JOURNAL OF MATHEMATICAL PHYSICS VOLUME 41, NUMBER 6 JUNE 2000

Functional integration

Pierre Cartier
Ecole Normale Supérieure, 45 rue d'Ulm, F-75230 Paris Cedex 05, France

Cécile DeWitt-Morette
Department of Physics and Center for Relativity, The University of Texas at Austin, Austin, Texas 78712-1081

(Received 5 November 1999; accepted for publication 28 January 2000)

Three approaches to functional integration are compared: Feynman's definition and the Feynman–Kac formula, Bryce DeWitt's formalism, and the authors' axiomatic scheme. They serve to highlight the evolution of functional integration in the second half of the twentieth century. © *2000 American Institute of Physics.* [S0022-2488(00)00306-6]

I. INTRODUCTION

Functional integration is a natural product of the twentieth century during which mathematics and mathematical physics have been dominated by the identification of useful infinite-dimensional spaces and the discovery of their powers:

- Marvelous connections between apparently disconnected subjects have been discovered thanks to infinite-dimensional spaces.
- Many disciplines, in particular quantum physics, cannot be formulated without infinite-dimensional spaces. Indeed, a synoptic table of physics subjects and mathematical theories, which enrich each other, brings together
  Newtonian Mechanics and Calculus (both Newton's achievements),
  General Relativity and Riemannian Geometry,
  Quantum Physics and Infinite-dimensional Spaces.
  As early as 1927, J. von Neumann clarified and unified the works of Heisenberg (1925) and Schrödinger (1926) in one simple statement: ''To each physical system there corresponds a complex Hilbert space whose one-dimensional subspaces define the states of the system.'' [1] And, nowadays much is expected from integration over function spaces in the development of quantum physics.

A function space is much richer, or less constrained, than the limit of $\mathbb{R}^m$ for $m=\infty$. Therefore a crucial landmark in the development of functional integration was the definition of path integrals which *do not resort to limits of integrals over* $\mathbb{R}^m$ when $m=\infty$. As pointed out by Feynman, replacing a functional integral by the limit of an integral over $\mathbb{R}^m$ is as crude a procedure as replacing an ordinary integral by the limit of a Riemann sum of areas of narrow rectangles.

The simplest functional integral is a path integral, i.e., an integral in which the variable of integration is a function (a path) defined on $\mathbb{R}$ or some time interval $\mathbf{T}\subset\mathbb{R}$. Functional integration in Quantum Field Theory is more than a formal transcription of path integration, and has not yet reached the degree of development of path integration.

We single out three approaches to functional integrals because they can serve as prototypes for many others; namely, the definitions proposed by Richard Feynman (Sec. II), Bryce DeWitt (Sec. III), and Pierre Cartier/Cécile DeWitt-Morette (Sec. IV). Each definition is the seed of computational techniques, and specific problems are best treated by an appropriate definition of functional integration.

We are not writing a review on functional integration in physics. For readers interested in Constructive Quantum Field Theory we recommend an introduction[2] written by J. C. Baez, I. E.

Segal, and Z. Zhou which includes, in particular, a glossary to clarify terminology, and a series of lexicons to correlate the mathematical formulation with the physical interpretation. For a bibliography of the subject up to 1987, we recommend the bibliography of the second edition of the classic *Quantum Physics, a functional integral point of view*[3] by J. Glimm and A. Jaffe. And to size up the explosion of the subject we quote from the Preface of another classic *Functional Integration and Quantum Physics*[4] ''It seemed (said B. Simon in 1979) that path integrals were an extremely powerful tool used as a kind of secret weapon by a small group of mathematical physicists.''

We do not follow the same approach to functional integration as do the constructivists. We do not approach Feynman path integrals by the Wiener route. On the other hand we will show, at the appropriate places, how our work presented in Sec. IV is related to the works of S. Bochner,[5] I. E. Segal,[6] P. Malliavin,[7] and the White Noise School.[8]

In Sec. V we relate the three definitions examined in this paper. The conclusion (Sec. VI) sketches avenues to explore so that functional integration will be as powerful a tool as ordinary integration is nowadays.

II. FEYNMAN'S DEFINITION AND KAC'S PROPOSAL[9]

Path integral as a limit $m=\infty$ in $\mathbb{R}^m$

Functional integration entered Quantum Physics in 1942 in the doctoral dissertation of Richard P. Feynman, ''The Principle of Least Action in Quantum Mechanics.'' The goal was a formulation of quantum electrodynamics based on direct interaction at a distance between charged particles. The problem was to find a ''generalization of quantum mechanics applicable to a system whose classical analogue is described by a principle of least action[10]''—and not necessarily by Hamiltonian equations of motion. Feynman solved the problem by writing[11] the probability amplitude $(q'_t|q'_T)$ for finding at time t in position q'_t a particle known to be at time T in position q'_T as follows:

$$(q'_t|q'_T)=\int\int\cdots\int (q'_t|q'_m)dq'_m(q'_m|q'_{m-1})dq'_{m-1}\cdots(q'_2|q'_1)dq'_1(q'_1|q'_T), \tag{II.1}$$

where the interval $[T, t]$ has been divided into a large number of small intervals

$$[T,t_1],\ldots,[t_m,t];$$

$$q'_k \text{ is an abbreviation for } q'_{t_k}\equiv q(t_k),$$

and, $L(\dot{q},q)$ being the Lagrangian for the classical system considered,

$$(q'_{t+\delta t}|q'_t) \text{ is ''often equal to } \exp\frac{i}{\hbar}\left[L\left(\frac{q'_{t+\delta t}-q'_t}{\delta t},q'_{t+\delta t}\right)\delta t\right]$$

$$\text{within a normalization constant in the limit as } \delta t \text{ approaches zero.''} \tag{II.2}$$

Feynman notes the ''vagueness[12]'' of the normalization constant as one of the difficulties of his equation. Its absolute value was obtained by Cécile Morette in 1951 by requiring that $(q'_{t+\delta t}|q'_t)$ satisfy a unitary condition. Its complex value is still a matter of debate. Each $q'_k\equiv q(t_k)$ is integrated over its full domain. The limit of (II.1) for large m is a sum over all continuous paths $q:[T,t]\to\mathbb{R}$ with fixed end points. It is a path integral. (See Fig. 1.)

The goal, action at a distance quantum electrodynamics, was not achieved by path integrals. But quantum mechanics had been formulated in terms of the action functional

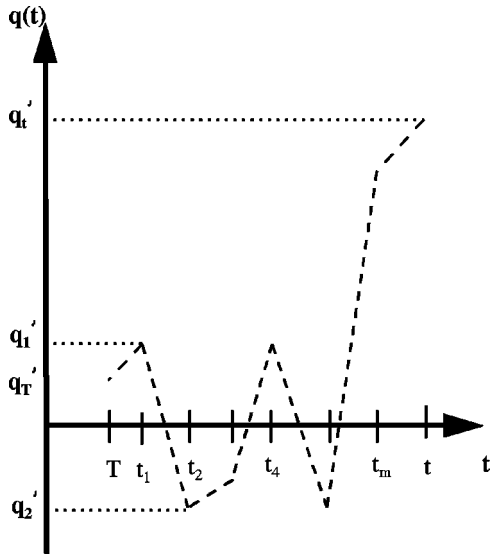

FIG. 1. A dotted line has been drawn for visual purpose from q'_T to q'_1, to $q'_2 \ldots$ to q'_t, but nothing is said in (II.1) about a path connecting $q'_T, q'_1, \ldots, q'_t$. The only relevant quantity is the short time propagator $(q'_{k+1}|q'_k)$. When $q'_1, \ldots, q'_m$ vary, the dotted line varies. The limit of the m-fold integral (II.1) for large m can then be said to be a ''sum over all possible paths'' with fixed end points q'_T and q'_t.

$$S(q) = \int_T^t L(\dot{q}(s), q(s))ds. \tag{II.3}$$

Feynman concluded his doctoral dissertation: ''The final test of any theory lies, of course, in experiment.... The author hopes to apply these methods to quantum electrodynamics.''

Quantum Electrodynamics

The opportunity came in 1947 when Hans Bethe made a nonrelativistic, somewhat heuristic but basically correct calculation of the energy difference between the $2S_{1/2}$ and $2P_{1/2}$ levels of the hydrogen atom recently discovered by Lamb, and brought to Feynman's attention the need to make a relativistic quantum field theoretic calculation of the Lamb shift, using a relativistic cutoff procedure. Feynman knew that the formalism beginning with (II.1) could do it, but he ''had to learn how to make a calculation.[13]'' He developed techniques based on his path integral formulation of probability amplitudes ''making diagrams to help analyze perturbation theory quicker.'' Path integration was ready to make its debut.

How was ''she[14]'' received? By physicists? By mathematicians? Apart from a handful,[15,16,17] physicists were either negative or uninterested. The tide began to turn when Freeman J. Dyson[18,19] made the connection between the radiation theories of Tomonaga, Schwinger, and Feynman. The theory of Feynman differs profoundly in its formulation from that of Tomonaga and Schwinger; but Dyson established their connections by constructing a series expansion to the basic Tomonaga–Schwinger equation,

$$i\hbar c[\partial\Psi/\partial\sigma(x_0)] = H_1(x_0)\Psi, \tag{II.4}$$

where Ψ is the state vector of the system, H_1 the photon–electron interaction, x_0 a point on a spacelike surface σ, and then showing that the rules for computing the series expansion are identical to Feynman's rules for computing the expansion of a functional integral in powers of the coupling constant of the interaction.

Feynman's rules are stated in terms of graphs: ''The graph corresponding to a particular matrix element is regarded, not merely as an aid to the calculation, but as a picture of the physical

process which gives rise to that matrix element.[18]'' These graphs, embodying both the physics and the mathematics of a problem, rapidly became popular. Computations of quantum electrodynamics processes were carried out furiously with this new technique. It was far more reliable than previous techniques for handling infinite terms, and far simpler. With the same amount of work one could compute terms one or two orders higher in a power expansion.

Path integration, from which the diagram technique had been developed, was for a number of years largely forgotten,[20,21] and much effort was spent in obtaining the graphical rules from formalisms other than a path integral representation of the solution.[22] These other methods were not safer mathematically than path integration, but path integration was too unusual a formulation to be easily accepted.

The Feynman–Kac formula

A typical path integral is the limit of an m-fold integral (II.1) when m is infinite, written symbolically. We come now to the gist of Feynman's method. Let us rewrite formula (II.1) by collecting the phase factors coming from the approximate formula (II.2). We get an approximation

$$(q_t'|q_T') \simeq \int_{\mathbb{R}^m} \Gamma_m \prod_{k=1}^{m} dq_k' e^{iS_m/\hbar}, \tag{II.5}$$

where Γ_m is the ''rather vague'' normalization constant, and where

$$S_m = \sum_{k=1}^{m} L\left(\frac{q_{k+1}'-q_k'}{t_{k+1}-t_k}, q_{k+1}'\right)(t_{k+1}-t_k), \tag{II.6}$$

with the conventions $t_{m+1}=t$, $q_{m+1}'=q_t'$. It is striking that S_m is the Riemann sum approximation to the action

$$S(q) = \int_T^t ds\, L(\dot{q}(s), q(s)) \tag{II.7}$$

evaluating along a path $q:[T,t]\to\mathbb{R}^D$ with given end points $q(T)=q_T'$, $q(t)=q_t'$. Feynman took the bold step to take the limit $m=\infty$ inside the integral in (II.5) and therefore to write an exact (meaningless) formula

$$(q_t'|q_T') = \int \mathcal{D}q\, e^{iS(q)/\hbar}. \tag{II.8}$$

The symbol $\mathcal{D}q$ is formally the limit of $\Gamma_m \Pi_{k=1}^m dq_k'$ incorporating the normalization constant Γ_m. The symbol $\mathcal{D}q$ is often written as $\Gamma\ \Pi_{T\leqslant s\leqslant t} dq(s)$ and $\int \mathcal{D}q \ldots$ is interpreted as ''sum over all possible paths as best as you can.'' The symbol $\mathcal{D}q$ is lacking not only a mathematical definition, but even an unambiguous heuristic one. Much effort was spent to give a direct meaning to both the normalization constant Γ and to the product $\Pi_s dq(s)$ over all time values s. To say that (II.8) is the limit of (II.5) for m infinite implies that the short time propagator $(q_{t+\delta t}'|q_t')$ is known, and that the limit exists. Our thesis is: *we should treat $\mathcal{D}q$ as a pure symbol, a black box not to be open, and all that matters is the set of rules for manipulating it.*

Meanwhile, Mark Kac in 1951 made the connection with the Wiener measure. Since the very beginning (in the 1920s) Wiener noticed that the integral of a functional $F(q)$ of a path q of the Brownian motion with respect to the Wiener measure could be calculated as a limit for $m=\infty$ of an integral

$$\int_{\mathbb{R}^m} C_m \prod_{k=1}^{m} dq_k' e^{-S_m(q_1'\cdots q_m')} F(q), \tag{II.9}$$

where the interpolating path is as in Fig. 1. Here the term in the exponential factor is given by

$$S_m = \frac{1}{2C} \sum_{k=1}^{m} \frac{(q'_{k+1} - q'_k)^2}{\tau_{k+1} - \tau_k}. \tag{II.10}$$

What produces the convergence for $m = \infty$ is the replacement of the imaginary exponential $e^{iS_m/\hbar}$ by a positive damping factor e^{-S_m}. The Wiener measure is a *bona fide* mathematical construction and has been around for seventy-five years. Following in the footsteps of Feynman, we could write it as

$$\int \mathcal{D}q e^{-S_c(q)} F(q) \tag{II.11}$$

instead of the customary $\mathbf{E}[F(q)]$ used by the probabilists. Here the ''action,'' derived from the kinetic energy alone, is given by

$$S_c(q) = \frac{1}{2C} \int_T^t \dot{q}(\tau)^2 d\tau, \tag{II.12}$$

which could also be written as

$$S_c(q) = \frac{1}{2C} \int_T^t \frac{(dq)^2}{d\tau}. \tag{II.13}$$

Writing the Wiener integral as

$$\mathbf{E}[F(q)] = \int \mathcal{D}q \cdot \exp\left\{ -\frac{1}{2C} \int_T^t \frac{(dq)^2}{d\tau} \right\} F(q) \tag{II.14}$$

brings many benefits; for instance, the so-called Cameron–Martin formula is just a reflection of the invariance of the ''volume element'' $\mathcal{D}q$ under a shift of q into $q + q_0$ (for q_0 fixed). But this point of view was never accepted by the ''standard'' probabilists.

