Q(\xi)\ \delta^n\xi \qquad (n \text{ even}) \quad . \tag{9.27}$$

The determinant of a square matrix $A = (a_{ij})$ of size n x n can also be interpreted as a Berezin integral. Namely introduce Grassman variables $\xi_1,\ldots,\xi_n$, $\eta_1,\ldots,\eta_n$ and the bilinear form $A(\xi,\eta) = \sum_{i,j} a_{ij}\xi_i\eta_j$. We claim

$$\iint \exp A(\xi,\eta)\ \delta^n\xi\delta^n\eta = (-1)^{n(n-1)/2} \cdot \det A \quad . \tag{9.28}$$

Indeed develop $A(\xi,\eta)$ as $\xi_1 u_1(\eta) + \ldots + \xi_n u_n(\eta)$ with $u_i(\eta) = \sum_{j=1}^{n} a_{ij}\eta_j$. Each of the terms $\xi_i u_i(\eta)$ has a square equal to 0, hence we have to calculate the coefficient of $\xi_1\ldots\xi_n$ $\eta_1\ldots\eta_n$ in $\prod_{i=1}^{n}(1 + \xi_i u_i(\eta))$. The relevant term in the product is $\xi_1 u_1(\eta) \ldots \xi_n u_n(\eta)$, which can be rearranged as

$$(-1)^{n(n-1)/2}\ \xi_1\ldots\ \xi_n\ u_1(\eta)\ \ldots\ u_n(\eta)$$

We remarked already that $u_1(\eta)\ \ldots\ u_n(\eta)$ is equal to $(\det A)\cdot\eta_1\ldots\eta_n$, hence formula (9.28) is proved. We could rewrite this formula as

$$\iint \exp\{\sum_{i,j}\xi_i a_{ij}\eta_j\}\ \delta\xi_1\delta\eta_1\ldots\delta\xi_n\delta\eta_n = \det A \tag{9.29}$$

in analogy with formula (4.22). To make the analogy more complete, some authors have proposed the normalization $\int\xi\delta\xi = (-2\pi i)^{1/2}$ instead of $\int\xi\delta\xi = 1$.

To get a formula analogous to (4.10), namely

$$\int\exp\tfrac{1}{2}Q(\xi)\delta^n\xi = (\det Q)^{1/2}\ , \tag{9.30}$$

we need only to prove the classical result that the determinant of the skew-symmetric matrix Q is the square of its Pfaffian (for n even). This can be done by a trick similar to the one used in subsection 4.1. Namely, from (9.12) and (9.27) we get

$$(-1)^{n/2}\ \mathrm{Pf}(Q)^2 = \iint\exp\tfrac{1}{2}(Q(\xi) - Q(\eta))\ \delta^n\xi\delta^n\eta\ . \tag{9.31}$$

Introduce the bilinear form $Q(\xi,\eta) = \sum_{i,j} q_{ij}\xi_i\eta_j$. From (9.28) one gets

$$(-1)^{n(n-1)/2}\ \det Q = \iint\exp Q(\xi,\eta)\delta^n\xi\delta^n\eta\ . \tag{9.32}$$

But the integral is transformed into the previous one, if one makes the substitution $(\xi,\eta) \to (\xi-\eta,\ \tfrac{1}{2}(\xi+\eta))$, of determinant 1 (Notice that n is even, hence $(-1)^{n^2/2} = 1$). The important fact about formula (9.30) is that it chooses one of the square roots of det Q.

9.7. We shall now mix ordinary variables with Grassmann variables. So consider vectors $x = (x_1,\ldots,x_p)$ in $\mathbb{R}^p$ and Grassmann variables $\xi = (\xi_1,\ldots,\xi_q)$. By a *superfunction* we mean a Grassmann polynomial in the variables ξ whose coefficients depend on x, namely

$$F(x;\xi) = \sum_I F_I(x)\cdot\xi_I \tag{9.33}$$

(summation over all subsets I of $\{1,\ldots,q\}$). On such a superfunction, we operate with ordinary derivatives $\partial_i = \partial/\partial x_i$ in the parameter x, and Grassmann derivatives $\delta_j = \delta/\delta\xi_j$. The rules (9.3) and (9.4) are supplemented by the classical ones

$$\partial_i\partial_{i'} = \partial_{i'}\partial_i \quad , \qquad x_i x_{i'} = x_{i'}x_i \tag{9.34}$$

$$\partial_i x_{i'} - x_{i'}\partial_i = \delta_{ii'} \quad . \tag{9.35}$$

Moreover any operator in the *bosonic family* $x_1,\ldots,x_p$, $\partial_1,\ldots,\partial_p$ commutes with any operator in the *fermionic family* $\xi_1,\ldots,\xi_q$, $\delta_1,\ldots,\delta_q$.