The transition amplitude $(q'_t | q'_T)$ is used in quantum mechanics to solve the Schrödinger equation. Let the Lagrangian be the standard one for a particle under an exterior force field, namely

$$L(\dot{q}, q) = \frac{m}{2} (\dot{q})^2 - V(q). \tag{II.15}$$

Then the Schrödinger equation takes the form

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + V\psi, \tag{II.16}$$

and the Feynman solution is given by

$$\psi(t, q'_t) = \int dq'_T (q'_t | q'_T) \psi(T, q'_T) \tag{II.17}$$

with a propagator given by (II.8). Let us introduce an imaginary time $\tau = it$. Then Schrödinger's equation goes into a diffusion equation

$$\partial_\tau \varphi = \frac{C}{2} \Delta \varphi - W\varphi \tag{II.18}$$

with $C=\hbar/m$ and $W=V/\hbar$. Let us now perform the change $\tau=it$ directly into the integral $\int \mathcal{D}q e^{iS(q)/\hbar}$ for the propagator $(q_t'|q_T')$. Formally $iS(q)/\hbar$ goes into $-S^I(q)$ (''imaginary time action'') given by[23]

$$S^I(q)=\int_{iT}^{it}\left\{\frac{1}{2C}\frac{(dq)^2}{d\tau}+W(q)d\tau\right\}, \tag{II.19}$$

and $(q_t'|q_T')$ goes into a kernel $K(q_{it}'|q_{iT}')$ given by

$$K(q_{it}'|q_{iT}')=\int \mathcal{D}q e^{-S^I(q)}. \tag{II.20}$$

Notice the imaginary parameters in $K(q_{it}'|q_{iT}')$. Kac's remark[24] is as follows: according to formulas (II.14) and (II.19), we can rewrite (II.20) in the form

$$K(q_t'|q_T')=\mathbf{E}\left[\exp-\int_T^t W(q(\tau))d\tau\right] \tag{II.21}$$

(for real parameters now) and indeed a solution to the diffusion equation (II.18) is given by

$$\varphi(t,q_t')=\int dq_T' K(q_t'|q_T')\varphi(T,q_T'). \tag{II.22}$$

To sum up, we get

$$\int \mathcal{D}q\cdot\exp\left\{\frac{i}{\hbar}S(q)\right\}=\mathbf{E}\left[\exp-\frac{1}{\hbar}\int_{iT}^{it}V(q(\tau))d\tau\right]. \tag{II.23}$$

The imaginary limits of integration mean that the two integrals on the r.h.s. (a functional integral over q, an ordinary integral over τ) have to be performed with τ real, then followed by an analytic continuation.

Kac concludes: ''Because of $i(\equiv\sqrt{-1})$ in the exponent, Feynman's theory is not easily made rigorous. On the other hand, [the left hand side of (II.23)] is most conveniently handled when transformed into the form [of the right hand side].'' The issues raised by Kac are:

(i) The validity of setting $\tau=it$; analytic continuation of the *integral* is relatively easy to justify; analytic continuation of the *integrand* and *integrator* is a different issue.

(ii) The use of stochastic calculus in setting up and computing Feynman integrals; whereas the integral is often the solution of a differential equation, the variable of integration is often a stochastic process.

(iii) The possibility that (II.21) makes better sense than (II.8). It is certainly less appealing physically: Wiener's integral ''spoils the physical unification of kinetic and potential parts of the action,[25]'' which is an important feature of Feynman's integrals when setting them up in quantum field theory.

In the 1972 Gibbs Lecture at Yale,[26] Dyson presented Feynman's integral as an ''open opportunity'' for mathematicians: The '' 'sum over histories' is mathematical nonsense[27–29] [but] if manipulated in a purely formal style without regard for rigorous justification, gives all the right answers.'' In particular, Dyson challenged his colleagues to find ''a rigorous definition of Feynman sums which are invariant under general coordinate transformations''—in brief, to find a rigorous path-integral formulation of quantum gravity.

Some later developments

Feynman's integral (also known as a path integral, a sum over fields, or a sum over histories) gained gradual acceptance in the 1960s.[30] It was presented[31,32] at the 1963 session of the Les Houches school, not at center stage but along with many of the techniques developed by B. DeWitt that are used in modern quantum field theory: background field method, effective action, presence of invariance groups, commuting and anticommuting variables.

The Kac formula was readily accepted in Quantum Statistical Mechanics where the basic quantity of interest is the statistical operator $\exp(-\beta H)$ given in terms of the Hamiltonian H of the system and its inverse temperature β. We refer the reader to the Lecture notes of J. Ginibre[33] for the situation in 1970. The use of the Wiener measure in Statistical Mechanics is so extensive and so rich that we can only give a glimpse into the subject by quoting the proceedings of a recent summer school on functional integration.[34]

In the 1970s, with the advent of supercomputers, K. Wilson initiated the study of gauge fields defined on lattices. Lattice quantum field theory is the most direct transcription of Feynman's original ideas that one can imagine. It is particularly important for field theories that cannot be handled by perturbation expansions: theories of strongly interacting fields, and theories in which renormalization cannot be achieved order by order but which may nevertheless possess consistent and nontrivial continuum limits. Supercomputers give, to problems cast in functional-integral form, answers that cannot be obtained from graphical expansions. In addition, they allow researchers direct access to topologically nontrivial configuration spaces and spaces of histories.

The topological power of functional integration appeared[35] in 1971 when the variable of integration was chosen to be a path taking its value in a multiply connected space. A basic theorem for systems whose configuration space is homotopically nontrivial was obtained from the path integral representation of its propagator. In the 1980s, E. Witten suggested and developed functional integrals of type (II.8) which yield finite-dimensional topological invariants:

(1) If the action is supersymmetric,[36] the functional integral can be used for proving the Atiyah–Singer index theorem, and for computing the index.
(2) If the action is a Chern–Simons action,[37] it can be used for computing Jones polynomials.

There is now a rich literature on the topological uses of functional integrals.

III. BRYCE DEWITT'S FORMALISM

In a chapter entitled ''The Heuristic Road to Quantization'' of a forthcoming book ''The Global Approach to Quantum Field Theory'' Bryce DeWitt gives a comprehensive description of the heuristic formal approach to functional integration which he has introduced and developed since the 1960s leading to the standard formal rules used by quantum field theorists.[31,32] It is elegantly set up in the framework of supermanifolds, so that both bosons and fermions are treated on an equal footing. Here we confine our attention to boson fields (i.e., *c*-type).[38]

Schwinger variational principle

The B. DeWitt formulation of functional integrals starts from the Schwinger variational principle; this principle assumes an action $\mathbf{S}$, functional of operators $\underline{\varphi}$ acting on a space of state vectors $|>$; it states that the variation of the transition amplitude $\langle A|B\rangle$ generated by the variation $\delta\mathbf{S}$ of the action functional is

$$\delta\langle A|B\rangle = i\langle A|\,\delta\mathbf{S}/\hbar\,|B\rangle, \tag{III.1}$$

or, equivalently, if one is interested in the amplitude of a transition from an in-state to an out-state

$$\delta\langle \text{out}|\text{in}\rangle = i\langle \text{out}|\,\delta\mathbf{S}/\hbar\,|\text{in}\rangle. \tag{III.2}$$

Henceforth the action **S** is measured in units of Planck's constant, $\hbar = 1$. To simplify the presentation we consider the case when the operators are vector-valued and when the action possesses no invariant flows. Let the variation of the action be bilinear in the field operator $\underline{\varphi}$ and a source J,

$$\delta \mathbf{S} = \delta J_j \underline{\varphi}^j, \tag{III.3}$$

then

$$\frac{\vec{\delta}}{i\,\delta J_j}\langle \text{out}|\text{in}\rangle = \langle \text{out}|\underline{\varphi}^j|\text{in}\rangle.$$

If subsequently the action suffers a second variation $\delta J_i \underline{\varphi}^i$, then

$$\frac{\vec{\delta}}{i\,\delta J_i}\frac{\vec{\delta}}{i\,\delta J_j}\langle \text{out}|\text{in}\rangle = \langle \text{out}|\underline{\varphi}^i \underline{\varphi}^j|\text{in}\rangle. \tag{III.4}$$

''Subsequently,'' symbolically written $i \geqslant j$, means that the time associated with the index i lies to the future of the time associated with the index j. Let T be a chronological order operator, then Eq. (III.4) can be written

$$\frac{\vec{\delta}}{i\,\delta J_i}\frac{\vec{\delta}}{i\,\delta J_j}\langle \text{out}|\text{in}\rangle = \langle \text{out}|T\underline{\varphi}^i \underline{\varphi}^j|\text{in}\rangle. \tag{III.5}$$

The T operation is required to commute with both differentiation and integration with respect to space–time coordinates. Equation (III.5) can be applied to the Taylor expansion of an arbitrary functional of the field operators.

Canonical commutation relations

The heuristic operator quantization rule which has been found to be of the widest application is to assume that the (super)commutator of quantum operators is equal to their Peierls' bracket, up to a factor i:

$$[\mathbf{A},\mathbf{B}] = i(\mathbf{A},\mathbf{B}), \quad (\hbar = 1), \tag{III.6}$$

or more explicitly $[\mathbf{A},\mathbf{B}] = i\hbar(\mathbf{A},\mathbf{B})$. The Peierls' bracket is a covariant formulation which generalizes the classical canonical Poisson bracket; its definition follows from the theory of measurement. Let A and B be two physical observables, let $D_A^- B$ be the retarded effect of A on B and $D_A^+ B$ be the advanced effect of A on B. The Peierls bracket is by definition

$$(A,B) := D_A^- B - D_A^+ B. \tag{III.7}$$

If the retarded/advanced effect of A and B is due to the change of the action functional S of the classical (unquantized) fields φ, then

$$(A,B) = \int dx \int dy \frac{\delta A}{\delta \varphi^i(x)} \tilde{G}^{ij}(x,y) \frac{\delta B}{\delta \varphi^j(y)} =: A_{,i} \cdot \tilde{G}^{ij} \cdot_{,j} B, \tag{III.8}$$

where $\tilde{G}$ is the difference between the advanced and retarded Green functions of the second variation of the action

$$\tilde{G}^{ij} := G^{+ij} - G^{-ij}, \tag{III.9}$$

and where

$$ {}_{j}S_{,i}\cdot G^{\pm ik}=-{}_{j}\delta^{k}, \tag{III.10}$$

with $G^{-ij}=0$ if $i\leqslant j, G^{+ij}=0$ if $i\geqslant j$ and ${}_{j}S_{,i}=\delta^{2}S/\delta\varphi^{i}(x)\,\delta\varphi^{j}(y)$.

In Eqs. (III.7)–(III.10) the fields φ are vector-valued functions. In Eq. (III.6) Peierls' bracket is extended to (functions of) quantum field operators $\underline{\varphi}$. The structure of (**A**, **B**) is assumed to be the same as the structure of (*A*, *B*):

$$ (\mathbf{A},\mathbf{B})=\mathbf{A}_{,i}\cdot\tilde{\mathbf{G}}^{ij}\cdot_{j,}\mathbf{B} \quad \text{with} \quad \tilde{\mathbf{G}}:=\mathbf{G}^{+}-\mathbf{G}^{-}, \tag{III.11}$$

which implies

$$ [\underline{\varphi}^{k},\underline{\varphi}^{j}]=i\tilde{\mathbf{G}}^{kj}. \tag{III.12}$$

The quantum $\tilde{\mathbf{G}}$ is called the (super)commutator operator and the classical $\tilde{G}$ the (super)commutator function. A key problem is the relationship between the classical Peierls' bracket (III.8) and the quantum Peierls' bracket (III.11), i.e., between $\tilde{G}$ and $\tilde{\mathbf{G}}$. We note right away, for later use, that

$$ i\tilde{\mathbf{G}}^{+kj}(x,x')=\underline{\varphi}^{k}(x)\,\underline{\varphi}^{j}(x')-T(\underline{\varphi}^{k}(x)\,\underline{\varphi}^{j}(x'))=i\tilde{G}^{-jk}(x',x). \tag{III.13}$$

The relationship between $\tilde{\mathbf{G}}$ and $\tilde{G}$ is encoded into the relationship between the operator formalism of quantum physics and its functional integral formalism; for instance one can look for a functional integral solution of the operator Schwinger variational equation (III.2); choose, for an ansatz, the Fourier transform of a functional *X*

$$ \langle \text{out}|\text{in}\rangle=\int_{\Phi}\mathcal{D}\varphi\cdot X(\varphi)\exp(iJ\varphi), \quad \varphi\in\Phi, \tag{III.14}$$

where the domain of integration Φ is determined in part by the chosen ''in'' and ''out'' states; by definition

$$ \mathcal{D}\varphi:=\Pi_{x,i}d\varphi^{i}(x). \tag{III.15}$$

A procedure for determining $X(\varphi)$ consists in finding a differential equation for *X*; the calculation begins by an integration by parts (which incidentally is a fundamental tool of Malliavin calculus[7]):

$$ \int_{\Phi}\mathcal{D}\varphi\cdot X(\varphi)\frac{\overleftarrow{\delta}}{i\,\delta\varphi^{i}}\exp(iJ\varphi)=-\int_{\Phi}\mathcal{D}\varphi\cdot X(\varphi)J_{i}\exp(iJ\varphi)=-\langle \text{out}|J_{i}|\text{in}\rangle; \tag{III.16}$$

J_i is obtained from the operator dynamical equation (III.20); the property (III.5) of matrix elements of chronologically ordered products gives the r.h.s. in the form of derivatives with respect to *J* of $\langle \text{out}|\text{in}\rangle$. Replacing $\langle \text{out}|\text{in}\rangle$ by the ansatz (III.14) yields a differential equation for *X*.

Operator dynamical equation

A stationary point of the action $S(\varphi)+J\varphi$ is a classical field φ which satisfies the following dynamical equation, in condensed notation [see (III.8)]:

$$ S_{,i}(\varphi)+J_{i}=0. \tag{III.17}$$

The operator dynamical equation is obtained from

$$ S_{,i}(\underline{\varphi})+J_{i}=0 \quad \text{with} \quad \underline{\varphi}=\varphi+\underline{\phi}, \tag{III.18}$$

where $\underline{\varphi}$ and $\underline{\phi}$ are quantum fields.

Taking the (super)commutator of (III.18) with $\underline{\varphi}^i$, after having Taylor-expanded it to order $O(\underline{\phi}^3)$ [i.e., to order $O(\hbar^2)$ when $\hbar$ is not set equal to 1] one obtains

$$ {}_{i,}S_{,k}(\varphi)\cdot\tilde{\mathbf{G}}^{kj}+\tfrac{1}{2i,}S_{,kl}(\varphi)\cdot(\tilde{\mathbf{G}}^{lj}\underline{\phi}^k+\underline{\phi}^l\tilde{G}^{kj})=0. \tag{III.19}$$

An easy manipulation using (III.9), (III.10), (III.13), and (III.17) yields the operator dynamical equation

$$T\cdot(S(\underline{\varphi})-i\ln\mu(\underline{\varphi}))\frac{\vec{\delta}}{\delta\underline{\varphi}^i}=-J_i, \tag{III.20}$$

where

$$\mu(\varphi):=|(\text{super})\text{determinant}G^+(\varphi)|^{-1/2}+\cdots\equiv|(\text{sdet}\ \ G^+(\varphi)|^{-1/2}+\cdots. \tag{III.21}$$

Having obtained an explicit expression for J_i, we return to the computation (III.16) of X

$$\int_\Phi\mathcal{D}\varphi\cdot X(\varphi)\frac{\vec{\delta}}{i\,\delta\varphi^i}\exp(iJ\varphi)=\langle\text{out}|T\cdot(S(\underline{\varphi})-i\ln\mu(\underline{\varphi}))\frac{\vec{\delta}}{\delta\underline{\varphi}^i}|\text{in}\rangle. \tag{III.22}$$

A straightforward generalization of (III.5) gives the r.h.s. in terms of a function $F(\vec{\delta}/i\,\delta J)$ acting on $\langle\text{out}|\text{in}\rangle$. Returning to the proposed ansatz (III.14)

$$F\left(\frac{\vec{\delta}}{i\,\delta J}\right)\langle\text{out}|\text{in}\rangle=\int_\Phi\mathcal{D}\varphi\cdot X(\varphi)F\left(\frac{\vec{\delta}}{i\,\delta J}\right)\exp(iJ\varphi)=\int_\Phi\mathcal{D}\varphi\cdot X(\varphi)F(\varphi)\exp(iJ\varphi). \tag{III.23}$$

Finally Eq. (III.22) establishes a differential equation for X:

$$\int_\Phi\mathcal{D}\varphi\cdot X(\varphi)\frac{\vec{\delta}}{i\,\delta\varphi^i}\exp(iJ\varphi)=\int_\Phi\mathcal{D}\varphi\cdot X(\varphi)(S(\varphi)-i\ln\mu(\varphi))\frac{\vec{\delta}}{\delta\varphi^i}\exp(iJ\varphi). \tag{III.24}$$

Its solution is

$$X(\varphi)=N\mu(\varphi)\exp[iS(\varphi)]. \tag{III.25}$$

N is an integration constant and the operator Schwinger variational equation (III.2) has been solved:

$$\langle\text{out}|\text{in}\rangle=N\int_\Phi\mu(\varphi)\mathcal{D}\varphi\cdot\exp[iS(\varphi)+iJ\varphi] \tag{III.26}$$

with $\mu(\varphi)$ given by (III.21). The term $\mu(\varphi)$ is often called the ''measure functional.'' Factor ordering, and chronological products played a key role in its computation. The term $\mu(\varphi)$ can be obtained by a totally different route, using Bogoliubov's relations, which we outline below.

The vacuum persistence amplitude

The vacuum persistence amplitude $\langle\text{out,vac}|\text{in,vac}\rangle=:\exp(iW)$ is given by setting $J=0$ in (III.26). Let us assume that in (III.26) the measure $\mu(\varphi)$ has not yet been determined by the operator dynamical equation.

The vacuum persistence amplitude can be computed independently of the functional integral (III.26) via the Bogoliubov's relations; the value of $\exp(iW)$ thus obtained is then set equal to (III.26) with $J=0$, and $\mu(\varphi)$ is found equal to (III.21).

We shall come back to $\mu(\varphi)$ in Sec. V when comparing the Bryce DeWitt's approach to functional integration and the approach of Cartier and DeWitt-Morette, in particular we shall explain why the Bryce DeWitt's functional integral solves a Schrödinger equation modified by a term, $\frac{1}{8}R$, where R the Riemann scalar curvature.

The ''measure functional'' $\mu(\varphi)$ plays an important role in the quantum theory of nonlinear fields as well as in quantum mechanics; for example, it plays a role in the Wick rotation, in the ghost formalism, in checking the consistency to 2-loop order of the functional proof of the index theorem.[39]

IV. CARTIER AND DEWITT-MORETTE'S AXIOMATIC SCHEME

A. Early steps

The first definition of Feynman path integrals to be given directly on function spaces,[40,41] was a direct generalization of the notion of *promeasures* due to Bourbaki. In Book VI, Chapter IX of his well-known treatise,[42] this author deals with the integration theory on a locally convex vector space $\mathbb{X}$; we refer the reader to the *Note historique* of this Bourbaki volume for an account of the development of ideas leading to the definition of promeasures.

A promeasure μ on a space $\mathbb{X}$ is a collection of bounded measures μ_V, where V runs over the closed vector subspaces of $\mathbb{X}$, such that $\mathbb{X}/V$ is of finite dimension, and where the measure μ_V lives on $\mathbb{X}/V$. There is a coherence relation between the μ_V's; as *S.* Bochner remarked in Ref. 5, the best way to express the coherence relation is via Fourier transform. More precisely, the projective system associated to $\mathbb{X}$ consists of spaces $\mathbb{X}/V$, where the subspaces V are determined by the topological dual $\mathbb{X}'$ of $\mathbb{X}$: such a space V consists of the elements x of $\mathbb{X}$ such that

$$\langle x_j', x\rangle = 0 \quad \text{for a finite number of } x_j' \in \mathbb{X}'. \tag{IV.1}$$

If $\mu=(\mu_V)$ is a promeasure, it is completely determined by its Fourier transform $\mathcal{F}\mu$, a function on the dual $\mathbb{X}'$ of $\mathbb{X}$ characterized by the relation (IV.5).

The most important class of promeasures are the Gaussians. On a Hilbert space $\mathcal{H}$, identified to its dual by F. Riesz's theorem, there is a canonical promeasure, with Fourier transform equal to $\exp(-\frac{1}{2}\|x\|^2)$. It has been first introduced by I. E. Segal in Ref. 6, under the name of ''weak canonical distribution'' or ''isonormal distribution.'' But, in quantum physics, we ought to consider objects with Fourier transform of the form $\exp(-i\|x\|^2)$. To achieve this, we need to define a *prodistribution* in a way similar to promeasures, by replacing the bounded measures μ_V above by more general distributions T_V [for instance, of the form $\exp i(x_1^2+,...,+x_D^2)d^Dx$ on $\mathbb{R}^D$].

We can consider a promeasure as an *integrator*, where the *integrands* are the bounded continuous *cylinder functions*: a cylinder function F on $\mathbb{X}$ is a function of the form $F(x)=\Phi(\langle x_1', x\rangle,\ldots,\langle x_D', x\rangle)$ for x_j' in $\mathbb{X}'$ and a function Φ and $\mathbb{R}^D$. By imposing suitable conditions on the derivatives of Φ, we can define suitable integrands for the prodistributions. Admittedly, integrating only cylinder functions is a severe restriction. The next step is the integration of limits of cylinder functions, like in ordinary integration we integrate limits of step functions. A lot of experimentation has been done with various classes of test functions on an infinite-dimensional space, and the dual classes of distributions. We refer the reader to the works of Malliavin,[7] Krée,[43] and also the White Noise School.[8]

Prodistributions, abstract as they may seem, are very practical; one can often use their techniques without knowing their definitions—let alone their name. One of the techniques exploits the transformation under linear mappings of Fourier transforms.

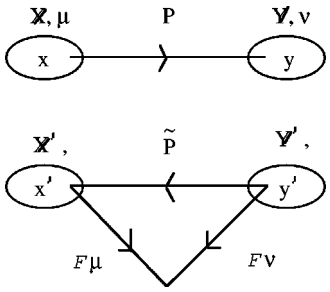

FIG. 2. A linear map on $\mathbb{R}^D$ is only a $D\times D$ matrix; a linear map on an infinite-dimensional space $\mathbb{X}$ has many more possibilities. Therefore the map $P:\mathbb{X}\to\mathbb{Y}$ (not necessarily finite-dimensional), and its consequence $\mathcal{F}_\nu=\mathcal{F}_\mu\circ\tilde{P}$, offer techniques of integration useful in many circumstances.

Linear maps

Linear maps $P:\mathbb{X}\to\mathbb{Y}$ on the infinite-dimensional space $\mathbb{X}$ are much more varied than the linear maps on $\mathbb{R}^n$, and a space $\mathbb{X}$ of paths $x:T\to\mathbb{R}^D$ can be linear even when x is not a linear map. (See Fig. 2.) On the dual $\mathbb{X}'$ of $\mathbb{X}$, the Fourier transform $\mathcal{F}\mu$ is given by

$$\mathcal{F}\mu(x')=\int_{\mathbb{X}} d\mu(x)\exp(-2\pi i\langle x',x\rangle). \tag{IV.2}$$

Let ν be the image of μ under the linear map $P:\mathbb{X}\to\mathbb{Y}$, and $\tilde{P}:\mathbb{Y}'\to\mathbb{X}'$ be the transposed map on their respective duals,

$$\langle x',x\rangle=\langle y',y\rangle \quad \text{with} \quad y=Px \quad \text{and} \quad x'=\tilde{P}y'; \tag{IV.3}$$

then

$$\mathcal{F}\nu=\mathcal{F}\mu\circ\tilde{P}. \tag{IV.4}$$

This formula is valid for a prodistribution μ because

$$\mathcal{F}\mu(x')=\mathcal{F}\mu_V(x')=\int_{\mathbb{X}/V} d\mu_V(u)\exp(-2\pi i\langle x',u\rangle) \tag{IV.5}$$

for $x'\in V^0$, where V^0 is the orthogonal of V, i.e., V^0 is the linear span of the set of points $\{x_j'\}$ used in (IV.1) to define V.

This simple technique has been used in great many applications. [See, for instance, two review articles (Refs. 44 and 45)]. A particularly illuminating one is the calculation of the glory scattering of waves by black holes.[46] It is too long to reproduce here because the process involves conservation laws and interfering degeneracies of the second variation of the action functional. We shall only quote the result, and extract from the calculation two applications of (IV.4) in a simpler but similar context.

An example: Polarized glory scattering

The WKB approximation of glory scattering cross sections (backward scattering) breaks down; but in a full semiclassical expansion, the dominating term of the intensity variation $d\sigma(\Omega)$ of a wave scattered in the solid angle $d\Omega$ by an axisymmetric potential, computed from first principles by functional integration, is

$$d\sigma(\Omega)=4\pi^2\lambda^{-1}B^2(\theta)\frac{dB(\theta)}{d\theta}J_{2s}^2(2\pi\lambda^{-1}B(\theta)\sin\theta)d\Omega, \tag{IV.6}$$

where λ is the wave length, $d\Omega = 2\pi \sin\theta d\theta$, $B(\theta)$ is the impact parameter of the eikonal rays of the wave, J_{2s} is the Bessel function of order $2s$, $s=0$ for scalar waves, $s=\frac{1}{2}$ for electromagnetic waves, and $s=1$ for gravitational waves. Equation (IV.6) is the polarized glory scattering cross section valid for θ close to π.

Discretization

In Fig. 2, any linear map $P:\mathbb{X}\to\mathbb{R}^n$ (n finite) is a discretization. For example, let $x\in\mathbb{X}$ be a path $x:\mathbf{T}\to\mathbb{R}$ with $\mathbf{T}=[t_a,t_b]$, and

$$P:x\to u=\{u^j\} \quad \text{where} \quad u^j=x(t_j)=\langle \delta_{t_j},x\rangle \tag{IV.7}$$

for a finite partition $\{t_1,\ldots,t_n\}$ of the interval $\mathbf{T}$. Equation (IV.4) gives the prodistribution ν on $\mathbb{R}^n$, off which one can read the short time propagator corresponding to μ on $\mathbb{X}$. The case $x:\mathbf{T}\to\mathbb{R}^D$ requires only a heavier notation.

The most general discretization is obtained by choosing

$$u^j=\langle x'_j,x\rangle \quad \text{where} \quad \{x'_j\} \quad \text{is a finite set in} \quad \mathbb{X}'. \tag{IV.8}$$

We shall work out explicitly an example in order to demystify prodistributions. Let

$$\begin{aligned} &\mathbb{X} \quad \text{be the space of continuous paths } x \text{ with one fixed end point, e.g.,}\\ &\qquad x(t_a)=a \quad \text{for every} \quad x\in\mathbb{X},\\ &\mu \quad \text{be defined by} \quad \mathcal{F}\mu(x')=\exp[-\pi W(x')] \quad \text{where } W \text{ is a quadratic form on } \mathbb{X}' \end{aligned} \tag{IV.9}$$

$$W(x')=\langle x',Gx'\rangle_{\mathbb{X}}; \tag{IV.10}$$

let

$$P:x\to u=\{u^j\} \quad \text{where} \quad u^j=\langle x'_j,x\rangle, \quad x'_j=\delta_{t_{j+1}}-\delta_{t_j}. \tag{IV.11}$$

The transposed $\tilde{P}$ of P is defined by (IV.3); let $\mathbb{R}_n$ be the dual of $\mathbb{R}^n$ and $u'=\{u'_j\}\in\mathbb{R}_n$, then (Fig. 2)

$$\langle x',x\rangle_{\mathbb{X}}=\langle u',u\rangle_{R^n},$$

i.e.,

$$\langle \tilde{P}u',x\rangle=\langle u',Px\rangle=\sum_j u'_j u^j=\sum_j u'_j\langle x'_j,x\rangle,$$

hence

$$\tilde{P}u'=\sum_j u'_j x'_j\,. \tag{IV.12}$$

Equation (IV.4) says that the Fourier transform of the prodistribution ν on $\mathbb{R}^n$ is

$$\mathcal{F}\nu(u')=\mathcal{F}\mu(\tilde{P}u')=\exp[-\pi W(\tilde{P}u')].$$

In the case of the Wiener measure

$$W(x')=\langle x',Gx'\rangle_{\mathbb{X}} \quad \text{with} \quad G(t,s)=\inf(t-t_a,s-t_a). \tag{IV.13}$$

Then, together with (IV.12) and (IV.11)

$$W(\tilde{P}u') = \sum u'_k u'_j \left\langle \delta_{t_{k+1}}(t) - \delta_{t_k}(t), \int ds \inf(t-t_a, s-t_a)\, \delta_{t_{j+1}}(s) - \delta_{t_j}(s) \right\rangle$$

$$= \sum u'_k u'_j \int dt \int ds (\delta(t-t_{k+1}) - \delta(t-t_k)) \inf(t-t_a, s-t_a)[\delta(s-t_{j+1}) - \delta(s-t_j)]$$

$$= \sum_k u'^2_k (t_{k+1} - t_k). \tag{IV.14}$$

Given the Fourier transform $\mathcal{F}\nu(u')$ on $\mathbb{R}_n$, a standard calculation gives ν on $\mathbb{R}^n$:

$$d\nu(u) = d\gamma_1(u^1) \cdots d\gamma_n(u^n), \tag{IV.15}$$

with $d\gamma_k(u^k) = (t_{k+1} - t_k)^{-1/2} \exp[-\pi (u^k)^2/(t_{k+1} - t_k)] du^k$.