Berezin's integral can be defined in the general case

$$\iint F(x;\xi)\, d^px\,d^q\xi = \int F_{12\ldots q}(x)\, d^px \ . \tag{9.36}$$

Integration by part with respect to ordinary variables as well as to Grassmann variables is now permitted.

In what sense if $F(x;\xi)$ a function ? The question has been much debated. Here is a simple answer. Consider an auxiliary Grassman algebra $\Lambda = \Lambda(\eta_1,\ldots,\eta_r)$ with *real* coefficients. Consider even elements $a_1,\ldots,a_p$ of Λ and odd elements $\alpha_1,\ldots,\alpha_q$ of Λ . For any subset I of $\{1,2,\ldots,q\}$ with elements $j_1 < j_2 < \ldots < j_s$, the substitution of α_j to ξ_j in $\xi_I = \xi_{j_1}\ldots\xi_{j_s}$ gives obviously $\alpha_I = \alpha_{j_1}\ldots\alpha_{j_s}$. We propose to define $F(a_1,\ldots,a_p;\alpha_1,\ldots,\alpha_q)$ as $\sum_I F_I(a_1,\ldots,a_p)\alpha_I$ provided we give a rule to substitute even elements $a_1,\ldots,a_p$ of Λ into an ordinary function $G(x_1,\ldots,x_p)$. Remark that a_j can be written as $a_j = a_j^0 + b_j$ where the constant term a_j^0 is a real number. Since $b_1,\ldots,b_p$ begin with Grassmann monomials of degree ≥ 2, any monomial in $b_1,\dots,b_p$ (which commute two by two !) of degree $d > \frac{r}{2}$ will vanish. Hence we can use a truncated Taylor series to define $G(a_1,\dots,a_p)$ as an element of Λ, namely

$$G(a_1^o+b_1,\dots,a_p^o+b_p) = \sum_{\lambda_1+\dots+\lambda_p\leq\frac{r}{2}} \partial_1^{\lambda_1}\dots\partial_p^{\lambda_p} G(a_1^o,\dots,a_p^o) \cdot b_1^{\lambda_1}\dots b_p^{\lambda_p}/\lambda_1! \dots \lambda_p! \tag{9.37}$$

9.8. Let us return to our discussion in subsection 3.5. Introduce a basis $e_1,\dots,e_n$ of V. The algebra ΣW, direct sum of the spaces $\Sigma^k W$, is also the sum of the spaces $\Sigma^{B,F}W = S^B V \otimes \Lambda^F V$. It follows that ΣW is an algebra of mixed polynomials in commuting variables $x_1,\dots,x_n$ and anticommuting variables $\xi_1,\dots,\xi_n$. They can be considered as superfunctions $F(x;\xi) = \sum_I F_I(x)\xi_I$ where each component $F_I(x)$ is a polynomial in $x = (x_1,\dots,x_n)$. On these mixed polynomials, we can operate with the operators x_i, ξ_j, $\partial/\partial x_i$, $\delta/\delta\xi_j$. We consider the two differential operators

$$d = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j} \quad , \qquad s = \sum_{j=1}^{n} x_j \frac{\delta}{\delta\xi_j} \quad . \tag{9.38}$$

We get the Leibnitz rules

$$d(FG) = dF\cdot G \pm F\cdot dG$$

$$s(FG) = sF\cdot G \pm F\cdot sG$$

(with a plus (minus) sign for F even (odd)). Since $\partial/\partial x_j$ is an even operator, and $\delta/\delta\xi_j$ and odd operator, both d *and* s *are odd operators*. Let us calculate sd + ds:

$$\begin{aligned} sd+ds &= \sum_{i,j}(x_i\delta_i\xi_j\partial_j + \xi_j\partial_j x_i\delta_i) \\ &= \sum_{i,j} x_i(\delta_{ij} - \xi_j\delta_i)\partial_j + \xi_j(x_i\partial_j + \delta_{ij})\delta_i \\ &= \sum_i x_i\partial_i + \sum_i \xi_i\delta_i \end{aligned}$$

(notice that $x_i\xi_j\delta_i\partial_j$ is equal to $\xi_j x_i\partial_j\delta_i$ since x_i commutes to ξ_j and ∂_j commutes to δ_i). An element H in $\Sigma^{B,F}W$ is a homogeneous polynomial of degree B in x, hence by Euler classical result, one gets $\sum_i x_i \partial_i H = B \cdot H$. Similarly, H is a homogeneous polynomial of degree F in ξ and the formula $\sum_i \xi_i \delta_i H = F \cdot H$ is easily proved. Hence sd + ds multiplies H by the total degree $k = B + F$.