Conclusion: The probability distribution (IV.15) shows that the random process $\{x(t); t \geqslant t_a\}$ defined by the Wiener measure is a Brownian path.

Semiclassical expansions

The linear map $P: \mathbb{X} \to \mathbb{Y}$ of interest when computing semiclassical expansions is the map from the space of vector fields along a classical path to the space of eigenvectors of the corresponding Jacobi operator (IV.25). We construct P and show its usefulness, in particular when the critical points of the action functional are degenerate (caustics, conservation laws).

Let $\mathbb{X}$ be a space of functions $x: \mathbf{T} \to M^D$ (a D-dimensional manifold not necessarily $\mathbb{R}^D$). The functional representation of the probability amplitude $\langle b, t_b | a, t_a \rangle$ of the transition of a system S from a state a at t_a to a state b at t_b is an integral over a domain $\mathbb{X}_{ab} \subset \mathbb{X}$ restricted by the states a and b. The domain $\mathbb{X}_{ab}$ of integration is such that the action functional on $\mathbb{X}_{ab}$ is finite:

$$S(x) < \infty \quad \text{for} \quad x \in \mathbb{X}_{ab}. \tag{IV.16}$$

Let $S(q)$ be a minimum of S for $q \in \mathbb{X}_{ab}$; the path q is a classical path (a critical point of S) defined by $2D$ constants of integration: D at t_a and D at t_b such that $q \in \mathbb{X}_{ab}$.

A semiclassical expansion begins with the functional Taylor expansion of S around q. Let $x(\alpha) \in \mathbb{X}$ be a one parameter family of paths such that $x(0) = q$. The functional Taylor expansion of $S(x(\alpha))$ in the direction ξ, where

$$\xi := \left. \frac{dx(\alpha)}{d\alpha} \right|_{\alpha = 0}, \tag{IV.17}$$

is an ordinary Taylor expansion in powers of α; at $\alpha = 1$,

$$S(x(1)) = S(q) + \tfrac{1}{2} S''(q) \cdot \xi\xi + \cdots. \tag{IV.18}$$

Explicitly

$$S''(q) \cdot \xi\xi = \int_{\mathbf{T}} dt\, \xi^\alpha(t) \mathcal{J}_{\alpha\beta}(q(t)) \xi^\beta(t), \tag{IV.19}$$

where $\mathcal{J}_{\alpha\beta}(q(t))$ is the differential Jacobi operator on the space of vector fields along q; moreover

$$\mathcal{J}_{\alpha\beta}(q,s,t) := \frac{1}{2} \frac{\delta^2}{\delta\xi^\alpha(s)\, \delta\xi^\beta(t)} [S''(q) \cdot \xi\xi] \tag{IV.20}$$

is the functional Jacobi operator on the tangent space $T_q \mathbb{X}_{ab}$.

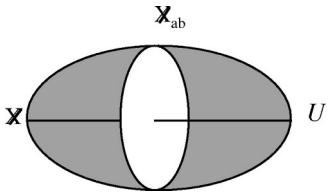

FIG. 3. The nature of the intersection of the space $\mathbb{X}_{ab}$ of paths with given boundary conditions (dictated by the domain of integration), and the space $\mathcal{U}$ of solutions of the Euler–Lagrange equations of the given system is the key information in semiclassical analysis.

Provided it is not degenerate $S''(q)\cdot\xi\xi$ is a quadratic form on $T_q\mathbb{X}_{ab}$ which can be used for constructing a Gaussian prodistribution μ on $T_q\mathbb{X}_{ab}$. See in Ref. 45 a study of the approximation of an integral on $\mathbb{X}_{ab}$ by an integral on $T_q\mathbb{X}_{ab}$.

If $S''(q)$ is degenerate one can either expand S around $x_0\neq q$ and/or keep higher terms in the Taylor expansion of $S(x)$. Whether $S''(q)$ is degenerate or not, the computation of semiclassical expansions is conveniently explained on the following diagram. Let $\mathcal{U}$ be the $2D$-dimensional space of classical paths $q(a,b)$ where $a\in\mathbb{R}^D$ and $b\in\mathbb{R}^D$ stand for the $2D$ constants of integration in the Euler–Lagrange solutions of $S'(q)=0$. (See Fig. 3.) Let

$$\mathcal{U}_{ab}=\mathcal{U}\cap\mathbb{X}_{ab}\subset\mathbb{X}. \tag{IV.21}$$

If $S''(q)$ is not degenerate, $\mathcal{U}_{ab}$ consists of isolated points.

If $S''(q)$ is degenerate, one needs to distinguish the case where $\mathcal{U}_{ab}$ is of dimension $l>0$ and the case where $q\in\mathcal{U}_{ab}$ is a multiple root of $S'(q)=0$. Finally, it can happen that $\mathcal{U}_{ab}$ is an empty set.

For quantitative purposes we choose a basis of $T_q\mathbb{X}_{ab}$ made of normalized eigenvectors of the Jacobi operator (IV.19)

$$\mathcal{J}(q)\psi_k(t)=\alpha_k\psi_k(t)\quad k\in\{0,1,...\}. \tag{IV.22}$$

There may be l zero eigenvalues $\alpha_k=0$ for $k=\{0,1,...l-1\}$. (See Refs. 47 and 48 for details of this case.) The orthonormalization holds

$$\int_{\mathrm{T}} dt(\psi_k(t)|\psi_j(t))=\delta_{kj}. \tag{IV.23}$$

We expand the vector field ξ in the eigenvector basis

$$\xi^\alpha(t)=\sum_{k=0}^{\infty} u^k\psi_k^\alpha(t). \tag{IV.24}$$

Let $\mathbb{Y}$ be the space of points u with coordinates u^k such that $\Sigma\alpha_k(u^k)^2<\infty$.

Equation (IV.24) defines a linear map

$$P:T_q\mathbb{X}_{ab}\to\mathbb{Y}\ \text{ by }\ \xi\mapsto\{u^k\}. \tag{IV.25}$$

Because the basis of $\mathbb{Y}$ is countable, it is straightforward to isolate its finite-dimensional subspace $\mathbb{Y}_0$ spanned by the eigenvectors ψ_k with zero eigenvalues $\alpha_k=0$. The functional integral on $\mathbb{Y}$ splits into an ordinary integral over $\mathbb{Y}_0$ and an integral over $\mathbb{Y}/\mathbb{Y}_0$. The split is already formulated in the second variation $S''(q)\cdot\xi\xi$ which is diagonal in the eigenvector basis,

$$S''(q)\cdot\xi\xi=\sum_{k=0}^{\infty}\alpha_k(u^k)^2=0+\sum_{k=l}^{\infty}\alpha_k(u^k)^2\ \text{ if }\ \alpha_k=0\ \text{ for }\ k\in\{0,1,...,l-1\}. \tag{IV.26}$$

Equation (IV.22) says that if $\alpha_k=0$, the eigenvector ψ_k is a Jacobi field, thus ψ_k is both in $T_q\mathbb{X}_{ab}$ and $T_g\mathcal{U}$ and the intersection $\mathcal{U}_{ab}=\mathbb{X}_{ab}\cap\mathcal{U}$ does not consist of isolated points.

B. An axiomatic scheme for path integration

Prodistributions are only a first step in defining functional integrals on function spaces. In order to develop a practical axiomatic scheme for functional integration, we have studied[45] many aspects of the subject and have identified a few simple basic ideas. Path integrals are introduced in Sec. II as representations of probability amplitudes $\langle\beta,t_b|\alpha,t_a\rangle$; the boundary informations α and β are, in Sec. II, the positions at t_a and t_b. In Sec. III, functional integrals are defined by means of $\langle\text{out}|\text{in}\rangle$ amplitudes; here we approach path integration as a mathematical operation in its own right, not as a representation of the matrix elements of an operator.

Domain of integration

The definition of a functional integral begins with the choice of its domain of integration $\mathbb{X}$. The importance of a domain of integration cannot be overestimated. Writing $\int_{\mathbb{X}}\gamma=\langle\mathbb{X},\gamma\rangle$ gives to the domain its *letters de noblesse*: it presents the domain as a linear functional on a space of functions, i.e., as a distribution. It is a landmark of new developments which began with F. Riesz in 1909 and culminated in the 1940s with the works of G. de Rham, A. Weil, I. M. Gelfand, L. Schwartz, and others.

In this section $\mathbb{X}$ is, most often but not exclusively, a space of paths,

$$x:\mathbf{T}\subset\mathbb{R}\to\mathbb{M}^D. \tag{IV.27}$$

T is a time interval, but not necessarily a fixed one; for instance it can be an interval (t_0, first exit time of x out of a domain $U\subset\mathbb{M}^D$). The range $\mathbb{M}^D$ can be $\mathbb{R}^D$, or a Riemannian manifold, or a multiply connected space, or a symplectic manifold, or a complex manifold, etc.

Spaces $\mathbb{X}$ of paths x are usually simpler than the paths:

- The space $\mathcal{P}_0\mathbb{M}^D$ of pointed paths on $\mathbb{M}^D$ (paths with *strictly one* fixed point) is contractible, even when $\mathbb{M}^D$ is a complicated manifold. Therefore one can map, biunivocally, $\mathcal{P}_0\mathbb{M}^D$ into the space $\mathcal{P}_0\mathbb{R}^D$ of pointed paths on $\mathbb{R}^D$, and carry out on $\mathcal{P}_0\mathbb{R}^D$ the computation of an integral over $\mathcal{P}_0\mathbb{M}^D$. Let P be a map on the space of paths

$$P:\mathcal{P}_0\mathbb{R}^D\to\mathcal{P}_0\mathbb{M}^D \text{ by } z\mapsto x \quad (\text{not } z(t)\mapsto x(t)) \tag{IV.28}$$

  such that writing $x(t,z)$ for the position at time t for the path x associated to z, one gets $x(t_0,z)=x_0$ and $z(t_0)=0$. In general P is not defined by a rule calculating $x(t)$ from $z(t)$ for every given t; the path x is a *functional* of z. Instead we can often write

$$x(t,z)=x_0\cdot\Sigma(t,z), \quad x_0\in\mathbb{M}^D, \tag{IV.29}$$

  where $\Sigma(t,z):\mathbb{M}^D\to\mathbb{M}^D$ is a map depending on t and z.
  A great many applications of (IV.28) and (IV.29) for computing path integrals over $\mathcal{P}_0\mathbb{M}^D$ can be found in Ref. 45. We only state below a general theorem (IV.55)–(IV.57) exploiting (IV.28) and (IV.29), and two applications of the theorem (IV.58)–(IV.63) and (IV.64)–(IV.66).
- Changes of variable of integration [e.g., (IV.11), (IV.25)] should be done on the space of paths, not in the range of the paths. For example, the change $x\mapsto y$ defined by

$$y(t)=\int_0^t dsK(t,s)x(s) \tag{IV.30}$$

  is a trivial linear change of variable on $\mathbb{X}$, but a nonlocal map on $\mathbb{M}^D$.
- There is an incredibly rich variety of function spaces $\mathbb{X}$ which can be used as domains of integrations. Too often path integrals are thought of as solutions of parabolic partial differential equations (diffusion or Schrödinger equations). They are also useful as solutions of

wave equations, Dirac equations, elliptic equations. For instance solutions of wave equations can be expressed as integral over spaces of Poisson paths. For use both in probability and in quantum physics, a Poisson path is characterized by a number of jumps and the jumplines during a given finite time interval $\mathbf{T}=[t_a,t_b]$:

$$\mathbb{X}=\cup\mathbb{X}_n, \quad \mathbb{X}_n \text{ the space of paths which jump } n \text{ times.} \tag{IV.31}$$

A path $x\in\mathbb{X}_n$ is characterized by the jump times

$$t_a\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t_b. \tag{IV.32}$$

See below (IV.38)–(IV.42) for a volume element defined on a space $\mathbb{X}$ of Poisson paths.
The domain of integration $\mathbb{X}$ contains the following information: $x\in\mathbb{X}$ is a function defined by its domain, its range, and its analytic properties [continuous, $S(x)<\infty$, etc.]. If a functional integral is a representation of a matrix element $\langle B|\text{operator}|A\rangle$, the functions x are restricted by the states A and B.

A domain of integration which is the proper arena for a large class of functional integrals is the Schwartz space $\mathcal{S}'$ of tempered distributions; it is extensively used in White Noise calculus. The basic volume element in White Noise calculus is defined by a Gaussian integral of type (IV.35) where the covariance G is a δ function. We often work with a Banach space $\mathbb{X}$ and its dual $\mathbb{X}'$, related to the Schwartz spaces $\mathcal{S}$ and $\mathcal{S}'$ as follows:

$$\mathcal{S}\subset\mathbb{X}'\underset{D}{\overset{G}{\rightleftarrows}}\mathbb{X}\subset\mathcal{S}',$$

where elements of $\mathbb{X}$ are fields and elements of $\mathbb{X}'$ are sources. A *rigged Hilbert space* (also known as a Gelfand triple)

$$\mathcal{S}\subset\mathcal{H}\subset\mathcal{S}'$$

is a particular case of the above quadruplet, corresponding to an identification of $\mathbb{X}=\mathcal{H}$ with $\mathbb{X}'$, hence $G=D=\mathbb{1}$, and used almost exclusively in White Noise calculus.

Volume elements

There is no universal definition of a volume element on a Banach space $\mathbb{X}$ but there is a definition of a class of volume elements appropriate to $\mathbb{X}$. A simple example shows the failure of the Lebesgue measure on an infinite-dimensional space and suggests a definition of volume elements. Let

$$I_D(a):=\int_{IR^D}d^Dx\cdot\exp\left(-\frac{\pi}{a}|x|^2\right)=a^{D/2}, \tag{IV.33}$$

where, as usual, $d^Dx:=dx^1dx^2,\ldots,dx^D$. In the limit $D=\infty$

$$I_\infty(a)=\begin{cases}0 & \text{if } 0<a<1\\ 1 & \text{if } a=1\\ \infty & \text{if } a>1\end{cases}$$

hence $I_\infty(a)$ is not a continuous function of a, as should reasonably be expected if $d^\infty x$ existed.

To begin with, this example suggests that we scale the Lebesgue measure d^Dx and introduce a dimensionless volume element (only dimensionless quantities can be generalized to infinite-dimensional spaces):

$$\mathcal{D}_ax=d^Dx/a^{D/2}. \tag{IV.34}$$

Equation (IV.33) now reads

$$\int_{IR^D} \mathcal{D}_a x \cdot \exp\left(-\frac{\pi}{a}|x|^2 \right) = 1$$

and suggests defining volume elements implicitly.

A volume element on $\mathbb{X}$ *is naturally defined implicitly by an integral over* $\mathbb{X}$. For example, Leibnitz might have had a better time defining implicitly dx by

$$\int_a^b dx = b - a$$

rather than defining dx by itself. In the context of functional integration it is sufficient to define $\mathcal{D}x$ by an implicit equation; we give two examples.

Example: Gaussian volume element

$$\int_{\mathbb{X}} \mathcal{D}_{s,Q,W} x \cdot \exp\left(-\frac{\pi}{s} Q(x) \right) \exp(-2\pi i \langle x', x \rangle) = \exp[-s\pi W(x')] \tag{IV.35}$$

with s positive or purely imaginary, $Q(x)$ a quadratic form on $\mathbb{X}$ positive if s is positive, real if s is purely imaginary, $W(x')$ a quadratic form on the dual $\mathbb{X}'$ of $\mathbb{X}$, inverse of Q in the following sense:

$$Q(x) = \langle Dx, x \rangle, \quad W(x') = \langle x', Gx' \rangle, \tag{IV.36}$$

$$DG = \mathbb{1}. \tag{IV.37}$$

Unless otherwise stated, we use s equal to 1 or i; another frequently used choice is $s \in \{2\pi, 2\pi i\}$.

D is often a differential operator [e.g., (IV.19)]; when restricted to $\mathbb{X}$ [e.g., $\mathbb{X}_{ab}$ in (IV.16)] it has usually a unique inverse; however, it is preferable to state explicitly W, even when determined implicitly by Q on $\mathbb{X}$. A complete specification of the volume element includes s, as well as Q and W. The specification of a volume element by formula (IV.35) is independent of the dimension of the domain of integration $\mathbb{X}$.

Example: Poisson paths

Another example is a volume element on the space $\mathbb{X}$ of Poisson paths (IV.31) and (IV.32). There a path $x \in \mathbb{X}_n$ is characterized by n jump times $\{t_1, \ldots, t_n\}$, and is interpreted as the sum $\delta_{t_1} + \delta_{t_2} + , \ldots, \delta_{t_n}$. The space $\mathbb{X}$ is the union of all $\mathbb{X}_n$.

Let a be a not necessarily real constant. Let $dv(t)$ be the dimensionless volume element on the time interval $\mathbf{T} = [t_a, t_b]$, $t_b - t_a = T$;

$$dv(t) = a\,dt, \quad \mathrm{vol}(\mathbf{T}) = aT, \tag{IV.38}$$

$$\mathrm{vol}(\mathbb{X}_n) = a^n T^n / n! \tag{IV.39}$$

$$\mathrm{vol}(\mathbb{X}) = \exp(\mathrm{vol}\,\mathbf{T}). \tag{IV.40}$$

We can even say

$$\mathbb{X} = \exp \mathbf{T} \tag{IV.41}$$

because addition of time intervals gives products of the corresponding spaces $\mathbb{X}$.

We can define the Fourier transform of a measure $\mathcal{D}_{a,T}x$ on $\mathbb{X}$ by

$$\int_{\mathbb{X}} \mathcal{D}_{a,T}x \cdot \exp(i\langle x,f\rangle) = \exp \int_{\mathbb{T}} dv(t) e^{if(t)} \tag{IV.42}$$

$$\text{Hint:} \langle x,f\rangle = f(t_1) + \cdots f(t_n). \qquad \blacksquare$$

If a is a real constant, the Poisson path can be described by a sequence of waiting times T_k between jumps: $\Pr(t_k \leqslant T_k \leqslant t_k + dt) = a \exp(-at_k)dt$. For the case where the decay rate a varies in time, see Ref. 49.

In general a volume element $\mathcal{D}_{\Theta,Z}$ on a Banach space $\mathbb{X}$ can be defined implicitly by

$$\int_{\mathbb{X}} \mathcal{D}_{\Theta,Z}x \cdot \Theta(x,x') = Z(x'), \tag{IV.43}$$

where $\Theta: \mathbb{X} \times \mathbb{X}' \to \mathbb{C}$ is continuous and bounded and $Z: \mathbb{X}' \to \mathbb{C}$ is continuous and bounded. Choosing a volume element is an art. In the integral

$$I = \int_{\mathbb{X}} \mathcal{D}_{\mathbb{X}}x \cdot F(x) \tag{IV.44}$$

one can choose first the volume element, then identify the functionals on $\mathbb{X}$ integrable with respect to $\mathcal{D}_{\mathbb{X}}x$, or one can choose a space $\mathcal{F}$ of functionals F, then identify $\mathcal{D}_{\mathbb{X}}x$ such that $I < \infty$.

Here are a few rules of thumb which can help in choosing a volume element:

- The probability requirements to be satisfied by a measure μ are not appropriate:

  $\mu \geqslant 0$ is irrelevant in quantum physics because quantum processes are expressed in terms of amplitudes and products of amplitudes. We have lost positive measures; we must go beyond Lebesgue integration.

  $\int_{\mathbb{X}} d\mu = 1$ is a normalization requirement which may be unwise to impose *a priori*. See, for instance, the normalization of (IV.67).

- A volume element must be dimensionless to be meaningful on an infinite-dimensional space. This requirement has dictated our normalizations of Gaussians and of Fourier transforms.
- Let $x: \mathbb{V} \to \mathbb{M}$, where $\mathbb{V}$ is not necessarily $\mathbf{T} \subset \mathbb{R}$. If $\mathbb{V}$ decomposes into subdomains

  $$\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2, \quad \mathbb{V}_1 \cap \mathbb{V}_2 = \varnothing$$

  and if x, x_1, x_2 are paths defined, respectively, on $\mathbb{V}, \mathbb{V}_1, \mathbb{V}_2$, then a desirable volume element should satisfy

  $$\mathcal{D}x = \mathcal{D}x_1 \mathcal{D}x_2.$$

  This is the substitute of independence, or Markov property, in probability.
- Invariance, or simple covariance, under action of some group, such as
  —diffeomorphism of $\mathbb{V}$ onto $\mathbb{V}$ (e.g., if $\mathbb{V} \equiv \mathbb{T}$, time reparametrization),
  —diffeomorphisms of $\mathbb{M}$ onto $\mathbb{M}$ (e.g., nonlinear σ-model),
  —any group action on $\mathbb{V}$ or $\mathbb{M}$,
  may help select an appropriate volume element.

- Usual integration techniques, such as integration by parts, change of variable of integration may suggest desirable volume elements. For instance, in a change of variable $y=\phi(x)$, one may require the volume element to be such that

$$\mathcal{D}y=\det(\delta\phi/\delta x)\mathcal{D}x \quad \text{for bosonic variables,} \tag{IV.45}$$

$$\mathcal{D}y=[\det(\delta\phi/\delta x)]^{-1}\mathcal{D}x \quad \text{for fermionic variables.} \tag{IV.46}$$

- If there is a term $\exp[-(\pi/s)Q(x)]$ in the integrand, then the Gaussian volume element (IV.35) is a natural choice.

Spaces of integrable functionals

The volume element $\mathcal{D}_{\Theta,Z}$ may be defined by (IV.43)

$$\int_{\mathbb{X}}\mathcal{D}_{\Theta,Z}x\cdot\Theta(x,x')=Z(x'). \tag{IV.47}$$

In quantum field theory, x is a field, x' is a source.

The space $\mathcal{F}_{\Theta,Z}$ of complex valued functionals F on $\mathbb{X}$ such that

$$F(x)=\int_{\mathbb{X}'}d\mu(x')\Theta(x,x'), \tag{IV.48}$$

where μ is a bounded measure on $\mathbb{X}'$, possibly complex, is a space of integrable functionals with respect to $\mathcal{D}_{\Theta,Z}$. This space is an easy generalization of the Albeverio–Hoegh–Krohn space.[50] Although (IV.48) does not necessarily define μ, it serves to prove the existence of

$$I:=\int_{\mathbb{X}}\mathcal{D}_{\Theta,Z}x\cdot F(x). \tag{IV.49}$$

Formally,

$$\int_{\mathbb{X}}\mathcal{D}_{\Theta,Z}x\cdot F(x)=\int_{\mathbb{X}}\mathcal{D}_{\Theta,Z}x\int_{\mathbb{X}'}d\mu(x')\Theta(x,x') \quad \text{by (IV.48)}$$

$$=\int_{\mathbb{X}'}d\mu(x')Z(x') \quad \text{by (IV.47).}$$

We note, moreover, that it is in practice not necessary to identify μ for computing (IV.49).

In integration theory, volume elements and spaces of integrable functions can be defined by integrals.

An A-type norm can be defined on $\mathcal{F}_{\Theta,Z}$ (like in Fourier analysis) by the equation

$$\|F\|_A:=\min_{\mu}\int_{\mathbb{X}'}|d\mu(x')\|Z(x')|. \tag{IV.50}$$

Other spaces of integrable functionals with respect to Gaussian volume elements include spaces of polynomials $F_n(x)=(\delta/\delta x')^n\exp(2\pi i\langle x',x\rangle)$, and the space L^2 ($\mathbb{X}$, Gaussian) of white noise. We shall come back to spaces of integrable functionals in Sec. IV C.

A general theorem

Definitions of domains of integration, volume elements, integrable functionals are only the scaffolding of functional integration; it serves to build theorems, applications, and give and take interactions with other branches of mathematics. Such constructions can be found in the Special

Issue of J. Math. Phys. on Functional Integration [Vol. **36**, No. 5, May (1995)]. Because of space and time limitations we mention only one theorem from our article in this issue, and only two of its many applications.

The goal of the theorem is to solve a generalized Schrödinger equation for functions $\Psi(T,x_0)$ when x_0 is on a manifold $\mathbb{M}^D$. The solution is a functional integral on the space $\mathcal{P}_0\mathbb{M}^D$ of pointed paths on $\mathbb{M}^D$ reduced, thanks to the theorem, to a functional integral on the space $\mathcal{P}_0\mathbb{R}^D$ of $L^{2,1}$ pointed paths[51] on $\mathbb{R}^D$ (IV.28) and (IV.29). The change of variable is provided by the map

$$P:\mathcal{P}_0\mathbb{R}^D\to\mathcal{P}_0\mathbb{M}^D, \tag{IV.51}$$

mapping z to x, where $x(t,z)$ is a function of t and a functional of z satisfying

$$dx(t,z)=X_{(\alpha)}(x(t,z))dz^\alpha+Y(x(t,z))dt,$$

$$x(t_0,z)=x_0, \quad z(t_0)=0. \tag{IV.52}$$

In general the vector fields $X_{(\alpha)},Y$ on $\mathbb{M}^D$ do not commute

$$[X_{(\alpha)},X_{(\beta)}]\neq 0, \quad [X_{(\alpha)},Y]\neq 0. \tag{IV.53}$$

The solution of (IV.52) is of the form

$$x(t,z)=x_0\cdot\Sigma(t,z). \tag{IV.54}$$

The transformation $\Sigma(t,z)$ on $\mathbb{M}^D$ can be expressed as a function of t and $z(t)$ only when the vector fields $X_{(\alpha)},Y$ commute.

Theorem

Let $\phi:\mathbb{M}^D\to\mathbb{R}$, let

$$Q_0(z)=\int_{\mathbb{T}} dt h_{\alpha\beta}\dot{z}^\alpha(t)\dot{z}^\beta(t), \tag{IV.55}$$

and $\mathcal{D}_{s,Q_0}$ be an abbreviation of $\mathcal{D}_{s,Q_0,W_0}$ defined by (IV.35). The functional integral

$$\Psi(T,x_0):=\int_{\mathcal{P}_0I\mathbb{R}^D}\mathcal{D}_{s,Q_0}(z)\cdot\exp\left(-\frac{\pi}{s}Q_0(z)\right)\phi(x_0\cdot\Sigma(t,z)) \tag{IV.56}$$

is a solution of

$$\frac{\partial\Psi}{\partial T}=\frac{s}{4\pi}h^{\alpha\beta}\mathcal{L}_{X_{(\alpha)}}\mathcal{L}_{X_{(\beta)}}\Psi+\mathcal{L}_Y\Psi,$$

$$\Psi(0,x_0)=\phi(x_0), \tag{IV.57}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X and $h^{\alpha\beta}h_{\beta\gamma}=\delta^\alpha_\gamma$.

Ψ and ϕ are functions on $\mathbb{M}^D$, but because the paths x on $\mathbb{M}^D$ are in a one-to-one correspondence with paths z on $\mathbb{R}^D$, the functional integral is an integral on $\mathcal{P}_0\mathbb{R}^D$.

Example (John LaChapelle): Paths in coordinates other than Cartesian

Let $z^1(t),z^2(t)$ be the Cartesian coordinates of $z(t)\in\mathbb{R}^2$, and $x^1(t)=r(t)$, $x^2(t)=\theta(t)$ be the polar coordinates of $x(t)\in\mathbb{R}^2\setminus\{0\}$. Equation (IV.52) is then

$$dx^i(t,z(t))=X^i_{(j)}[x(t,z(t))]dz^j(t) \quad i,j\in\{1,2\}, \tag{IV.58}$$

$$[X^i_{(j)}(x)]=\begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}$$

and the Lie derivatives with respect to $X_{(1)}$ and $X_{(2)}$ are

$$\begin{aligned} \mathcal{L}_{X_{(1)}} &= \cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \\ \mathcal{L}_{X_{(2)}} &= \sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}. \end{aligned} \tag{IV.59}$$

Define also $h^{\alpha\beta}=h_{\alpha\beta}=\delta_{\alpha\beta}$. According to the theorem, $\Psi(T,x_0)$ defined by (IV.56) is a solution of

$$\frac{\partial\Psi}{\partial T}=\frac{s}{4\pi}(\mathcal{L}^2_{X_{(1)}}+\mathcal{L}^2_{X_{(2)}})\Psi \tag{IV.60}$$

with

$$\mathcal{L}^2_{X_{(1)}}+\mathcal{L}^2_{X_{(2)}}=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}=\text{Laplacian in polar coordinates.}$$

This example is particularly simple because in the general expression

$$x(t)=x_0\cdot\Sigma(t,z)$$

$\Sigma(t,z)$ is not a functional of z but a function of $z(t)$. Because of this very simplicity, many publications have been, and still are, treating path integrals in non-Cartesian coordinates by relating $x(t)$ to $z(t)$.