It remains to prove that our operators satisfy the rules (3.20) and (3.21). Any element a in W^+ is a linear combination with complex coefficients of $x_1, \ldots, x_n$, hence $da = \pi a$. From Leibnitz rule one derives

$$d(a_1 \ldots a_B) = \sum_{i=1}^{B} a_1 \ldots a_{i-1} \, da_i a_{i+1} \ldots a_B$$

and moreover elements of W^- act as scalars with respect to the derivation d. Formula (3.20) follows at once, and the proof of (3.21) is similar.

9.9. The basic idea in supersymmetry is to consider transformations mixing ordinary variables $x_1, \ldots, x_p$ (hereafter called bosonic variables) with Grassmann variables $\xi_1, \ldots, \xi_q$ (the "fermionic" variables). A linear transformation will take the matrix form

$$\begin{pmatrix} x' \\ \xi' \end{pmatrix} = T \begin{pmatrix} x \\ \xi \end{pmatrix}$$

where T is written in block form

$$T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

For instance, one gets $x_i' = \sum_{j=1}^{p} m_{ij} x_i + \sum_{k=1}^{q} n_{ik} \xi_k$.

If we insist that x_i' should be bosonic (that is even), we cannot achieve such a transformation with ordinary numbers unless the n_{ik}'s are 0, that is N = 0, P = 0. If T is a matrix with complex coefficients, it will have the form $T = \begin{pmatrix} M & 0 \\ 0 & Q \end{pmatrix}$ and we shall get no mixing

$$x' = Mx \quad , \quad \xi' = Q\xi \quad .$$

The trick is to introduce an auxiliary Grassmann algebra $\Lambda = \Lambda(\eta_1, \ldots, \eta_r)$ and to assume the following parity rule:

the elements of M and Q are even
the elements of N and P are odd.

We shall define the (Berezin) determinant of such a matrix as an even element of Λ .

Let us preface the definition by a few remarks:

(a) Let us denote by a^o the constant term of an element a of Λ; we can say that a^o is obtained by putting $\eta_1 = \ldots = \eta_r = 0$ in a. One checks that $(a+b)^o = a^o + b^o$, and $(ab)^o = a^o b^o$ for a,b in Λ. So if for instance $P(t_1,\ldots,t_s)$ is a polynomial and $a_1,\ldots,a_s$ are even elements of Λ , the constant term of $P(a_1,\ldots,a_s)$ will be equal to $P(a_1^o,\ldots,a_s^o)$.

(b) Suppose an element a of Λ has an inverse b, that is $ab = 1$; hence $a^o b^o = 1$ holds. Therefore, a^o is not zero. Conversely, if this is so, write $a = a^o(1-\lambda)$ where λ has a zero constant term. Then one gets $\lambda^{r+1} = 0$ and $(a^o)^{-1}(1+\lambda+\ldots+\lambda^r)$ is an inverse of a.

Let us assume that $T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is as before and that the constant terms of detM and detQ are not zero. Then the matrix M has an inverse, and the matrices Q and $Q - PM^{-1}N$ have the same constant term (since P and N have odd elements, and the constant term of an odd element is 0). Hence the determinant of $Q - PM^{-1}N$ has the same constant term as detQ, which is not 0, and our determinant has an inverse in Λ. We are justified to define the *superdeterminant* (or Berezin determinant) of T as

$$\operatorname{sdet} T = \det M \cdot \det(Q - PM^{-1}N)^{-1} \quad . \tag{9.39}$$

This formula should be compared to formula (1.24).

9.10. The justification of the previous definition comes from the possibility to extend to superfunctions a number of classical formulas.