In order to compute the point-to-point transition amplitude in polar coordinates $\langle t_b,r_b,\theta_b|t_a,r_a,\theta_a+2n\pi\rangle$ we set $x_0=(r_b,\theta_b)$ and

$$\phi(x(t_a,z))=\delta(r\cos\theta-r_a\cos\theta_a)\,\delta(r\sin\theta-r_a\sin\theta_a) \tag{IV.61}$$

$$=\frac{1}{r_a}\sum_{n\in\mathbb{Z}}\delta(r-r_a)\,\delta(\theta-\theta_\alpha+2n\pi). \tag{IV.62}$$

At this point we recognize that $\mathbb{R}^2\backslash\{0\}$ is multiply connected; by completing the above calculation, we obtain not the path integral representation of $\langle t_b,r_b,\theta_b|t_a,r_a,\theta_a+2n\pi\rangle$ but the path integral representation of

$$\frac{1}{r_a}\sum_{n\in\mathbb{Z}}\langle t_b,r_b,\theta_b|t_a,r_a,\theta_a+2n\pi\rangle, \tag{IV.63}$$

which can be shown to be equal to the point-to-point transition amplitude in Cartesian coordinates. For integrals over $\mathcal{P}_0\mathbb{M}^D$ when $\mathbb{M}^D$ is multiply connected, see Refs. 35 and 45. For extracting $\langle t_b,r_b,\theta_b|t_a,r_a,\theta_a+2n\pi\rangle$ from (IV.63), see Ref. 45.

Example: $\mathbb{M}^D$ is a Riemannian manifold

The general theorem (IV.56) and (IV.57) was formulated after we had studied the work of Eells and Elworthy.[52] They use the Cartan development map which maps $z\in\mathcal{P}_0\mathbb{R}^D$ into x

$\in \mathcal{P}_0\mathbb{M}^D$ as follows. Let $\mathbb{N}$ be the orthonormal frame bundle on a compact Riemannian manifold $\mathbb{M}^D$. The parametrization of $\mathcal{P}_0\mathbb{M}^D$ by $\mathcal{P}_0\mathbb{R}^D$ is the result of a composition of two parametrizations:

$$\begin{gathered}\mathcal{P}_0\mathbb{R}^D \leftrightarrows \mathcal{P}_0^H\mathbb{N} \leftrightarrows \mathcal{P}_0\mathbb{M}^D, \quad byz \leftrightarrows \rho \leftrightarrows x, \\ \text{the fixed point in } \mathcal{P}_0\mathbb{M}^D \text{ is } x(t_b)=x_b, \\ \text{the fixed point in } \mathcal{P}_0^H\mathbb{N} \text{ is } \rho(t_b)=\rho_b, \\ \text{the fixed point in } \mathcal{P}_0\mathbb{R}^D \text{ is } z(t_b)=0.\end{gathered} \tag{IV.64}$$

The space $\mathcal{P}_0^H\mathbb{N}$ consists of paths $\rho:\mathbf{T}\to\mathbb{N}$ with velocity $\dot{\rho}(t)$ in the horizontal tangent space $T^H_{\rho(t)}\mathbb{N}$ for every $t\in\mathbf{T}$. The projection $\pi:\mathbb{N}\to\mathbb{M}^D$ induces a bijection between $\mathcal{P}_0^H\mathbb{N}$ and $\mathcal{P}_0\mathbb{M}^D$. The parametrization of $\mathcal{P}_0^H\mathbb{N}$ by $\mathcal{P}_0\mathbb{R}^D$ is defined by

$$d\rho(t)=X_{(\alpha)}(\rho(t))dz^\alpha(t). \tag{IV.65}$$

The vector fields $X_{(\alpha)}$ are defined by the chosen connection on the frame bundle: the horizontal lift of $\dot{x}(t)$ by the connection σ is

$$\dot{\rho}(t)=\sigma(\rho(t))\dot{x}(t). \tag{IV.66}$$

Given the frame $\rho(t):\mathbb{R}^D\to T_{x(t)}\mathbb{M}^D$, and inserting $\rho(t)\circ[\rho(t)]^{-1}$ in (IV.66), one gets

$$\begin{aligned}\dot{\rho}(t)&=[\sigma(\rho(t))\circ\rho(t)][(\rho(t))^{-1}\dot{x}(t)] \\ &=X_{(\alpha)}(\rho(t))\dot{z}^\alpha(t).\end{aligned}$$

In Elworthy's work, we note that in (IV.65) ρ and z are stochastic processes, but here ρ and z are $L^{2,1}$ paths.

By definition the Cartan development map of z is $\mathrm{Dev}z=\pi\circ\rho$.

Using (IV.65) in the general theorem gives a parabolic equation on $\mathbb{N}$ whose projection on $\mathbb{M}^D$ is the parabolic equation with the Laplace–Beltrami operator.

C. An approach to functional integration

Path integration is a prototype for a functional integration in quantum field theory, but functional integration for systems with an infinite number of degrees of freedom is not simply a formal transcription of path integration. In this section we consider (Gaussian) functional integrals, and examine two problems which have no counterparts in Gaussian path integrals:

- Definition of the volume elements (effective actions).
- One-parameter family of scale dependent fields (renormalization).

Volume elements

The functional integration definition corresponding to (IV.43) can be written as

$$\int \mathcal{D}_\varphi \cdot \exp\left(\frac{i}{\hbar}S(\varphi)-i\langle J,\varphi\rangle\right)=\exp\left(\frac{i}{\hbar}W(J)\right)=Z(J)/Z(0), \tag{IV.67}$$

where φ is either a self-interacting field, or a collection of interacting fields. But the generating functional $Z(J)$ is difficult to ascertain *a priori* for the following reason. Let $\Gamma(\overline{\varphi})$ be the Legendre transform of $W(J)$:

$$\hbar\overline{\varphi}:=\frac{\delta W(J)}{\delta J}, \quad \Gamma(\overline{\varphi}):=W(J(\overline{\varphi}))-\hbar\langle J(\overline{\varphi}),\overline{\varphi}\rangle. \tag{IV.68}$$

Then $\Gamma(\overline{\varphi})$ is the inverse of $W(J)$ in the same sense as Q and W are inverse of each other [(IV.36)], but $\Gamma(\overline{\varphi})$ is the *effective action* which has to be used for computing observables. If $S(\varphi)$ is quadratic, the *bare action* $S(\varphi)$ and the effective action $\Gamma(\varphi)$ are identical, the fields do not interact. But in the case of interacting fields, the exact relation between bare and effective action is the main difficulty embodied in the model.

In the following we break up the action S into a quadratic form Q and a remainder $U=S-Q$. We define $\mathcal{D}_Q\varphi$ as in Eq. (IV.35) by

$$\frac{\delta^2 W(J)}{\delta J_a \delta J_b} \quad \text{and} \quad \frac{\delta^2 \Gamma(\overline{\varphi})}{\delta \overline{\varphi}^c \delta \overline{\varphi}^d}, \tag{IV.69}$$

but $\Gamma(\overline{\varphi})$ is the *effective action* which has to be used for computing observables, and cannot be chosen arbitrarily.

$$\int_{\mathbb{X}} \mathcal{D}_Q\varphi \cdot \exp\left(-\frac{\pi}{s}Q(\varphi)\right)\exp(-2\pi i\langle J,\varphi\rangle) := \exp[-\pi s W(J)], \tag{IV.70}$$

or more conveniently $d\mu_G$ by

$$\int_{\mathbb{X}} d\mu_G(\varphi)\exp(-2\pi i\langle J,\varphi\rangle) := \exp[-\pi s W(J)]. \tag{IV.71}$$

The integral $I=\int_{\mathbb{X}}\mathcal{D}_Q\varphi\cdot\exp[(i/\hbar)S(\varphi)]$ becomes (under the substitution $s=\pi\hbar i$)

$$I=\int_{\mathbb{X}} d\mu_G(\varphi)\exp\left(\frac{i}{\hbar}U(\varphi)\right)\equiv\left\langle \mu_G, \exp\frac{i}{\hbar}U\right\rangle. \tag{IV.72}$$

Breaking up the action is not always desirable, but we shall learn interesting properties of functional integration from (IV.71). The covariance G is defined by

$$W(J)=\langle J,GJ\rangle; \tag{IV.73}$$

it is the inverse of the operator D defined by

$$Q(\varphi)=\langle D\varphi,\varphi\rangle; \tag{IV.74}$$

it is also the two-point function

$$\frac{s}{2\pi}G(x,y)=\int_{\mathbb{X}} d\mu_G(\varphi)\varphi(x)\varphi(y). \tag{IV.75}$$

We shall contrast covariances in quantum mechanics and quantum field theory on two simple examples:

Qm Let $D=-\dfrac{d^2}{dt^2}$; its inverse on the space $\mathbb{X}_{ab}$ of paths with two fixed end points is

$$G(t,s)=\theta(s-t)(t-t_a)(t_b-s)\frac{1}{T}+\theta(t-s)(t_b-t)(s-t_a)\frac{1}{T}$$

with $T:=t_b-t_a$. (IV.76)

QFT Let $D=-\Delta$ on $\mathbb{R}^D$; then

$$G(x,y)=\frac{C_D}{|x-y|^{D-2}}, \tag{IV.77}$$

$$\text{with a constant } C_D \text{ equal to } \Gamma\left(\frac{D}{2}-1\right)/4\pi^{D/2}. \tag{IV.78}$$

Notice that $G(t,s)$ is a continuous function, $G(x,y)$ is singular at the origin for Euclidean fields, and singular on the lightcone for Minkowskian fields. However, we note that the quantity of interest is not the covariance G but the variance W:

$$W(J)=\langle J,GJ\rangle,$$

which is singular only if J is a pointlike source $\langle J,\varphi\rangle=\varphi(x)$.

Scale dependent covariances and fields

The beauty of covariances is that they can be decomposed into scale dependent contributions, and the decomposition of the covariance carries with it a field decomposition. A field is then expressed as an integral over a scaling variable $l\in[0,\infty[$ of scale dependent fields. A scale-parameter family of fields provides a simple formulation of renormalization. We follow here the work of D. C. Brydges, J. Dimock, and T. R. Hurd.[53]

Notation: $\hbar=c=1$; physical dimensions are physical length dimensions. Physical scaling:

$$S_l u(x):=l^{[u]}u(x/l),\quad [u]:=\text{physical dimension of } u. \tag{IV.79}$$

Multiplicative differentials:

$$\begin{aligned} d^{\times}l&=dl/l,\\ \partial^{\times}/\partial l&=l\partial/\partial l. \end{aligned} \tag{IV.80}$$

Objects defined by a covariance:

$$\text{functional Laplacian}\quad \Delta_G:=\frac{s}{2\pi}\int_{\mathbb{R}^D}dx\int_{\mathbb{R}^D}dy\,G(x,y)\,\frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)},\quad s\in\{1,i\}, \tag{IV.81}$$

$$\text{convolution}\quad (\mu_G * F)(\varphi):=\int_{\mathbb{X}}d\mu_G(\psi)F(\varphi+\psi), \tag{IV.82}$$

$$\text{hence}\quad \mu_G * F=\exp(\tfrac{1}{2}\Delta_G)F, \tag{IV.83}$$

$$\text{Bargmann–Segal transform}\quad B_G:=\mu_G *=\exp(\tfrac{1}{2}\Delta_G), \tag{IV.84}$$

$$\text{Wick transform}\quad :\ :_G:=\exp(-\tfrac{1}{2}\Delta_G). \tag{IV.85}$$

A covariance can be written, in Minkowski or Euclidean space as

$$G(x,y)=\int_0^{\infty}d^{\times}l\cdot S_l u((x-y)^2), \tag{IV.86}$$

where $[u]=[G]=2-D$. If $G(x,y)$ is given by (IV.77), then the only requirement on u is

$$\int_0^{\infty}d^{\times}k\cdot k^{-[u]}u(k^2)=C_D \tag{IV.87}$$

in the Euclidean case, and moreover $u(-k^2)=i^{D-2}u(k^2)$ in the Minkowski case. The domain of integration $[0, \infty[$ of the scaling variable can be broken up into a union of subdomains,

$$[0,\infty[=\bigcup_{-\infty}^{\infty}[2^j l_0,2^{j+1}l_0[, \tag{IV.88}$$

which expresses the possibility of separating different scale contributions. The corresponding decompositions of covariance and field is

$$G=\sum_{j=-\infty}^{+\infty} G_{[2^j l_0,2^{j+1}l_0[}, \tag{IV.89}$$

$$\varphi=\sum_{j=-\infty}^{+\infty} \varphi_{[2^j l_0,2^{j+1}l_0[}. \tag{IV.90}$$

Henceforth the suffix G in $\mu_G,\Delta_G,B_G,:\ :_G$ is replaced by the interval defining the scale dependent covariance; for example

$$\mu_{[l_0\infty[}=\mu_{[l_0,l[}*\mu_{[l,\infty[}. \tag{IV.91}$$

The goal is to compute (IV.72), which we can write

$$\lim_{l_0=0} I(l_0)=\lim_{l_0=0}\langle \mu_{[l_0,\infty[},Z\rangle, \quad Z \text{ is an abbreviation for } \exp\frac{i}{\hbar}U(\varphi). \tag{IV.92}$$

With the decomposition (IV.91)

$$I(l_0)=\langle \mu_{[l,\infty[},\mu_{[l_0,l[}*Z\rangle. \tag{IV.93}$$

The convolution $\mu_{[l_0,l[}*Z$ integrates out the contributions of the scale dependent fields in the range $[l_0,l[$; it substitutes to Z an l-dependent effective integrand.

An important operator P_l introduced by Brydges provides the construction of a parabolic equation in l satisfied by an effective integrand P_lZ:

$$P_l:=S_{l/l_0}B_{[l_0,l[}, \tag{IV.94}$$

with S the physical scaling (IV.79) and B the Bargmann transform (IV.84). The Brydges operator (IV.94) rescales the Bargmann–Segal transform so that all integrals are performed with an *l-independent* Gaussian. Indeed, it can be shown that

$$I(l_0)=\langle \mu_{[l_0,\infty[},Z\rangle=\langle \mu_{[l_0,\infty[},P_lZ\rangle. \tag{IV.95}$$

The scaling evolution equation satisfied by P_lZ is readily obtained from the definition of P_l

$$\frac{\partial^\times}{\partial l}P_lZ=(\dot S+\tfrac{1}{2}\dot\Delta)(P_lZ), \tag{IV.96}$$

where

$$\dot S=\frac{\partial^\times}{\partial l}\bigg|_{l=l_0} S_{l/l_0} \quad \text{and} \quad \dot\Delta=\frac{\partial^\times}{\partial l}\bigg|_{l=l_0}\Delta_{[l_0,l[}. \tag{IV.97}$$

The evolution equation (IV.96) determines the renormalization group flow equation.

To a given space $\mathcal{F}(\mathbb{X})$ of functionals F on $\mathbb{X}$, integrable with respect to a Gaussian μ_G, the pair of mappings (Bargmann, Wick) associates a corresponding Fock space Fock $(\mathbb{X},G)$. There are several examples of the pattern

$$\mathrm{Fock}(\mathbb{X},G) \underset{:\ :_G}{\overset{B_G}{\underset{\rightarrow}{\leftarrow}}} \mathcal{F}(\mathbb{X}).$$

The largest Fock space consists of all entire functions on the complexified $\mathbb{X}_{\mathbb{C}}$. The corresponding $\mathcal{F}(\mathbb{X})$ consists of suitable distributions on $\mathbb{X}$.

V. FEYNMAN, DEWITT, AND CARTIER AND DEWITT-MORETTE COMPARED

How does the axiomatic scheme presented in Sec. IV account for the Feynman definition and the Bryce DeWitt formalism?

(1) The Feynman definition is obtained by discretization of the time interval $\mathbf{T}=[t_a,t_b]$, and a linear map (IV.7):

$$P:\mathbb{X}\rightarrow\mathbb{R}^n \text{by} x\mapsto u=\{\langle \delta_{t_j},x\rangle\}_j. \tag{V.1}$$

By projection in $\mathbb{R}^n$, one obtains the short time propagator

$$\langle \beta,t_{j+1}|\alpha,t_j\rangle.$$

Examples:

- The first short time propagator proposed[21] for the action $S(x)=\int_{\mathrm{T}}dt[(m/2)(\dot{x}(t))^2-V(x(t))]$ was

$$K(j+1,j)=\left(\frac{m}{2\pi i\hbar}\right)^{D/2}|\mathcal{D}(j+1,j)|^{1/2}\exp\left(\frac{i}{\hbar}S(j+1,j)\right), \tag{V.2}$$

  where $S(j+1,j)$ is the value of the action S computed along a linear path from $x(t_j)$ to $x(t_{j+1})$, and $\mathcal{D}(j+1,j)$ is the Van Vleck determinant for the action $S(j+1,j)$, that is

$$\mathcal{D}(j+1,j)=(\Delta_j t)^{-D}\det[\partial^2 S(j+1,j)/\partial x_{j+1}\partial x_j], \tag{V.3}$$

  with $\Delta_j t=t_{j+1}-t_j$.

The method used for computing (V.2) was not the projection P because in 1950 there was no definition of Feynman path integrals on function spaces $\mathbb{X}$. The condition used for computing (V.2) was a unitarity condition which could not give more than the absolute value of the short time propagator. Unfortunately (V.2) was used without the absolute value sign in the determinant, and Pauli remarked that if $x(t)\in\mathbb{M}^D$ ($\mathbb{M}^D$ Riemannian), Feynman path integral with

$$K(j+1,j)=\left(\frac{m}{2\pi i\hbar}\right)^{D/2}\mathcal{D}(j+1,j)^{1/2}\exp\left(\frac{i}{\hbar}S(j+1,j)\right) \tag{V.4}$$

is a solution of the Schrödinger equation modified by a term proportional to the scalar curvature R of the Riemannian manifold $\mathbb{M}^D$, namely

$$i\hbar\,\partial\psi/\partial t=\left(H+\alpha\frac{\hbar^2}{m}R\right)\psi; \tag{V.5}$$

α is a numerical constant whose value depends on the chosen short time propagator $K(j+1,j)$. Several proposals (see Refs. 54 and 55) have been made for the short time propagator when $x(t)\in\mathbb{M}^D$.

The short time propagator obtained from the general theorem (IV.56) and (IV.57) applied to the example (IV.65) of paths taking their values in $\mathbb{M}^D$ is

$$K(j+1,j)=\left(\frac{m}{2\pi i\hbar}\Delta_j t\right)^{D/2}\mathcal{D}(j+1,j)\exp\left(\frac{i}{\hbar}S(j+1,j)\right), \tag{V.6}$$

where $S(j+1,j)$ is evaluated along a geodesic, and where the Van Vleck determinant (V.3) is now multiplied by $g^{-1/2}(x_{j+1})g^{-1/2}(x_j)$, where g is the absolute value of the metric determinant. We note that both $K(j+1,j)$ given by (V.2) and $K(j+1,j)$ given by (V.6) are proportional to $(\Delta_j t)^{-D/2}$ because $\mathcal{D}(j+1,j)$ carries a different exponent in these two formulas. If computed with the short time propagator (V.6), the Schrödinger equation has no additional term proportional to $R(\alpha=0)$, as expected from (IV.57).

(2) In order to compare as simply as possible the functional integrals in Secs. III and IV, set

$$I^B=\int_\Phi [d\varphi]\mu[\varphi]\exp i(S[\varphi]), \quad B \text{ for B. DeWitt} \tag{V.7}$$

$$=\int_\Phi [d\varphi]\exp i(S[\varphi]-i\ln\mu[\varphi]) \tag{V.8}$$

and

$$I^C=\int_{\mathbb{X}}\mathcal{D}_\varphi\cdot\exp\frac{i}{\hbar}S(\varphi), \quad C \text{ for Cartier and DeWitt-Morette.} \tag{V.9}$$

The domain of integration Φ of I^B as it stands is the limit of $\mathbb{R}^n$ when $n=\infty$ [see (V.11)]. If I^B is the solution $\langle \text{out}|\text{in}\rangle$ of

$$\delta\langle \text{out}|\text{in}\rangle = i\langle \text{out}|\langle \delta J,\varphi\rangle|\text{in}\rangle \tag{V.10}$$

then, in general, I^B includes ordinary integrals over two sets of parameters α and β associated with the ''in'' and ''out'' regions respectively, and the action $S[\varphi]$ includes an additional term $J\varphi$. By definition

$$\langle d\varphi\rangle := \prod_{x,\alpha} d\varphi^\alpha(x). \tag{V.11}$$

The so-called ''measure functional'' $\mu[\varphi]$ has been obtained by requiring I^B to be a solution of (V.10), and has been found proportional to the inverse square root of the determinant of the advanced Green function G^+ of the Jacobi operator of the action functional $S[\varphi]$

$$\mu(\varphi)\approx|(\text{super})\det G^+[\varphi]|^{-1/2}. \tag{V.12}$$

The domain of integration $\mathbb{X}$ in I^C is defined by the range, the domain, and the analytic properties of the variable of integration φ. If I^C represents a matrix element $\langle A|\text{operator}|B\rangle$, the domain $\mathbb{X}_{AB}\subset\mathbb{X}$ is restricted by the states A and B. The volume element $\mathcal{D}\varphi$ is defined in general by (IV.43). In the example used for comparing I^B and I^C, the volume element $\mathcal{D}_{Q_0}\varphi$ is defined by

$$\int_{\mathbb{X}}\mathcal{D}_{Q_0}\varphi\cdot\exp[i\pi Q_0(\varphi)]\exp(-2\pi i\langle J,\varphi\rangle)=\exp[-i\pi W_0(J)] \tag{V.13}$$

and the action S is set equal to $Q_0+(S-Q_0)$:

$$S=Q_0+(S-Q_0)=:Q_0+U. \tag{V.14}$$

Both formalisms have been developed along different research plans. I^B has been developed for use in Quantum Field Theory. I^C has been worked out, so far, primarily for path integration in order to provide simple or robust tools useful for a great variety of problems.

In comparing I^B and I^C, we note that the actions in (V.8) and (V.9) are different. Different actions give different Schrödinger equation. From I^B one obtains

$$i\hbar\,\partial\psi/\partial t=\left(H+\frac{1}{8m}\hbar^2R\right)\psi,$$

the term $(1/8m)\hbar^2R$ is absent if one works with I^C.

For comparing the use of $\mathcal{D}\varphi$ defined by (V.13) and the use of $\mu[\varphi]$ given by (V.12), we shall work out the simplest possible example, namely, the WKB approximation of the point-to-point amplitude $\langle b,t_b|a,t_a\rangle$ for $a,b\in\mathbb{R}^D$. Let

$$x:\mathbf{T}=[t_a,t_b]\to\mathbb{R}^D, \tag{V.15}$$

$$S(x)=\int_{\mathrm{T}}dt\left(\frac{m}{2}|\dot{x}(t)|^2-V(x(t))\right). \tag{V.16}$$

The computation of $I:=\langle b,t_b|a,t_a\rangle$ using I^B can be found in Ref. 56, its computation using I^C can be found in Ref. 44. I (no B, no C) will be found to be the same in both cases.

(i) Computing I^C with the techniques of Sec. IV

$$I^C=\langle b,t_b|a,t_a\rangle=\int_{\mathbb{X}_b}\mathcal{D}x\cdot\exp\left(\frac{i}{\hbar}S(x)\right)\delta(x(t_a)-a). \tag{V.17}$$

$\mathbb{X}_b$ is the space of pointed paths $x(t_b)=b$. The fixed point is chosen to be the end point so that the functional integral represents a wave function $\Psi(t_b,b)$ rather than a ''backward'' wave function $\Psi(t_b,a)$.

According to the general theorem (IV.56) the integral I^C can be restated as an integral over the space Z_b of pointed paths with $z(t_b)=0$. In this nearly trivial case (IV.52) reduces to

$$\begin{aligned} dx(t)&=\lambda\,dz(t),\\ x(t,z)&=b+\lambda z(t). \end{aligned} \tag{V.18}$$

We set $\lambda=(2\pi\hbar/m)^{1/2}$ so that Q_0 in (V.22) is dimensionless.

Let $z\mapsto\xi$ be a reparametrization in Z_b defined by

$$x(t,\xi)=x_{cl}+\lambda\xi(t),\quad x_{cl}\ \text{is the classical path},\ x_{cl}(t_a)=a,x_{cl}(t_b)=b. \tag{V.19}$$

It is simply a translation in the path space Z_b, hence $\mathcal{D}z=\mathcal{D}\xi$ by a fundamental property of our $\mathcal{D}x$.

According to the general theorem, we get therefore

$$I=\int_{Z_b}\mathcal{D}\xi\cdot\exp\left(\frac{i}{\hbar}S(x_{cl}+\lambda\xi)\right)\delta(\lambda\xi(t_a)). \tag{V.20}$$

The Taylor expansion of the action functional is

$$\frac{i}{\hbar}S(x_{cl}+\lambda\xi)=\frac{i}{\hbar}S(x_{cl})+i\pi(Q_0(\xi)+Q_V(\xi))+O(\hbar^{1/2}), \tag{V.21}$$

where

$$Q_0(\xi)=\int_{\mathrm{T}} dt|\dot{\xi}(t)|^2 \tag{V.22}$$

and the parameter λ has been chosen so that $Q_0(\xi)$ is dimensionless. Moreover

$$Q_V(\xi)=-\frac{1}{m}\int_{\mathrm{T}} dt\partial_\alpha\partial_\beta V(x_{cl}(t))\xi^\alpha(t)\xi^\beta(t). \tag{V.23}$$

The volume element $\mathcal{D}\xi=\mathcal{D}_{Q_6}\xi$ on Z_b is characterized by the invariance under translation and the normalization

$$\int_{Z_b}\mathcal{D}_{Q_0}\xi\cdot\exp[i\pi Q_0(\xi)]=1. \tag{V.24}$$

The computation of the WKB approximation I_{WKB} of I reduces to the computation of

$$I_V:=\int_{Z_b}\mathcal{D}_{Q_0}\xi\cdot\exp[i\pi(Q_0(\xi)+Q_V(\xi))]\delta(\lambda\xi(t_a)). \tag{V.25}$$

Three easy steps conclude the calculation of I_v:

- Changing by a linear map the volume element $\mathcal{D}_{Q_0}$ into $\mathcal{D}_{Q_0+Q_V}$
$$I_V=I_1I_2 \tag{V.26}$$
with
$$I_1=\left|\det\frac{Q_0}{Q_0+Q_V}\right|^{1/2} i^{-Ind(Q_0+Q_V)/2}, \tag{V.27}$$
$$I_2=\int_{Z_b}\mathcal{D}_{Q_0+Q_V}\xi\cdot\exp[i\pi(Q_0(\xi)+Q_V(\xi))]\delta(\lambda\xi(t_a)). \tag{V.28}$$
- Another linear map $\xi\mapsto\xi(t_a)$ gives I_2 as an integral over $\mathbb{R}^D$, i.e., I_2 as the square root of a finite-dimensional determinant.
- The ratio of infinite-dimensional determinants in I_1 can be expressed as a finite-dimensional determinant.[57,58]
Gathering the terms which contribute to the WKB approximation of (V.20) one gets
$$I_{\mathrm{WKB}}=e^{\pi i(p-q)/4}(2\pi\hbar)^{-D/2}\left|\det\frac{\partial^2\mathcal{S}(a,b)}{\partial a^\alpha\partial b^\beta}\right|^{1/2}\exp\left(\frac{i}{\hbar}\mathcal{S}(a,b)\right), \tag{V.29}$$
where S is the action function equal to $S(x_{cl})$, and p and q are the number of positive and negative eigenvalues of the Van Vleck–Morette matrix $\partial^2S/\partial a^\alpha\partial b^\beta$.

(ii) Computing I^B with the techniques of Sec. III

Superficially, computing I^B is very similar to computing I^C, expansion of the action S around $S(x_{cl})$, Gaussian integrations of quadratic terms, ratio of finite-dimensional determinants equal to finite-dimensional determinants. The differences, small, when computing $\langle b,t_b|a,t_a\rangle$, are nevertheless useful for comparing I^B and I^C. Equation (V.19) has become

$$x(t)=x_{cl}(t)+\xi(t), \tag{V.30}$$

that is x is no longer identified as a functional of ξ, and $\lambda=1$ hides the fact that WKB is an expansion in $\hbar^{1/2}$. The change of variable of integration from x to ξ is admittedly trivial in this case. In I^B, $G^+(x_{cl}+\xi)$ and $S(x_{cl}+\xi)$ are expanded around their values at x_{cl}. The only integral to be computed for the WKB approximation I_{WKB} of $\langle b,t_b|a,t_a\rangle$ is

$$\int \mathcal{D}\xi \cdot \exp\left(\frac{i}{2}S''(x_{cl})\cdot\xi\xi\right)=\mathrm{const}(\det G)^{1/2},$$

where G is the Green function of the Jacobi operator (IV.19) and (IV.20), $\mathcal{J}(x_{cl})$ for $x_{cl}(t_a)=a$, $x_{cl}(t_b)=b$. The constant is determined independently from the combination law of probability amplitudes. The Green function G is said to be boundary adapted.[59] Gathering all the terms contributing to I_{WKB}, one obtains

$$I_{\mathrm{WKB}}=\mathrm{const}\det\left(\frac{G}{G^+}\right)^{1/2}\exp iS(x_{cl}). \tag{V.31}$$

The ratio of infinite-dimensional determinants can also be expressed as a finite-dimensional determinant and (V.31) is found identical to (V.29) where the constant has been obtained explicitly.

(iii) Comparing I^B and I^C

In the WKB approximation of a very simple example I^B and I^C give the same result; on the other hand, the Schrödinger operators for a wave function on a Riemannian manifold differ by $(1/8m)\hbar^2R$.

In the definition (V.7) the term $\mu[\varphi]$ is ubiquitous in Quantum Field Theory and justifies many expected results derived without concern for undefined terms. The action $S[\varphi]$ stays as a whole.