We begin with Gaussian integrals. Consider a *mixed quadratic form* in the variables $(x,\xi) = (x_1,\ldots,x_p,\xi_1\ldots\xi_q)$ namely

$$Q(x,\xi) = \sum_{i,i'} a_{ii'} x_i x_{i'} + 2\sum_{i,j} b_{ij} x_i \xi_j + \sum_{j,j'} c_{jj'} \xi_j \xi_{j'} \quad . \tag{9.40}$$

Here we assume $a_{ii'} = a_{i'i}$ and $c_{jj'} = -c_{j'j}$. We assume furthermore that $a_{ii'}$ and $c_{jj'}$ are even and that b_{ij} is odd. Hence $Q(x,\xi)$ will take even values when we replace $x_1,\ldots,x_p$ by even elements in Λ, and $\xi_1,\ldots,\xi_q$ by odd elements in Λ. Introduce the matrix of coefficients of $Q(x,\xi)$, namely $Q = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$ where $A = (a_{ii'})$ etc. Assume that the constant term of $\sum_{i,i'} a_{ii'}x_i x_{i'}$ has a strictly positive real part when $x_1,\ldots,x_p$ are real. Furthermore, assume the matrix C has an inverse (that is $\det(C)^0 \neq 0$). Notice that this implies that the integer q is even. Under these hypothesis, one gets

$$\iint \exp - \frac{1}{2} Q(x,\xi) d^p x \delta^q \xi = (2\pi)^{p/2} (-1)^{q/2} (\mathrm{sdet}\ Q)^{-1/2} \tag{9.41}$$

a common generalization of formulas (4.11) and (9.30). The integration variables $x_1,\ldots,x_p$ are *real* .

In a similar way associate to the matrix $T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ a bilinear form

$$S(x^*,\xi^*;x,\xi) = \sum_{i,i'} x_i^* m_{ii'} x_{i'} + \tag{9.42}$$

$$+ \sum_{i,j} x_i^* n_{ij} \xi_j + \sum_{j,i} \xi_j^* p_{ji} x_i + \sum_{j,j'} q_{jj'} \xi_j^* \xi_{j'} \quad .$$

Assume that the selfadjoint part $\frac{1}{2}(M + M^*)$ of M be positive definite and that Q be invertible as a matrix. Then we get the following generalization of formulas (4.22) and (9.29)

$$\int \exp -S(x^*,\xi^*;x,\xi) dx_1^* dx_1 \ldots dx_p^* dx_p \delta\xi_1^* \delta\xi_1 \ldots \delta\xi_q^* \delta\xi_q =$$

$$= (2\pi i)^p (s \det T)^{-1} \quad . \tag{9.43}$$

In this integral, $x_1,\ldots,x_p$ are considered as complex variables, with x_j^* complex conjugate to x_j, and $dx_j^* dx_j$ interpreted as $2i du_j dv_j$ if $x_j = u_j + iv_j$ and u_j, v_j are real (see formula (4.23)).

We can also generalize the formula for nonlinear changes of coordinates. We consider again variables $x_1,\ldots,x_p$, $\xi_1,\ldots,\xi_q$ where the x_i's are even (or bosonic) and the ξ_j's odd (or fermionic). A superchange of variables is of the form

$$x_i = U_i(y;\eta) \quad , \qquad \xi_j = V_j(y;\eta) \qquad .$$

Here $y = (y_1,\dots,y_p)$ are even variables and $\eta = (\eta_1,\dots,\eta_q)$ are odd variables. Moreover, the superfunction $U_i(y;\eta)$ is even containing only terms with a product of an even number of odd variables η_j, and $V_j(y;\eta)$ is odd with a similar definition.

Denote by $\partial U/\partial y$ the matrix $M = (m_{ii'})$ with entries $m_{ii'} = \partial U_i(y;\eta)/\partial y_{i'}$ and use similar notations in the case of the other partial derivatives. The matrix

$$T = \begin{pmatrix} \partial U/\partial y & \delta U/\delta\eta \\ \partial V/\partial y & \delta V/\delta\eta \end{pmatrix}$$

has the required properties: $\partial U/\partial y$ and $\delta V/\delta\eta$ are even and $\delta U/\delta\eta$, $\partial V/\partial y$ are odd. Assume furthermore that the matrices $\partial U/\partial y$ and $\delta V/\delta\eta$ are invertible. Then the superjacobian determinant is defined as the superfunction $J(y;\eta) = \text{s det } T$. Given any superfunction $G(x;\xi)$ we have the following integration formula

$$\iint G(x;\xi)d^px\delta^q\xi = \iint G(U(y;\eta);V(y;\eta))J(y;\eta)\, d^py\delta^q\eta. \quad (9.44)$$

Symbolically, we have

$$d^px\xi^q = J(y;\eta)d^py\delta^q\eta \qquad \text{for} \quad x = U(y;\eta),\ \xi = V(y;\eta). \quad (9.45)$$

9.11. As with any definition of determinant, there arises the question of the validity of the multiplicative rule. The definition of sdet T can be recast as follows (see our calculations in subsection 1.8). Write T as a product

$$T = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix} . \quad (9.46)$$

Then we get

$$\text{s det }(T) = (\det A)\cdot(\det B)^{-1} \quad . \quad (9.47)$$

In particular, the matrices of the form $\begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix}$ or $\begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}$ have a superdeterminant equal to 1, and the formula

$$\text{s det}(TT') = \text{s det}(T) \cdot \text{s det}(T') \tag{9.48}$$

holds if T is of the form $\begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix}$ or $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, or else if T' is of the form $\begin{pmatrix} I_p & Y' \\ 0 & I_q \end{pmatrix}$ or $\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$. To prove the multiplicative rule (9.48) in general, we need only to settle the case where

$$T = \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix} \qquad T' = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} .$$

It then reduces to the proof of the identity

$$\det(I_p + YX) = \det(I_q - X(I_p + YX)^{-1}Y) . \tag{9.49}$$

Using the identity

$$I_p - (I_p + YX)^{-1}YX = (I_p + YX)^{-1} , \tag{9.50}$$

it suffices to prove the relation

$$\det(I_p + VU) = \det(I_q + UV)^{-1} , \tag{9.51}$$

if U and V are matrices of respective sizes q x p and p x q with odd elements from Λ. This should be compared to the standard formula

$$\det(I_p + VU) = \det(I_q + UV) , \tag{9.52}$$

where U and V are matrices with elements from a commutative ring, for instance *even* elements from Λ. We leave it to the reader to prove formula (9.51) directly.

9.12. Let us sketch an "invariant" definition of the superdeterminant, due to I.Manin. Consider first polynomials $P(x;\xi)$ in one even variable x and one odd variable ξ . Such a polynomial can be written as $P(x;\xi) = A(x) + B(x)\xi$. Denote by δ the odd polynomial $x\xi$. Then the product $\delta P(x;\xi)$ is equal to $xA(x)\xi$ because $\xi^2 = 0$. Hence if $\delta P = 0$, we get $xA(x) = 0$, hence $A(x) = 0$, hence $P(x;\xi) = B(x)\xi$. In turn, $B(x)$ can be written as $B(0) + xC(x)$, where $C(x)$ is a polynomial, and finally

$$P(x;\xi) = B(0)\xi + \delta C(x) \quad . \tag{9.53}$$

This can be generalized to any number of even variables $x_1,\dots,x_n$ and odd variables $\xi_1,\dots,\xi_n$. Consider the odd polynomial $\delta = x_1\xi_1 + \dots + x_n\xi_n$. Setting $\delta_j = x_j\xi_j$, we get $\delta = \delta_1 + \dots + \delta_n$ and $\delta_i\delta_j = -\delta_j\delta_i$ since $x_ix_j = x_jx_i$ and $\xi_i\xi_j = -\xi_j\xi_i$. Therefore $\delta^2 = 0$. Then by induction on n, one proves that a polynomial $P = P(x;\xi)$ satisfies the equation $\delta P = 0$ iff it can be written as

$$P(x;\xi) = c\,\xi_1\cdots\xi_n + \delta Q(x,\xi) \tag{9.54}$$

for some polynomial $Q(x;\xi)$. The constant c is uniquely defined by P since $c\xi_1\dots\xi_n = P(0;\xi)$.

Consider again variables $x_1,\dots,x_p$, $\xi_1,\dots,\xi_q$ and a linear transformation with matrix $T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ as before. Introduce new variables $y_1,\dots,y_q$, $\eta_1,\dots,\eta_p$ with y_j even and η_k odd , and the odd polynomial

$$\delta = x_1\eta_1 + \dots + x_p\eta_p + y_1\xi_1 + \dots + y_q\xi_q \quad . \tag{9.55}$$

Let us denote by $\begin{pmatrix} M' & N' \\ P' & Q' \end{pmatrix}$ the inverse of the transpose of T and define

$$T^* = \begin{pmatrix} Q' & P' \\ N' & M' \end{pmatrix} \quad . \tag{9.56}$$

If we act simultaneously on (x,ξ) by the linear transformation with matrix T, and on (y,η) by the matrix T^*, we see that δ remains invariant.