The example (V.18) and (V.29) is a particular application of the general theorem (IV.56) and (IV.57) Equation (V.18) is a very trivial case of (IV.52). The restriction of Z_b to paths with both ends fixed is achieved by replacing $\phi(x_0\cdot\Sigma(t,z))$ in (IV.56) by $\delta(\lambda\xi(t_a))$ in (V.20). The power of the general theorem lies in its focus on the domain of integration and in the variety of its great many applications.

VI. CONCLUSION: FEYNMAN, THE MODERN LEIBNITZ

Functional integration is still really in its infancy. To make a comparison, let us recall the main steps in the development of ordinary calculus. It all started in the 17th century with explicit solutions to old geometrical problems: lengths, areas, volumes of various geometrical objects. There were basically two methods at hand:

- A rigorous, but very cumbersome one, in the hands of people like Fermat and Pascal. Only geniuses could handle it.
- A more intuitive, but much less rigorous one, the ''indivisibles'' of Cavalieri.

With the help of these two methods, a large body of knowledge was developed (Huygens, Roberval,...).

At the end of the 17th century, Leibnitz invented a beautiful algorithm, the ''*dx*'' notation. Using this algorithm, many of the previously difficult results were reduced to a rather simple algebraic manipulation, and many more results were found. For the practitioner of mathematics, who is not interested in logical foundations, this step is still today all one needs. This is not the place to compare the contributions of Leibnitz and Newton, but together they provided the background for a century of astonishing discoveries.

The third period, to be dubbed ''the critical one,'' began with Cauchy around 1820 and lasted well into the middle of the 20th century. Step by step, all the logical questions connected with the

basic notions were raised and solved: nature of numbers and functions, continuity, derivability, integrability,.... The modern ''Functional Analysis'' with its norms, Hilbert spaces, Banach spaces, is the crowning of this period.

The last word has not been said. For example, Non-standard Analysis has not yet been fully exploited.

The three periods ''pre-Leibnitz, Leibnitz and post-Leibnitz'' have their counterparts in the development of functional integration in Quantum Physics. Before Feynman introduced path integration and its offspring, the diagram technique, computations of Quantum Field Theory processes were arduous and cumbersome, particularly when plagued by infinities—and best left in the hands of (near) geniuses.

Feynman—the modern Leibnitz—created the algorithm symbolized by his $\int \mathcal{D}q\ldots$ symbol. From there a very useful technology was developed. More and more often, it is realized that functional integration provides the shortest route for solutions. Its very formulation encodes simultaneously the dynamics of the system and the boundary conditions which characterize the system. Not in-frequently the functional integration provides, or at least suggests, solutions to unsolved problems. Just like the Leibnitz' formulation is all what a practitioner needs, Feynman's formulation is sufficient for many purposes. See for instance the classic book of L. S. Schulman[60] and more recently the Table of Integrals compiled by Grosche and Steiner.[61]

We are now entering a period similar to the post-Leibnitz' period. Several explorations, particularly in the areas of the topology of low-dimensional manifolds, quantum gauge field theory, and renormalization, have given new results which have been later on justified by other methods. We quote only three papers[62–64] whose references lead to other investigations. Such explorations create a fruitful environment for the next period—a period similar to the ''critical period'' ushered in by Cauchy–Weierstrass–Lebesgue during which ordinary integration received its complete logical foundation. And so far ''It works as advertised.[62],,

Note added. Quantum Fields and Strings: A Course for Mathematicians, Vols. 1 and 2, edited by P. Deligne *et al*. (American Mathematical Society, Institute for Advanced Study, 1999) contains fifteen hundred pages, not devoted to functional integration, but with functional integrals throughout.

[1] We have not been able to locate the reference to this well known sentence.
[2] J. C. Baez, I. E. Segal, and Z. Zhou, *Introduction to Algebraic and Constructive Quantum Field Theory* (Princeton University Press, Princeton, NJ, 1992).
[3] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed. (Springer-Verlag, New York, 1987).
[4] B. Simon, *Functional Integration and Quantum Physics* (Academic, New York, 1979).
[5] S. Bochner, *Harmonic Analysis and the Theory of Probability* (University of California Press, Berkeley, 1960).
[6] I. E. Segal, ''Distributions in Hilbert space and canonical systems of operators,'' Trans. Am. Math. Soc. **88**, 12–42 (1958).
[7] P. Malliavin, *Stochastic Analysis* (Springer-Verlag, Berlin, 1997).
[8] T. Hida, H.-H. Kuo, J. Potthof, and L. Streit, *White Noise, An Infinite Dimensional Calculus* (Kluwer Academic, Dordrecht, 1993).
[9] Section II is based on the article ''Functional integration; a semi-historical perspective,'' by one of us (C. DeW.) in *Symposia Gaussiana*, Conf. A, edited by Behara, Fritsch, and Lintz (Walter de Gruyter, Berlin, 1995).
[10] R. P. Feynman, ''The Principle of Least Action in Quantum Mechanics,'' Princeton University Publication No. 2948, p. 73 (1942).
[11] See Ref. 10, Eq. (24).
[12] See Ref. 10, p. 72.
[13] R. P. Feynman, ''The development of the space-time view of quantum electrodynamics,'' Phys. Today No. 8, 31–44 (1966).
[14] Path integration is referred to as ''she'' not because integration is feminine in French but because Feynman, when accepting the Nobel Prize, referred to it as a lady: ''So what happened to the old theory that I fell in love with as a youth? Well, I would say it's become an old lady who has very little attractive left in her, and the young today will not have their hearts pound when they look at her anymore. But, we can say the best we can for any old woman, that she has been a very good mother and has given birth to some very good children. And I thank the Swedish Academy of Sciences for complimenting one of them. Thank you.'' (Feynman, 1966, p. 44).
[15] According to subsequent accounts, only four or five physicists appreciated path integration from the very beginning. See, for instance, Ref. 13; Dyson, 1989; and Mehra, 1988.
[16] F. J. Dyson, ''Feynman at Cornell,'' Phys. Today **42** (2), 32–38 (1989).

[17] J. Mehra, ''My Last Encounter with Richard P. Feynman,'' memorial talk delivered on 24 February 1988, Cornell University.
[18] F. J. Dyson, ''The radiation theories of Tomonaga, Schwinger, and Feynman,'' Phys. Rev. **75**, 486–502 (1949).
[19] J. Gleick, *Genius* (New York, 1992) pp. 269–270. (See the interoffice memorandum from Dyson to J. Robert Oppenheimer in the early Fall of 1948 and the interoffice reply a couple of months later ''*Nolo contendre*. R.O.'')
[20] The semiclassical approximation of a path integral (Morette, 1951) had been constructed but had not yet proved its usefulness.
[21] C. Morette, ''On the definition and approximation of Feynman's path integral,'' Phys. Rev. **81**, 848–852 (1951).
[22] See, for instance, the state of the art in G. t'Hooft and M. Veltman, ''Diagrammar,'' CERN preprint 78–9, 1–114 (1973), subsequently expanded into M. Veltman, *Diagrammatica: the path to Feynman rules* (Cambridge University Press, Cambridge, 1994).
[23] This can also be written as $\int_T^t\{\dot{q}^2/2C+W(q)\}d\tau$. Hence the Hamiltonian and not the Lagrangian!
[24] M. Kac, *Probability and Related Topics in Physical Sciences Vol. I*, 167 (Interscience, New York, 1957–1959).
[25] Feynman, Letter to Cécile DeWitt-Morette, 10 December 1971.
[26] F. J. Dyson, ''Missed opportunities,'' Bull. Am. Math. Soc. **78** (5), 635–652 (1972).
[27] See for instance Cameron, 1960, p. 126; but see also Cameron and Storvick, 1980.
[28] R. H. Cameron, ''A family of integrals serving to connect the Wiener and Feynman integrals,'' J. Math. Phys. **39**, 126–140 (1960).
[29] R. H. Cameron and D. A. Storvick, *Some Banach Algebras of Analytic Feynman Integrable Functionals*, Springer-Verlag Lecture Notes in Mathematics, Vol. 798 (Springer-Verlag, Berlin, 1980).
[30] E.g., references to sum over histories sprinkle J. A. Wheeler's discussions at the 1957 Chapel Hill conference as a possible approach to quantum gravity. Recorded in ''On the Role of Gravitation in Physics,'' Chapel Hill conference, Astia Document No. AD 11 81 80, Wright Air Development Center (1957).
[31] B. S. DeWitt, ''Dynamical Theory of Groups and Fields,'' in *Relativity Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1963–1964), pp. 585–820, developed further in Ref. 32.
[32] B. S. DeWitt, ''The Spacetime Approach to Quantum Field Theory,'' in *Relativity, Groups and Topology II*, edited by B. DeWitt and R. Stora (North-Holland, Amsterdam, 1983–1984), pp. 382–738.
[33] J. Ginibre, ''Some Applications of Functional Integration in Statistical Mechanics,'' in *Statistical Mechanics and Quantum Field Theory*, edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971), pp. 327–427.
[34] *Functional Integration, Basics and Applications*, edited by C. DeWitt-Morette, P. Cartier, and A. Folacci (Plenum, New York, 1997).
[35] M. G. G. Laidlaw and C. DeWitt-Morette, ''Feynman functional integrals for systems of indistinguishable particles,'' Phys. Rev. D **3**, 1375–1378 (1971).
[36] L. Alvarez-Gaumé, ''Supersymmetry and Index Theorem,'' *Supersymmetry, Proceedings of the 1984 NATO School at Bonn*, edited by K. Dietz, R. Flume, G. V. Gehlen, and V. Rittenberg (1984).
[37] E. Witten, ''Quantum field theory and the Jones polynomial,'' Commun. Math. Phys. **121**, 351–399 (1989).
[38] For the convenience of the reader interested in studying B. DeWitt's work, we introduce in this section the author's condensed notation, including the different labelling for right and left derivatives as required for fermion fields [see, e.g., Eqs. (III.8)].
[39] A. Mostafazadeh, ''Supersymmetry and the Atiyah–Singer index theorem. I. Peierls Brackets, Green's functions, and a proof of the Index theorem via Gaussian superdeterminants,'' J. Math. Phys. **35**, 1095–1124 (1994); **35**, 1125–1138 (1994). (The scalar curvature factor in the Schrödinger equation.)
[40] C. Morette DeWitt, ''Feynman's path integral; definition without limiting procedure,'' Commun. Math. Phys. **28**, 47–67 (1972).
[41] C. DeWitt-Morette, ''Feynman path integrals I. Linear and affine techniques. II. The Feynman Green function,'' Commun. Math. Phys. **37**, 63–81 (1973).
[42] N. Bourbaki, *Intégration*, (Hermann, Paris, 1969), Chap. IX.
[43] P. Krée, ''Introduction aux théories des distributions en dimension infinie,'' Bull. Soc. Math. France **46**, 143–162 (1976), and references therein, in particular, Seminar P. Lelong, Springer-Verlag Lecture Notes in Mathematics, Vols. 410 and 474 (Springer-Verlag, Berlin, 1972–1974).
[44] C. DeWitt-Morette, A. Maheswari, and B. Nelson, ''Path integration in non-relativistic quantum mechanics,'' Phys. Rep. **50**, 266–372 (1979).
[45] P. Cartier and C. DeWitt-Morette, ''A new perspective on functional integration,'' J. Math. Phys. **36**, 2237–2312 (1995). (Located on the World Wide Web at http://godel.ph.utexas.edu/Center/Papers.html and http://babbage.sissa.it/list/funct-an/9602.)
[46] C. DeWitt-Morette, ''Feynman path integrals. From the Prodistribution Definition to the Calculation of Glory Scattering,'' in *Stochastic Methods and Computer Techniques in Quantum Dynamics*, edited by H. Mitter and L. Pittner, Acta Phys. Austriaca, Suppl. **26**, 101–170 (1984). Reviewed in Zentralblatt für Mathematik 1985.
[47] P. Cartier and C. DeWitt-Morette, ''Physics on and near caustics,'' *NATO–ASI Proceedings*, ''Functional Integration: Basics and Applications'' (Cargèse 1996, Plenum, NY, 1997).
[48] P. Cartier and C. DeWitt-Morette, ''Physics on and near caustics. A simpler version,'' RCTP (Jagna, January 1998).
[49] See Kit Foong, ''Functional integration and wave propagation,'' in Ref. 34, pp. 97–180.
[50] S. A. Albeverio and R. J. Hoegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Springer-Verlag Lecture Notes in Mathematics, Vol. 523 (Springer-Verlag, Berlin, 1974–1976).
[51] Continuous, first derivative (in the sense of distributions) square integrable, one fixed end point x_0 either at t_a, or at t_b, or at any $t_0 \in [t_a, t_b]$.

[52] K. D. Elworthy, *Stochastic Differential Equations on Manifolds* (Cambridge University Press, Cambridge, MA, 1982).
[53] D. Brydges, J. Dimock, and T. R. Hurd, ''A non-Gaussian fixed point for ϕ^4 in $4-\epsilon$ dimensions,'' Commun. Math. Phys. **198**, 111–156 (1998).
[54] C. Morette, ''On the definition and approximation of Feynman's path integral,'' Phys. Rev. **81**, 848–852 (1951).
[55] C. DeWitt-Morette, K. D. Elworthy, B. L. Nelson, and G. S. Sammelmann, ''A stochastic scheme for constructing solutions of the Schrödinger equation,'' Ann. Inst. Henri Poincaré, Sect. A **32**, 327–341 (1980).
[56] B. S. DeWitt, *Supermanifolds*, 2nd ed. (Cambridge University Press, Cambridge, MA, 1992), pp. 281 and 334.
[57] See Ref. 45, pp. 2260–2263.
[58] See Ref. 56, pp. 358.
[59] Note that G is not the Green function of Q_0+Q_V which is defined on Z_b, because G is boundary-adapted to paths with two fixed points.
[60] L. S. Schulman, ''Introduction to the Path Integral,'' in *Path Summation: Achievements and Goals*, edited by S. Lundquist, A. A. Ranfagni, V. Sa-yakanit, and L. S. Schulman (World Scientific, Singapore, 1987–1988), pp. 3–46.
[61] C. Grosche and F. Steiner, *Handbook of Feynman Path Integrals* (Springer-Verlag, Berlin, 1998).
[62] D. S. Freed and R. E. Gompf, ''Computer calculation of Witten's three-manifold invariant,'' Commun. Math. Phys. **147**, 563–604 (1992).
[63] L. H. Kauffman, ''Witten's Integral and the Kontsevich Integral'' (preprint).
[64] M. Blau and G. Thompson, ''Localization and diagonalization: A review of functional integral techniques for low-dimensional gauge theories and topological field theories,'' J. Math. Phys. **36**, 2192–2236 (1995).