Put $\omega = \xi_1\cdots\xi_q\ \eta_1\cdots\eta_p$. Then $\delta\omega = 0$, and any solution $P = P(x,\xi,y,\eta)$ of $\delta P = 0$ is of the form $c\cdot\omega + \delta Q$ for some constant c and some polynomial $Q = Q(x,\xi,y,\eta)$. The constant c is equal to $P(0,\xi,0,\eta)$. Now transform simultaneously x,ξ by T and y,η by T^*. Let φ be the transform of ω. From $\delta\omega = 0$ and the invariance of δ, we get $\delta\varphi = 0$, hence

$$\varphi = c\omega + \delta Q(x,\xi,y,\eta) \quad . \tag{9.57}$$

The constant c is obtained by putting x and y equal to 0, that is neglecting the terms containing these variables. But then, ξ transforms into $Q\xi$ and η into $M'\eta$, hence by the Grassmann definition of the determinant (formula (9.1)), one gets $c = \det Q \cdot \det M'$. We want to show that *the constant* c, which we denote for a moment by D(T), *is the inverse of the superdeterminant of* T.

By definition, T transforms ω in $D(T)\cdot\omega + \delta Q$ for some Q. Acting now by the transformation associated to a matrix T', we transform ω in $D(T')\omega + \delta Q'$ for some Q' and Q into Q", hence we transform $D(T)\omega + \delta Q$ into $D(T)D(T')\omega + \delta Q'''$ for a suitable Q'''. The multiplicative rule follows

$$D(TT') = D(T) \cdot D(T') \quad . \tag{9.58}$$

It remains to consider the elementary cases:

(a) If $T = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix}$ then $T^* = \begin{pmatrix} I_q & 0 \\ -{}^tX & I_p \end{pmatrix}$, hence $Q = I_q$, $M' = I_p$ and $D(T) = 1$.

(b) If $T = \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}$ then $T^* = \begin{pmatrix} I_q & -{}^tY \\ 0 & I_p \end{pmatrix}$, hence $Q = I_q$, $M' = I_p$ and $D(T) = 1$.

(c) If $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ then $T^* = \begin{pmatrix} {}^tB^{-1} & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, hence $Q = B$, $M' = {}^tA^{-1}$, hence $D(T) = (\det B)\cdot(\det A)^{-1}$.

The formula $D(T) = (\operatorname{s\,det} T)^{-1}$ follows now from the remarks at the beginning of subsection 9.11.

Since s det T is equal to $D(T)^{-1}$, the multiplicative rule follows at once from formula (9.58).

9.13. We conclude by a few remarks.

(a) For $T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ as before, put $T^\pi = \begin{pmatrix} Q & P \\ N & M \end{pmatrix}$. Then

$\operatorname{s\,det}(T^\pi)$ is the inverse of s det (T). This is the general symmetry rule for changing the parity. Indeed T acting on variables x_i, ξ_j, replace the even variables x_i by odd variables

η_i and the odd variables ξ_j by even variables y_j. Then T is transformed into T^{π}.

(b) Same notations as before. Consider the vector space of polynomials in x,ξ homogeneous of degree B in x and F in ξ; it is the same as the space denoted $\Sigma^{B,F}V$ in subsection 2.9, where V is the vector space generated by $x_1,\ldots,x_p$, $\xi_1,\ldots,\xi_q$ with obvious subspaces V^+ and V^-. Introduce an auxiliary variable t. We get the following general form of the Master theorem

$$s\det(1 + tT) = \sum_{B,F} (-1)^F t^{B+F}\, \mathrm{Tr}(T|\Sigma^{B,F}V) \quad , \tag{9.59}$$

where the last term is the trace of the operator induced on $\Sigma^{B,F}V$ by the transformation of the variables x,ξ by T in the polynomials $P(x;\xi)$ in $\Sigma^{B,F}V$. The Master theorem in the form (3.16) reads as follows

$$s\det\left(1 + t\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\right) = 1. \tag{9.60}$$

(c) For a matrix $T = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$, we define its supertrace as $\mathrm{str}(T) = \mathrm{Tr}(M) - \mathrm{Tr}(Q)$. The following formulas

$$d\log s\det T = s\,\mathrm{tr}(T^{-1}dT) \quad , \tag{9.61}$$

$$s\det(1 + zT) = \exp\{\sum_{n\geq 1} (-1)^{n-1}\, s\,\mathrm{tr}(T^n)z^n/n\}$$

are the analogues of formulas for determinants considered before.

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